数学分析习题集题解

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（五）

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出版说明

吉米多维奇(В.П.ДЕМИДОВИЧ)著《数学分析习题集》一书的中译本，自五十年代初在我国翻译出版以来，引起了全国各大专院校广大师生的巨大反响。凡从事数学分析教学的师生，常以该校该习题集中的习题，作为检验掌握数学分析基础知识和基本技能的一项重要手段。二十多年来，对我因数学分析的教学工作是甚为有益的。

该书四十多道习题，数量多，内容丰富，由浅入深，部分题目难度大。涉及的内容有函数与极限，单变量函数的微分学，不定积分，定积分，级数，多变量函数的微分学，带参变量积分，以及积分与曲线积分，曲面积分等等，概括了数学分析的全部主题。当前，我国广大读者，特别是专心刻苦自学的广大数学爱好者，在为四个现代化而勤奋学习的热涟中，迫切需要对一些疑难习题有一个较明确的回答。有鉴于此，我们特约作者，将全书4462题的所有解答汇释成书，共分六册出版。本书可以作为高等院校的教学参考用书，同时也可作为广大读者在自学微积分过程中的参考用书。

众所周知，原习题集题多难度大，其中不少习题如果认真习作的话，既可以大大地巩固我们所学到的基本概念，又可以有效地提高我们的运算能力，特别是有些难题还可以促使我们学会综合分析的思维方法。正由于这样，我们殷切期望初学数学分析的青年读者，一定要刻苦钻研，千万不要轻易
查抄本书的解答，因为任何削弱独立思索的作法，都是违背我们出版此书的本意。何况所作解答并非一定标准，仅作参考而已。如有某些误解、差错也在所难免，一经发觉，恳请指正，不胜感谢。

本书蒙潘承洞教授对部分难题进行了审校。特请郭大钧教授、邵品琮教授对全书作了重要仔细的审校。其中相当数量的难度大的题，都是郭大钧、邵品琮亲自作的解答。

参加本册审校工作的还有张乾先、徐沅同志。

参加编审工作的还有黄春潮同志。

本书在编审过程中，还得到山东大学、山东工学院、山东师范学院和曲阜师范学院的领导和同志们的大力支持，特在此一并致谢。

1979年4月
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第六章  多变量函数的微分法

§1. 多变量函数的极限、连续性

1° 多变量函数的极限 设函数 \( f(P) = f(x_1, x_2, \ldots, x_n) \) 在以 \( P_0 \) 为聚点的集合 \( E \) 上有定义。若对于任何的 \( e > 0 \) 存在有 \( \delta = \delta(e, P_0) > 0 \)，使得只要 \( P \in E \) 及 \( 0 < \rho(P, P_0) < \delta \) (其中 \( \rho(P, P_0) \) 为 \( P \) 和 \( P_0 \) 二点间的距离)，则

\[
|f(P) - A| < e,
\]

我们就说

\[
\lim_{P \to P_0} f(P) = A.
\]

2° 连续性 若

\[
\lim_{P \to P_0} f(P) = f(P_0),
\]

则称函数 \( f(P) \) 于 \( P_0 \) 点是连续的。

若函数 \( f(P) \) 于已知域内的每一点连续，则称函数 \( f(P) \) 于此域内是连续的。

3° 一致连续性 若对于每一个 \( e > 0 \) 都存在有仅与 \( e \) 有关的 \( \delta > 0 \)，使得对于域 \( G \) 中的任何点 \( P', P'' \)，只要是

\[
\rho(P', P'') < \delta,
\]

便有不等式

\[
|f(P') - f(P'')| < e
\]

成立，则称函数 \( f(P) \) 于域 \( G \) 内是一致连续的。
于有界闭域内的连续函数于此域内是一致连续的。

确定并绘出下列函数存在的域。

3136. \( u = x + \sqrt{y} \).

解：存在域为半平面，
\( y \geq 0 \).

如图 6.1 阴影部分所示，包括整个 \( Ox \) 轴在内。

![图 6.1](image)

3137. \( u = \sqrt{1 - x^2} + \sqrt{y^2 - 1} \).

解：存在域为满足不等式
\( |x| \leq 1, |y| \geq 1 \)
的点集，如图 6.2 阴影部分所示，包括边界（粗实线）在内。

3138. \( u = \sqrt{1 - x^2 - y^2} \).

解：存在域为圆

![图 6.2](image)
\[ x^2 + y^2 \leq 1, \]

如图 6.3 阴影部分所示，包括圆周在内。

3139. \[ u = \frac{1}{\sqrt{x^2 + y^2 - 1}}. \]

解 存在域为满足不等式

\[ x^2 + y^2 \geq 1 \]

的点集，即圆 \( x^2 + y^2 = 1 \) 的外面，如图 6.4 所示，不包括圆周（虚线）在内。

3140. \[ u = \frac{1}{\sqrt{(x^2 + y^2 - 1)(4 - x^2 - y^2)}}. \]

解 存在域为满足不等式

\[ 1 \leq x^2 + y^2 \leq 4 \]

的点集，如图 6.5 所示的环，包括边界在内。

3141. \[ u = \sqrt{\frac{x^2 + y^2 - x}{2x - x^2 - y^2}}. \]

解 存在域为满足不等式

\[ x \leq x^2 + y^2 \leq 2x \]

的点集。由 \( x^2 + y^2 \)
得出

\[(x - \frac{1}{2})^2 + y^2 \geq \left(\frac{1}{2}\right)^2,\]

由 \(x^2 + y^2 < 2x\) 得出

\[(x - 1)^2 + y^2 < 1,\]

两者组成一月形，

如图 6·7 阴影部分所示。

3142. \(u = \sqrt{1 - (x^2 + y)^2}.\)

解 存在域为满足

不等式

\[-1 \leq x^2 + y \leq 1\]

的点集，如图 6·7 阴影部分所示，包括边界在内。

3143. \(u = \ln(-x - y).\)

解 存在域为半平面

\[x + y < 0,\]

如图 6·8 阴影部分所示，不包括直线 \(x + y = 0\) 在内。

3144. \(u = \arcsin\frac{y}{x}.\)

解 存在域为满足
不等式

\[ \left| \frac{y}{x} \right| \leq 1 \]

或 \( |y| \leq |x| \quad (x \neq 0) \)

的点集，这是一对对顶的直角，如图6.9阴影部分所示，不包括原点在内。

3.145. \( u = \arccos \frac{x}{x+y} \).

解 存在域为满足不等式

\[ \left| \frac{x}{x+y} \right| \leq 1 \]

的点集。由 \( \left| \frac{x}{x+y} \right| \leq 1 \) 得 \( x \leq |x+y| \quad (x \neq -y) \)，即 \( x^2 \leq x^2 + 2xy + y^2 \) 或 \( y(y+2x) \geq 0 \)，也即

\[
\begin{cases}
y \geq 0, \\
y \geq -2x,
\end{cases} \quad \text{或} \quad \begin{cases}
y \leq 0, \\
y \leq -2x.
\end{cases}
\]

但 \( x, y \) 不能同时为零。这是由直线：\( y = 0 \) 和 \( y = -2x \) 所围成的一对对顶的角，如图6.10阴影部分所示，包括边界在内，但不限于包括公共顶点 \( O (0,0) \) 在内。
3146. \( u = \arcsin \frac{x}{y^2} + \arcsin(1 - y) \).

解  存在域为满足不等式

\[
\left| \frac{x}{y^2} \right| \leq 1 \text{ 及 } |1 - y| \leq 1 \quad (y \neq 0)
\]

的点集，即

\[
\begin{align*}
\begin{cases}
y^2 \geq x, \\
0 < y \leq 2 
\end{cases}
\quad \text{和} \\
\begin{cases}
y^2 \geq -x, \\
0 < y \leq 2.
\end{cases}
\end{align*}
\]

这是由抛物线：

\( y^2 = x, \quad y^2 = -x \)

和 直线 \( y = 2 \) 所围成的曲边三角形。如图6.11阴影部分所示，不包括原点在内。

3147. \( u = \sqrt{\sin(x^2 + y^2)} \).

解  存在域为满足不等式

\[\sin(x^2 + y^2) \geq 0\]

或

\[2k\pi \leq x^2 + y^2 \leq (2k + 1)\pi \quad (k = 0, 1, 2, \ldots)\]

的点集，如图6.12所示的同心环簇。
3148. \( u = \arccos \frac{z}{\sqrt{x^2 + y^2}} \). 

解：存在域为满足不等式

\[
\left| \frac{z}{\sqrt{x^2 + y^2}} \right| \leq 1
\]

(\( x, y \) 不同时为零)

或

\[
x^2 + y^2 - z^2 \geq 0
\]

(\( x, y \) 不同时为零)

的点集，这是圆锥 \( x^2 + y^2 - z^2 = 0 \) 的外边，如图6.13 阴影部分所示，包括边界在内，但要除去圆锥的顶点。

3149. \( u = \ln(xy) \).

解：存在域为满足不等式

\[
xyz > 0
\]

的点集，即

\[
x > 0, \ y > 0, \ z > 0; \text{ 或 } x > 0, \ y < 0, \ z < 0;
\]

\[
x < 0, \ y < 0, \ z > 0; \text{ 或 } x < 0, \ y > 0, \ z < 0.
\]

其图形为空间第一、第三、第六及第八卦限的总体，但不包括坐标面。由于图形为读者所熟知，故省略。以下有类似情况，不再说明。

3150. \( u = \ln(-1 - x^2 - y^2 + z^2) \).
解 存在域为满足不等式
\[-x^2 - y^2 + z^2 \geq 1\]
的点集。这是双叶双曲面
\[x^2 + y^2 - z^2 = -1\]
的内部，如图6.14所示，包括界面以内。
作出下列函数的等位线:

3151. \( z = x + y \)。
解 等位线为平行直线族
\[ x + y = k, \]
其中 \( k \) 为一切实数，如图6.15所示。

3152. \( z = x^2 + y^2 \)。
解 等位线为曲线族
\[ x^2 + y^2 = a^2 \]
\[ (a \geq 0). \]
当 \( a = 0 \) 时为原点，当 \( a > 0 \) 时，等位线为以原点为圆心的同心圆族。

3153. \( z = x^2 - y^2 \)。
解 等位线为曲线族
\[ x^2 - y^2 = k. \]
当 \( k = 0 \) 时为两条互相垂直的直线: \( y = x, y = -x \).
当$k \neq 0$时为以$y = \pm x$为公共渐近线的等边双曲线族，
其中当$k > 0$时顶点为$(-\sqrt{k}, 0), (\sqrt{k}, 0)$，当$k < 0$时顶点为$(0, -\sqrt{-k}), (0, \sqrt{-k})$。

3154. $z = (x + y)^2$。
解 等位线为曲线族
\[(x + y)^2 = a^2 \quad (a \geq 0),\]
当$a = 0$为直线$x + y = 0$。当$a \neq 0$时为与直线$x + y = 0$平行的且等距的直线$x + y = \pm a$。

3155. $z = \frac{y}{x}$。
解 等位线为以坐标原点为束心的直线束
\[y = kx \quad (x \neq 0),\]
不包括$Oy$轴在内。

3156. $z = \frac{1}{x^2 + 2y^2}$。
解 等位线为椭圆族
\[x^2 + 2y^2 = a^2 \quad (a \geq 0),\]
长半轴为$a$，短半轴为$\frac{a}{\sqrt{2}}$，焦点为$(-a\sqrt{\frac{3}{2}}, 0)$及$(a\sqrt{\frac{3}{2}}, 0)$。

3157. $z = \sqrt{xy}$。
解 等位线为曲线族
\[xy = a^2 \quad (a \geq 0),\]
当$a = 0$时为坐标轴$x = 0$及$y = 0$。当$a \neq 0$时为以两坐标轴为公共渐近线且位于第一、第三象限内的等
3158. \( z = |x| + y \).

解 等位线为曲线族
\[ |x| + y = k, \]
其中 \( k \) 为一切实数，当 \( x \geq 0 \) 时为 \( x + y = k \);
当 \( x < 0 \) 时为 \( -x + y = k \)。这是顶点在 \( Oy \)
轴上两支互相垂直的射线所构成的折线族，如图6·16所示。

3159. \( z = |x| + |y| - |x+y| \).

解 等位线为曲线族
\[ |x| + |y| - |x+y| = a. \]
因为恒有 \( |x| + |y| \geq |x+y| \)，所以 \( a \geq 0 \)。
当 \( a = 0 \) 时，由 \( |x| + |y| = |x+y| \) 两边平方即得
\[ x^2 + y^2 = 0, \]
即为整个第一、第三象限，包括两坐标轴为在内。
当 \( a > 0 \) 时，\( xy < 0 \)，分下面四组求解：
(1) \( x > 0, y < 0, x + y \geq 0, |x| + |y| - |x+y| = a \)，解之得 \( y = -\frac{a}{2} \);
(2) \( x > 0, y < 0, x + y \leq 0, |x| + |y| - |x+y| = a \)，解之得 \( x = \frac{a}{2} \);
(3) \( x=0, \ y>0, \ x+y \geq 0, \ |x|+|y|-|x+y| = a \), 解之得 \( x = -\frac{a}{2} \);

(4) \( x<0, \ y>0, \ x+y \leq 0, \ |x|+|y|-|x+y| = a \), 解之得 \( y = \frac{a}{2} \).

这是顶点位于直线 \( x+y=0 \) 上的两支互相垂直的折线族，它的各射线平行于坐标轴，如图 6.17 所示。

3160. \( z = e^{\frac{2x}{x^2+y^2}} \).

解  等位线为曲线族

\[ \frac{2x}{x^2+y^2} = k \ (x, \ y \ 不同时为零) \]

其中 \( k \) 为异于零的一切实数，上式可变形为

\[ (x - \frac{1}{k})^2 + y^2 = \left( \frac{1}{k} \right)^2 \ (k \neq 0) \]

当 \( k=0 \) 时，即得 \( e^{\frac{2x}{x^2+y^2}} = 1 \)，从而等位线为 \( x=0 \) 即 \( Oy \) 轴，但不包括原点。
当 $k \neq 0$ 时为中心在 $Ox$ 轴上且经过坐标原点（但不包括原点在内）的圆束，圆心在 $(\frac{1}{k}, 0)$，半径为 $|\frac{1}{k}|$，如图6.18所示。

3161. $z = x^a (x > 0)$。
解 等位线为曲线族 $x^a = a (a > 0)$。图 6.18
当 $a = 1$ 时为直线 $x = 1$及 $Ox$ 轴的正向半射线，但不包括原点在内。
当 $0 < a < 1$ 与 $a > 1$ 时的图象如图6.19所示。

3162. $z = x^a e^{-x} (x > 0)$。
解 等位线为曲线族
$x^y e^{-x} = \alpha \quad (\alpha > 0)$，
即
$y \ln x - x = \ln \alpha$。
当 $a = e^{-1}$ 时为直线 $x = 1$
和曲线 $y = \frac{x - 1}{\ln x}$；当 $0 < a < \frac{1}{e}, \frac{1}{e} < a < 1$ 或 $a \geq 1$ 时
图象布满整个右半平面，
如图6·20所示，不包括 $Oy$轴。

3163.  $z = \ln \sqrt{\frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}} \quad (\alpha > 0)$。

解   等位线为曲线族

$$\frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = k^2 \quad (k > 0).$$

整理得

$$(1-k^2)x^2 - 2a(1+k^2)x + (1-k^2)a^2 + (1-k^2)y^2 = 0.$$  
当 $k = 1$ 时得 $x = 0$，即 $Oy$ 轴。当 $k \neq 1$ 时，上述方程可变形为

$$\left[ x - \frac{a(1+k^2)}{1-k^2} \right]^2 + y^2 = \left( \frac{2ak}{1-k^2} \right)^2,$$

这是以点 $\left( \frac{a(1+k^2)}{1-k^2}, 0 \right)$ 为圆心，半径为 $\frac{2ak}{1-k^2}$
的圆族。当 \(0 < k < 1\) 时，圆分布在右半平面；当 \(k = 1\) 时，圆分布在左半平面。

如果注意到圆心与原点距离的平方为

\[
\left( \frac{a(1 + k^2)}{1 - k^2} \right)^2 = \frac{a^2((1 - k^2)^2 + 4k^2)}{(1 - k^2)^2} = a^2 + \left( \frac{2ak}{1 - k^2} \right)^2,
\]

即等位线圆族与圆 \(x^2 + y^2 = a^2\) 在交点处的半径互相垂直（或圆心距与两圆的半径构成直角三角形），便知等位线圆族与圆 \(x^2 + y^2 = a^2\) 成正交。如图6.21所示。

3164. \(z = \arctan \frac{2ay}{x^2 + y^2 - a^2} (a \geq 0)\)。

解 等位线为曲线族

\[
\frac{2ay}{x^2 + y^2 - a^2} = k,
\]

其中 \(k\) 为一切实数，但要除去点 \((-a, 0)\) 及 \((a, 0)\)。
当 \(k = 0\) 时，\(y = 0\)，即为 \(Ox\) 轴，但不包含上述两点；
当 \(k \neq 0\) 时，方程可变形为
\[x^2 + \left(y - \frac{a}{k^2}\right)^2 = a^2 \left(1 + \frac{1}{k^2}\right),\]

这是圆心在 Oy 轴上且经过点 \((-a,0)\) 及 \((a,0)\) 但不包括这两点在内的圆族，如图 6.22 所示。

§165. \(z = \text{sgn}(\sin x \sin y)\).

解 若 \(z = 0\)，则 \(\sin x \cdot \sin y = 0\)，此即直线族

\[x = m\pi \text{和} y = n\pi \ (m, n = 0, \pm 1, \pm 2, \cdots);\]

若 \(z = -1\) 或 \(z = 1\)，则 \(\sin x \cdot \sin y < 0\) 或 \(\sin x \cdot \sin y > 0\)，此即正方形系

\[m\pi < x < (m + 1)\pi, \ n\pi < y < (n + 1)\pi,\]

其中 \(z = (-1)^{n+m}\).

如图 6.23 所示，\(z = 0\) 时为图中网格外直；\(z = 1\) 为图中带斜线的正方形；
\(z = -1\) 为图中空白正方形，但后两者都不包括边界。

求下列函数的等位
面。
3166. \( u = x + y + z \).
解  等位面为平行平面族
\[ x + y + z = k, \]
其中 \( k \) 为一切实数。
3167. \( u = x^2 + y^2 + z^2 \).
解  等位面为中心在原点的同心球族
\[ x^2 + y^2 + z^2 = a^2 \quad (a \geq 0), \]
其中当 \( a = 0 \) 时即为原点。
3168. \( u = x^2 + y^2 - z^2 \).
解  当 \( u = 0 \) 时等位面为圆锥 \( x^2 + y^2 - z^2 = 0 \); 当
\( u > 0 \) 时等位面为单叶双曲面族 \( x^2 + y^2 - z^2 = a^2 \quad (a \geq 0) \); 当 \( u < 0 \) 时等位面为双叶双曲面族 \( -x^2 - y^2 + z^2 = a^2 \quad (a \geq 0) \).
3169. \( u = (x + y)^2 + z^2 \).
解  等位面为曲面簇
\[ (x + y)^2 + z^2 = a^2 \quad (a \geq 0). \]
当 \( a = 0 \) 时为 \( x + y = 0 \) 和 \( z = 0 \)。当 \( a > 0 \) 时作坐标变换
\[
\begin{align*}
x' &= x \cos \frac{\pi}{4} + y \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} (x + y), \\
y' &= -x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} (-x + y), \\
z' &= z,
\end{align*}
\]
这是旋转变换。在新坐标系中原等位面方程转化为
\[ 2x'^2 + z'^2 = a^2, \]
即
\[ \frac{x'^2}{a^2} + \frac{z'^2}{a^2} = 1. \]

这是以 $y'$ 轴为公共轴的椭圆柱面，母线的方向平行于 $y'$ 轴，准线为 $y' = 0$ 平面上的椭圆
\[ \frac{x'^2}{a^2} + \frac{z'^2}{a^2} = 1. \]

长半轴为 $a$ ($z'$ 轴方向)，短半轴为 $\frac{a}{\sqrt{2}}$ ($x'$ 轴方向)。

$y'$ 轴在新系 $O-x'y'z'$ 中的方程为
\[ \begin{cases} x' = 0, \\ z' = 0, \end{cases} \]

而在旧系 $O-xyz$ 中的方程为
\[ \begin{cases} x + y = 0, \\ z = 0, \end{cases} \]

即为所求的椭圆柱面族的公共对称轴。

3170. \( u = \text{sgn} \sin(x^2 + y^2 + z^2) \).

解 当 $u = 0$ 时等位面为球心在原点的同心球族
\[ x^2 + y^2 + z^2 = n\pi \quad (n = 0, 1, 2, \ldots) \].

当 $u = -1$ 或 $u = 1$ 时等位面为球层族
\[ n\pi < x^2 + y^2 + z^2 < (n + 1)\pi \quad (n = 0, 1, 2, \ldots) , \]
其中 $u = \langle -1 \rangle$.

根据曲面的已知方程研究其性质。

3171. $z = f(y - ax)$.

解 引入参数 $t, s$, 将曲面方程 $z = f(y - ax)$ 表成参数方程

$$
\begin{align*}
  x &= t, \\
  y &= at + s, \\
  z &= f(s).
\end{align*}
$$

今固定 $s$, 得到以 $t$ 为参数的直线方程, 其方向数为 $1, a, 0$. 因此, 曲面为以 $1, a, 0$ 为母线方向的一个柱面. 令 $t = 0$, 可得

$$
\begin{align*}
  x &= 0, \\
  y &= s, \quad \text{或} \quad x = 0, \\
  z &= f(s), \\
  \end{align*}
$$

这是 $x = 0$ 平面上的一条曲线, 也是柱面

$z = f(y - ax)$

的一条准线.

3172. $z = f(\sqrt{x^2 + y^2})$.

解 这是绕 $Oz$ 轴旋转的旋转曲面的标准形式. 令 $y = 0$, 得曲线

$$
\begin{align*}
  y &= 0, \\
  z &= f(x) \quad (x \geq 0),
\end{align*}
$$

它是旋转曲面的一条母线.

3173. $z = xf\left(\frac{y}{x}\right)$.
解  引入参数 $t, s$, 将曲面方程 $z = x f\left(\frac{y}{x}\right)$ 表成参数方程

$$
\begin{align*}
\begin{cases}
  x = t, \\
  y = st (t \neq 0), \\
  z = tf(s).
\end{cases}
\end{align*}
$$

今固定 $s$, 这是以 $t$ 为参数的一条过原点的直线。因此，所给曲面为顶点在原点的一锥面，但不包括原点在内。令 $t = 1$, 得曲线

$$
\begin{align*}
\begin{cases}
  x = 1, \\
  y = s, \\
  z = f(s),
\end{cases}
\end{align*}
\quad \text{或} \quad
\begin{align*}
\begin{cases}
  x = 1, \\
  y = f(y), \\
  z = f(s).
\end{cases}
\end{align*}
$$

这是 $x = 1$ 平面上的一条曲线，也是锥面 $z = x f\left(\frac{y}{x}\right)$ 的一条准线。

3174*，$z = f\left(\frac{y}{x}\right)$。

解  引入参数 $t, s$, 将曲面方程 $z = f\left(\frac{y}{x}\right)$ 表成参数方程

$$
\begin{align*}
\begin{cases}
  x = t, \\
  y = st, \\
  z = f(s),
\end{cases}
\end{align*}
$$

* 题号右上角“*”号表示题解答案与原习题集中译本所附答 案不一致，以后不再说明。中译本基本是按俄文第二版翻译的。俄文第二版中有一些错误已在俄文第三版中改正。
今固定 $s$，这是一条过点 $(0, 0, f(s))$ 的直线，方向数为 $1, s, 0$。因此，它与 $Oz$ 轴垂直，与 $Oxy$ 平面平行，且其方向与 $s$ 有关。从而得出，曲面 $z = f\left(\frac{y}{x}\right)$ 表示一个直纹面。一般说来，它既不是柱面，又不是锥面。令 $t = 1$，得到直纹面的一条准线

$$
\begin{cases}
  x = 1, \\
  z = f(y).
\end{cases}
$$

从此曲线上每一点引一条与 $Oz$ 轴垂直且相交的直线，这样的直线的全体，便构成由 $z = f\left(\frac{y}{x}\right)$ 所表示的直纹面。

3175. 作出函数

$$F(t) = f(\cos t, \sin t)$$

的图形，式中

$$f(x, y) = \begin{cases}
  1, & \text{若 } y \geq x, \\
  0, & \text{若 } y < x.
\end{cases}$$

解 按题设，当 $\sin t \geq \cos t$，即 $\frac{\pi}{4} + 2k\pi \leq t \leq \frac{5\pi}{4} + 2k\pi$ ($k = 0, \pm 1, \pm 2, \cdots$) 时，$F(t) = 1$；而当
\[
\sin t < \cos t, \quad \text{即} \quad -\frac{3}{4} \pi + 2k\pi < t < \frac{\pi}{4} + 2k\pi \quad \text{时}, \quad F(t) = 0. \quad \text{如图 6.24 所示.}
\]

3176. 若
\[
f(x, y) = \frac{2xy}{x^2 + y^2},
\]
求 \( f\left(1, \frac{y}{x}\right) \).

解 \quad f\left(1, \frac{y}{x}\right) = \frac{2 \cdot 1 \cdot \frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2} = \frac{2xy}{x^2 + y^2} = f(x, y).

3177. 若
\[
f\left(\frac{y}{x}\right) = \frac{\sqrt{x^2 + y^2}}{x} \quad (x > 0),
\]
求 \( f(x) \).

解 \quad 由 \( f\left(\frac{y}{x}\right) = \sqrt{1 + \left(\frac{y}{x}\right)^2} \) 知 \( f(x) = \sqrt{1 + x^2} \).

3178. 设
\[
z = \sqrt{y + f(\sqrt{x - 1})}.
\]
若当 \( y = 1 \) 时 \( z = x \), 求函数 \( f \) 和 \( z \).

解 \quad 因为当 \( y = 1 \) 时 \( z = x \), 所以
\[
f(\sqrt{x - 1}) = x - 1 = (\sqrt{x - 1})(\sqrt{x + 1}) = (\sqrt{x + 1})(\sqrt{x - 1} + 2),
\]
从而得
\[ f(t) = t(t+2) = t^2 + 2t, \]

且
\[ z = \sqrt{y} + x - 1 \quad (x > 0). \]

3179. 设
\[ z = x + y + f(x - y). \]
若当 \( y = 0 \) 时，\( z = x^2 \)，求函数 \( f \) 及 \( z \)。
解 因为当 \( y = 0 \) 时 \( z = x^2 \)，所以
\[ x^2 = x + f(x), \]
即
\[ f(x) = x^2 - x. \]
且
\[ z = x + y + (x - y)^2 - (x - y) = 2y + (x - y)^2. \]

3180. 若 \( f(x + y, \frac{y}{x}) = x^2 - y^2 \)，求 \( f(x, y) \)。
解 因为
\[ f(x + y, \frac{y}{x}) = x^2 - y^2 = (x + y)(x - y) \]
\[ = (x + y)^2 \frac{x - y}{x + y} = (x + y)^2 \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}, \]
所以
\[ f(x, y) = x^2 \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}. \]

3181. 证明：对于函数
\[ f(x, y) = \frac{x - y}{x + y} \]
由
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = 1; \quad \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = -1, \]
从而 \( \lim_{x \to 0} f(x,y) \)不存在。
证
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x-y}{x+y} \right\} = \lim_{x \to 0} \frac{x}{x} = 1, \]
\[ \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x,y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x-y}{x+y} \right\} \]
\[ = \lim_{y \to 0} \frac{-y}{y} = -1. \]
由于两个单极限都存在，而累次极限不等，故 \( \lim f(x,y) \)不存在。

3182. 证明：对于函数
\[ f(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} \]
有
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x,y) \right\} = 0, \]
然而 \( \lim f(x,y) \)不存在。
证
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} \right\} \]
\[ = \lim_{x \to 0} 0 = 0, \]
\[
\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x^2 y^2}{x^2 y^2 + (x - y)^2} \right\} = \lim_{x \to 0} 0 = 0.
\]

如果按 \( y = kx \to 0 \) 的方向取极限，则有

\[
\lim_{x \to 0} f(x, y) = \lim_{x \to 0} \frac{x^4 k^2}{x^4 k^2 + x^2 (1 - k)^2}.
\]

特别地，分别取 \( k = 0 \) 及 \( k = 1 \)，便得到不同的极限 0 及 1。因此，\( \lim_{x \to 0} f(x, y) \)不存在。

3183. 证明：对于函数

\[
f(x, y) = (x + y) \sin \frac{1}{x} \sin \frac{1}{y}
\]

累次极限

\[
\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} 和 \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\}
\]

不存在，然而 \( \lim_{x \to 0} f(x, y) = 0 \)。

证：由不等式

\[
0 \leq \left| (x + y) \sin \frac{1}{x} \sin \frac{1}{y} \right| \leq |x + y| \leq |x| + |y|
\]

知 \( \lim_{x \to 0} f(x, y) = 0 \)。

但当 \( x \neq \frac{1}{k \pi} \)，\( y \to 0 \) 时，\( (x + y) \sin \frac{1}{x} \sin \frac{1}{y} \) 的极限不存在，因此累次极限

\[
\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}
\]

不存在，同法可证累次极限

\[
\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}
\]

也不存在。

3184. 求

\[
\lim_{x \to a} \left\{ \lim_{y \to b} f(x, y) \right\} 及 \lim_{y \to b} \left\{ \lim_{x \to a} f(x, y) \right\}
\]

设：
(a) \( f(x, y) = \frac{x^2 + y^2}{x^2 + y^4} \), \( a = \infty \), \( b = \infty \);

(\text{b}) \( f(x, y) = \frac{x^b}{1 + x^a} \), \( a = + \infty \), \( b = + 0 \);

(\text{c}) \( f(x, y) = \sin \frac{\pi x}{2x + y} \), \( a = \infty \), \( b = \infty \);

(\text{d}) \( f(x, y) = \frac{1}{xy} - \frac{x^y}{1 + x^y} \), \( a = 0 \), \( b = \infty \);

(\text{e}) \( f(x, y) = \log_x (x + y) \), \( a = 1 \), \( b = 0 \).

解

(a) \( \lim_{x \to \infty} \left( \lim_{y \to \infty} f(x, y) \right) = \lim_{x \to \infty} \left( \lim_{y \to \infty} \frac{x^2 + y^2}{x^2 + y^4} \right) \)

= \lim_{x \to \infty} 0 = 0 ,

\( \lim_{y \to \infty} \left( \lim_{x \to \infty} f(x, y) \right) = \lim_{y \to \infty} \left( \lim_{x \to \infty} \frac{x^2 + y^2}{x^2 + y^4} \right) \)

= \lim_{y \to \infty} 1 = 1 ;

(b) \( \lim_{x \to +\infty} \left( \lim_{y \to 0} f(x, y) \right) = \lim_{x \to +\infty} \left( \lim_{y \to 0} \frac{x^y}{1 + x^y} \right) \)

= \lim_{x \to +\infty} \frac{1}{2} = \frac{1}{2} ,

\( \lim_{y \to 0} \left( \lim_{x \to +\infty} f(x, y) \right) = \lim_{y \to 0} \left( \lim_{x \to +\infty} \frac{x^y}{1 + x^y} \right) \)

= \lim_{y \to 0} 1 = 1 ;

(\text{c}) \( \lim_{x \to \infty} \left( \lim_{y \to \infty} f(x, y) \right) = \lim_{x \to \infty} \left( \lim_{y \to \infty} \sin \frac{\pi x}{2x + y} \right) \)

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\[
\lim_{x \to \infty} 0 = 0 ,
\]
\[
\lim_{y \to \infty} \left\{ \lim_{x \to \infty} f(x, y) \right\} = \lim_{y \to \infty} \left\{ \lim_{x \to \infty} \sin \frac{\pi x}{2x+y} \right\}
\]
\[
= \lim_{y \to \infty} 1 = 1 ;
\]
\[
\lim_{x \to 0} \left\{ \lim_{y \to \infty} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to \infty} \frac{1}{\tan \frac{xy}{1+xy}} \right\}
\]
\[
= \lim_{x \to 0} \left\{ \frac{1}{\tan \frac{xy}{1+xy}} \right\} = 0 ,
\]
\[
\lim_{y \to \infty} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to \infty} \left\{ \lim_{x \to 0} \tan \frac{xy}{1+xy} \right\}
\]
\[
= \lim_{y \to \infty} \left\{ \frac{1}{\tan \frac{xy}{1+xy}} \cdot \lim_{x \to 0} \tan \frac{xy}{1+xy} \right\} = 1 
\]
\[
\lim_{y \to \infty} 1 = 1 ;
\]
\[
\lim_{x \to 1} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 1} \left\{ \lim_{y \to 0} \log_{x} (x+y) \right\}
\]
\[
= \lim_{x \to 1} \left\{ \lim_{y \to 0} \frac{\ln(x+y)}{\ln x} \right\} = \lim_{x \to 1} \frac{\ln x}{\ln x} = 1 ,
\]
\[
\lim_{y \to 0} \left\{ \lim_{x \to 1} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 1} \frac{\ln(x+y)}{\ln x} \right\} = \infty .
\]

求下列极限；

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3185. \( \lim_{x \to 0} \frac{x + y}{x^2 - xy + y^2} \).

解 由不等式 \( x^2 + y^2 \geq 2 |xy| \) 得

\[ 0 \leq \left| \frac{x + y}{x^2 - xy + y^2} \right| \leq \frac{|x + y|}{x^2 + y^2 - |xy|} \leq \frac{|x + y|}{|xy|} \leq \frac{1}{|x|} + \frac{1}{|y|} , \]

而 \( \lim_{x \to 0} \left( \frac{1}{|x|} + \frac{1}{|y|} \right) = 0 \)，故有

\[ \lim_{x \to 0} \frac{x + y}{x^2 - xy + y^2} = 0 . \]

3186. \( \lim_{x \to \infty} \frac{x^2 + y^2}{x^4 + y^4} \).

解 由不等式

\[ 0 \leq \frac{x^2 + y^2}{x^4 + y^4} \leq \frac{x^2 + y^2}{2x^2y^2} = \frac{1}{2} \left( \frac{1}{x^2} + \frac{1}{y^2} \right) \]

及 \( \lim_{x \to \infty} \frac{1}{2} \left( \frac{1}{x^2} + \frac{1}{y^2} \right) = 0 \)，即得

\[ \lim_{x \to \infty} \frac{x^2 + y^2}{x^4 + y^4} = 0 . \]

3187. \( \lim_{x \to 0} \frac{\sin xy}{x} \).

解 \( \lim_{x \to 0} \frac{\sin xy}{x} = \lim_{x \to 0} \left( \frac{\sin xy}{xy} \cdot y \right) = a . \)
3188. \[ \lim_{y \to +\infty} \frac{x^2 + y^2}{e^{x+y}}. \]

解
\[ \lim_{y \to +\infty} \frac{x^2 + y^2}{e^{x+y}} = \lim_{x \to +\infty} \left[ \frac{(x+y)^3}{e^{x+y}} - 2 \cdot \frac{x}{e^x} \cdot \frac{y}{e^y} \right] = 0. \]

**）利用564题的结果。

3189. \[ \lim_{y \to +\infty} \left( \frac{xy}{x^2 + y^2} \right)^2. \]

解
由不等式
\[ 0 \leq \left( \frac{xy}{x^2 + y^2} \right)^2 \leq \left( \frac{1}{x^2} \right)^2 \]

及 \[ \lim_{x \to +\infty} \left( \frac{1}{x^2} \right)^2 = 0, \]

即得
\[ \lim_{x \to +\infty} \left( \frac{xy}{x^2 + y^2} \right)^2 = 0. \]

3190. \[ \lim_{y \to 0} \frac{x^2 + y^2}{x^2 y^2}. \]

解
由不等式
\[ |x^2 y^2 \ln(x^2 + y^2)| \leq \frac{(x^2 + y^2)^2}{4} |\ln(x^2 + y^2)| \]

及 \[ \lim_{y \to 0} \frac{(x^2 + y^2)^2}{4} \ln(x^2 + y^2) = \lim_{t \to 0} \frac{1}{4} t^2 \ln(t) = 0, \]

即得
\[ \lim_{y \to 0} \frac{x^2 + y^2}{x^2 y^2} = \lim_{y \to 0} \frac{x^2 y^2 \ln(x^2 + y^2)}{x^2 y^2} = e^0 = 1. \]
3191. \( \lim_{x \to a} \left(1 + \frac{1}{x}\right)^{\frac{x^2}{x+y}} \).

解
\[
\lim_{x \to a} \left(1 + \frac{1}{x}\right)^{\frac{x^2}{x+y}} = \lim_{x \to a} \left(1 + \frac{1}{x}\right)^{\frac{x}{x+y}}
= \lim_{x \to a} e^{\frac{x}{x+y}}
= \lim_{x \to a} \left[ \lim_{x \to a} \ln\left(1 + \frac{1}{x}\right) \right] \cdot \lim_{x \to a} \frac{x}{x+y}
= e^1 \cdot 1 = e.
\]

3192. \( \lim_{\rho \to 0} \frac{\ln(\rho + \rho e^\rho)}{\sqrt{x^2 + y^2}} \).

解
\[
\lim_{\rho \to 0} \frac{\ln(\rho + \rho e^\rho)}{\sqrt{x^2 + y^2}} = \frac{\ln(1 + e^0)}{1} = \ln 2.
\]

3193. 若 \( x = \rho \cos \varphi \), \( y = \rho \sin \varphi \), 问下列极限沿怎样的方向 \( \varphi \) 有确定的极限值存在:

(a) \( \lim_{\rho \to 0} e^{\frac{x}{x^2 + y^2}}; \)  (b) \( \lim_{\rho \to \infty} e^{\frac{x^2 - y^2}{2} \sin 2x y} \).

解
(a) \( \lim_{\rho \to 0} e^{\frac{x}{x^2 + y^2}} = \lim_{\rho \to 0} e^{\frac{\cos \varphi}{\rho}} = \begin{cases} 0, & \text{当} \cos \varphi \leq 0; \\ 1, & \text{当} \cos \varphi = 0; \\ +\infty, & \text{当} \cos \varphi > 0. \end{cases} \)

于是，仅当 \( \cos \varphi \leq 0 \) 即 \( \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \) 时，所给的极限
才有确定的值。
(6) \(e^{x^2-y^2} \sin 2xy = e^{\rho^2 \cos^2 \varphi} \sin (\rho^2 \sin 2\varphi)\)。

当 \(\rho \to +\infty\) 时，\(\sin (\rho^2 \sin 2\varphi)\) 有界，除 \(\varphi = \frac{k\pi}{2}\) \((k = 0, 1, 2, 3)\) 外无极限，且

\[
\lim_{\rho \to +\infty} e^{\rho^2 \cos 2\varphi} = \begin{cases} 
0, & \text{当} \cos 2\varphi < 0; \\
1, & \text{当} \cos 2\varphi = 0; \\
+\infty, & \text{当} \cos 2\varphi > 0.
\end{cases}
\]

于是，仅当 \(\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}\) 及 \(\frac{5\pi}{4} \leq \varphi \leq \frac{7\pi}{4}\) 以及 \(\varphi = 0, \varphi = \pi\) 时才有确定的极限。

求下列函数的不连续点：

3194. \(u = \frac{1}{\sqrt{x^2 + y^2}}\).

解  函数 \(u = \frac{1}{\sqrt{x^2 + y^2}}\) 在点 \((0, 0)\) 无定义，故原点 \((0, 0)\) 为此函数的不连续点。以下各题类似情况，不再说明。

3195. \(u = \frac{xy}{x+y}\).

解  直线 \(x+y = 0\) 上的一切点均为 \(u = \frac{xy}{x+y}\) 的不连续点。

3196. \(u = \frac{x+y}{x^8 + y^8}\).

解  对于任意不等于零的实数 \(a\)，有
\[
\lim_{(x,y)\to(0,0)} \frac{x+y}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{1}{x^2-xy+y^2} = \frac{1}{3a^2}.
\]

于是，对于直线 \(x+y=0\) 上除原点 \(O\) 外的一切点均为移去的不连续点。而原点 \(O(0,0)\) 为无穷型不连续点。

3197. \(u = \sin \frac{1}{xy}\).

解 \(xy = 0\) 上的一切点即两坐标轴上的诸点均为 \(u = \sin \frac{1}{xy}\) 的不连续点。

3198. \(u = \frac{1}{\sin x \sin y}\).

解 直线 \(x = m\pi\) 及 \(y = n\pi\) \((m, n = 0, \pm 1, \pm 2, \ldots)\) 上的各点均为 \(u = \frac{1}{\sin x \sin y}\) 的不连续点。

3199. \(u = \ln(1-x^2-y^2)\).

解 圆周 \(x^2+y^2 = 1\) 上各点是 \(u = \ln(1-x^2-y^2)\) 的不连续点。

3200. \(u = \frac{1}{xyz}\).

解 坐标面：\(x = 0, y = 0, z = 0\) 上各点均为 \(u = \frac{1}{xyz}\) 的不连续点。

3201. \(u = \ln \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}\).
解点 \((a, b, c)\) 为 \(u = \ln \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}\) 的不连续点。

3202. 证明：函数

\[ f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & \text{若 } x^2 + y^2 \neq 0; \\ 0, & \text{若 } x^2 + y^2 = 0, \end{cases} \]

分别对于每一个变数 \(x\) 或 \(y\) (当另一变数的值固定时) 是连续的，但并非对这些变数的总体是连续的。

证 先固定 \(y = a \neq 0\)，则得 \(x\) 的函数

\[ g(x) = f(x, a) = \begin{cases} \frac{2ax}{x^2 + a^2}, & x \neq 0; \\ 0, & x = 0, \end{cases} \]

即 \(g(x) = \frac{2ax}{x^2 + a^2}\) （\(-\infty < x < +\infty\)），它是处处有定义的有理函数。又当 \(y = 0\) 时，\(f(x, 0) = 0\)，它显然是连续的。于是，当变数 \(y\) 固定时，函数 \(f(x, y)\) 对于变数 \(x\) 是连续的。同理可证，当变数 \(x\) 固定时，函数 \(f(x, y)\) 对于变数 \(y\) 是连续的。

作为二元函数，\(f(x, y)\) 虽在除点 \((0, 0)\) 外的各点均连续，但在点 \((0, 0)\) 不连续。事实上，当动点 \(P(x, y)\) 沿射线 \(y = kx\) 趋于原点时，有

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{2kx^2}{x^2(1 + k^2)} = \frac{2k}{1 + k^2}, \]

对于不同的 \(k\) 可得不同的极限值，从而知 \(\lim_{(x, y) \to (0, 0)} f(x, y)\) 不存在。因此，函数 \(f(x, y)\) 在原点不是二元连续。
3203. 证明：函数

\[ f(x, y) = \begin{cases} \frac{-x^2y}{x^4 + y^2}, & \text{若} x^2 + y^2 \neq 0, \\ 0, & \text{若} x^2 + y^2 = 0, \end{cases} \]

在点 \(O(0,0)\) 沿着过此点的每一射线

\[ x = t \cos \alpha, \quad y = t \sin \alpha \quad (0 \leq t < +\infty) \]

连续，即

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0,0); \]

但此函数在点 \((0,0)\) 并非连续的。

证 当 \(\sin \alpha = 0\) 时，\(\cos \alpha = 1\) 或 \(-1\)。于是，当 \(t \neq 0\) 时，\(f(t \cos \alpha, t \sin \alpha) = \frac{t^2 \cdot 0}{t^4 + 0} = 0\)，而 \(f(0,0) = 0\)，故有

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0,0). \]

当 \(\sin \alpha \neq 0\) 时，有

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = \lim_{t \to 0} \frac{t^3 \cos^2 \alpha \sin \alpha}{t^4 \cos^4 \alpha + t^2 \sin^2 \alpha} \]

\[ = \lim_{t \to 0} \frac{t \cos^2 \alpha \sin \alpha}{t^2 \cos^4 \alpha + \sin^2 \alpha} = 0, \]

故

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0,0). \]

其次，设动点 \(P(x, y)\) 沿抛物线 \(y = x^2\) 趋于原点，得

\[ \lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \neq f(0,0). \]

因此，函数 \(f(x, y)\) 在点 \((0,0)\) 不连续。
证明：函数

\[ f(x, y) = x \sin \frac{1}{y}, \text{若 } y \neq 0 \text{ 及 } f(x, 0) = 0 \]

的不连续点的集合不是封闭的。

证 当 \( y_0 \neq 0 \) 时，函数 \( f(x, y) \) 在点 \((x_0, y_0)\) 是连续的，即 \( f(x, y) \) 在除去 \( O x \) 轴以外的一切点均连续。

又因 \(|f(x, y) - f(0, 0)| = |f(x, y)| \leq |x|\)，故知

\( f(x, y) \) 在原点也是连续的。

考虑当 \( x_0 \neq 0 \) 时，对于点 \((x_0, 0)\)，由于极限

\[ \lim_{y \to 0} f(x_0, y) = \lim_{y \to 0} x_0 \sin \frac{1}{y} \]

不存在，故知 \( f(x, y) \) 在点 \((x_0, 0)\) 不连续。

这样，我们证明了，函数 \( f(x, y) \) 的全部不连续点为 \( O x \) 轴上除去原点外的一切点。显然，原点是不连续点集合的一个聚点，但它本身却不是 \( f(x, y) \) 的不连续点。因此，\( f(x, y) \) 的不连续点的集合不是封闭的。

3205. 证明：若函数 \( f(x, y) \) 在某域 \( G \) 内对变数 \( x \) 是连续的，而关于 \( x \) 对变数 \( y \) 是一致连续的，则此函数在所考虑的域内是连续的。

证 任意固定一点 \( P_0(x_0, y_0) \in G \)。

由于 \( f(x, y) \) 关于 \( x \) 对变数 \( y \) 一致连续，故对任给的 \( \varepsilon > 0 \)，存在 \( \delta_1 = \delta_1(\varepsilon) > 0 \)，使当 \((x, y') \in G, (x, y'') \in G \) 且 \(|y' - y''| \leq \delta_1\) 时，就有

\[ |f(x, y') - f(x, y'')| \leq \frac{\varepsilon}{2} \]
又因 \( f(x, y) \) 在点 \((x_0, y_0)\) 关于变量 \(x\) 是连续的，
故对上述的 \(\varepsilon\)，存在 \(\delta_2 > 0\)，使当 \(|x - x_0| < \delta_2\) 时，
就有

\[
|f(x, y_0) - f(x_0, y_0)| < \frac{\varepsilon}{2}.
\]

取 \(0 < \delta \leq \min \{\delta_1, \delta_2\}\)，并使点 \((x_0, y_0)\) 的 \(\delta\) 邻域
全部包含在区域 \(G\) 内，则当点 \(P(x, y)\) 属于点 \((x_0, y_0)\)
的 \(\delta\) 邻域，即 \(|PP_0| < \delta\) 时，
\(|x - x_0| < \delta \leq \delta_2，|y - y_0| < \delta \leq \delta_1\).

从而有

\[
|f(x, y) - f(x_0, y_0)| \leq |f(x, y) - f(x, y_0)|
+ |f(x, y_0) - f(x_0, y_0)|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

因此，\(f(x, y)\) 在点 \(P_0\) 连续。由 \(P_0\) 的任意性知，函数
\(f(x, y)\) 在 \(G\) 内是连续的。

3206. 证明：若在某域 \(G\) 内函数 \(f(x, y)\) 对变量 \(x\) 是连续的，
并满足对变量 \(y\) 的里普什兹条件，即

\[
|f(x, y') - f(x, y'')| \leq L |y' - y''|,
\]

式中 \((x, y') \in G, (x, y'') \in G\) 而 \(L\) 为常数，则此函数在
已知域内是连续的。

证 由于 \(f(x, y)\) 在 \(G\) 内满足对 \(y\) 的里普什兹条件，
故知 \(f(x, y)\) 在 \(G\) 内关于 \(x\) 对变量 \(y\) 是一致连续的。
因此，由 3205 题的结果，即知 \(f(x, y)\) 在 \(G\) 内是连续的。

3207. 证明：若函数 \(f(x, y)\) 分别地对每一个变量 \(x\) 和 \(y\) 是
证明不仿设 \( f(x, y) \) 关于 \( x \) 是单调的。

设 \( (x_0, y_0) \) 为函数 \( f(x, y) \) 的定义域 \( G \) 内的任一点。由于 \( f(x, y) \) 关于 \( x \) 连续，故对任给的 \( \varepsilon > 0 \)，存在 \( \delta_1 > 0 \)（假定 \( \delta_1 \) 足够小，使我们所考虑的点都在 \( G \) 内），使当 \( |x - x_0| \leq \delta_1 \) 时，就有

\[
|f(x, y_0) - f(x_0, y_0)| \leq \varepsilon \frac{1}{2}.
\]

对于点 \( (x_0 - \delta_1, y_0) \) 及 \( (x_0 + \delta_1, y_0) \)，由于 \( f(x, y) \) 关于 \( y \) 连续，故对上述的 \( \varepsilon \)，存在 \( \delta_2 > 0 \)（也要求 \( \delta_2 \) 足够小，使考虑的点都在 \( G \) 内），使当 \( |y - y_0| \leq \delta_2 \) 时，就有

\[
|f(x_0 - \delta_1, y) - f(x_0 - \delta_1, y_0)| \leq \varepsilon \frac{1}{2}.
\]

及

\[
|f(x_0 + \delta_1, y) - f(x_0 + \delta_1, y_0)| \leq \varepsilon \frac{1}{2}.
\]

令 \( \delta = \min \{\delta_1, \delta_2\} \)，则当 \( |\Delta x| \leq \delta, |\Delta y| \leq \delta \) 时，由于 \( f(x, y) \) 关于 \( x \) 单调，故有

\[
|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)| \\
\leq \max \{ |f(x_0 + \delta_1, y_0 + \Delta y) - f(x_0, y_0)|, \\
|f(x_0 - \delta_1, y_0 + \Delta y) - f(x_0, y_0)| \}.
\]

但是

\[
|f(x_0 \pm \delta_1, y_0 + \Delta y) - f(x_0, y_0)| \\
\leq |f(x_0 \pm \delta_1, y_0 + \Delta y) - f(x_0 \pm \delta_1, y_0)| \\
+ |f(x_0 \pm \delta_1, y_0) - f(x_0, y_0)|
\]
\[ \varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \]

故当 \(|\Delta x| \leq \delta, |\Delta y| \leq \delta\) 时，就有

\[ |f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)| \leq \varepsilon, \]

即 \(f(x, y)\) 在点 \((x_0, y_0)\) 是连续的。由点 \((x_0, y_0)\) 的任意性知，\(f(x, y)\) 是 \(G\) 内的二元连续函数。

3208. 设函数 \(f(x, y)\) 于域 \(a \leq x \leq A, b \leq y \leq B\) 上是连续的，而函数列 \(\varphi_n(x)\) \((n = 1, 2, \ldots)\) 在 \([a, A]\) 上一致收敛并满足条件 \(b \leq \varphi_n(x) \leq B\)。证明：函数列

\[ F_n(x) = f(x, \varphi_n(x)) \]

也在 \([a, A]\) 上一致收敛。

证：由于 \(b \leq \varphi_n(x) \leq B\)，故 \(F_n(x) = f(x, \varphi_n(x))\) 有意义。

由题设 \(f(x, y)\) 在域 \(a \leq x \leq A, b \leq y \leq B\) 上连续，故在该域上一致连续，即对任给的 \(\varepsilon > 0\)，存在 \(\delta = \delta(\varepsilon) > 0\)，使对于此域中的任两点 \((x_1, y_1), (x_2, y_2)\)，只要 \(|x_1 - x_2| \leq \delta, |y_1 - y_2| \leq \delta\) 时，就有

\[ |f(x_1, y_1) - f(x_2, y_2)| \leq \varepsilon. \]

特别地，当 \(|y_1 - y_2| \leq \delta\) 时，对于一切的 \(x \in [a, A]\)，均有

\[ |f(x, y_1) - f(x, y_2)| \leq \varepsilon. \]

对于上述的 \(\delta > 0\)，因为 \(\varphi_n(x)\) 在 \([a, A]\) 上一致收敛，故存在自然数 \(N\)，使当 \(m > N, n > N\) 时，对于一切的 \(x \in [a, A]\)，均有

\[ |\varphi_n(x) - \varphi_m(x)| \leq \delta. \]

于是，对任给的 \(\varepsilon > 0\)，存在自然数 \(N\)，使当 \(m > N, n > N\) 时，对于一切的 \(x \in [a, A]\)，均有

\[ |f(x, \varphi_n(x)) - f(x, \varphi_m(x))| \leq \varepsilon. \]

\[ F_n(x) = f(x, \varphi_n(x)) \]

也在 \([a, A]\) 上一致收敛。
$N$, $n\geq N$时，对于一切的 $x \in [a, A)$，均有

$$|F_n(x) - F(x)| = |f(x, \varphi_n(x)) - f(x, \varphi(x))| \leq \varepsilon.$$ 

因此，$F_n(x)$在$(a, A)$上一致收敛。

3209. 设：1）函数 $f(x, y)$ 于域 $R(a \leq x < A; b \leq y < B)$ 内是连续的；2）函数 $\varphi(x)$ 于区间 $(a, A)$ 内连续并有属于区间 $(b, B)$ 内的值。证明：函数 $F(x) = f(x, \varphi(x))$ 于区间 $(a, A)$ 内是连续的。

证 设点 $(x_0, y_0)$ 为域 $R$ 中的任一点。由题设知函数 $f(x, y)$ 于域 $R$ 中连续，故对任给的 $\varepsilon > 0$，存在 $\delta > 0$，使得 $|x - x_0| < \delta, |y - y_0| < \delta((x, y) \in R)$ 时，就有

$$|f(x, y) - f(x_0, y_0)| \leq \varepsilon.$$ 

再由 $\varphi(x)$ 在 $(a, A)$ 中的连续性可知，对上述的 $\delta > 0$，存在 $\eta > 0$（可取 $\eta = \delta$），使当 $|x - x_0| < \eta$ $(x \in (a, A))$ 时，恒有

$$|\varphi(x) - \varphi(x_0)| = |y - y_0| < \delta.$$ 

于是，

$$|f(x, \varphi(x)) - f(x_0, \varphi(x_0))| \leq \varepsilon,$$

即

$$|F(x) - F(x_0)| \leq \varepsilon.$$ 

因此，$F(x)$ 在点 $x_0$ 处连续。由 $x_0$ 的任意性知函数 $F(x)$ 在 $(a, A)$ 内是连续的。

3210. 设：1）函数 $f(x, y)$ 于域 $R (a \leq x < A; b \leq y < B)$ 内是连续的；2）函数 $x = \varphi(u, v)$ 及 $y = \psi(u, v)$ 于域 $R'$
\( a' < u < A' \); \( b' < v < B' \)内是连续的并有分别属于区间\((a, A)\)和\((b, B)\)的值．证明：函数
\[ F(u, v) = f(\varphi(u, v), \psi(u, v)) \]
于域\( R' \)内连续．
证 以下假定所取的\( \delta \)或\( \eta \)足够小，使点的\( \delta \)或\( \eta \)邻域都在给的域内．
设点\((x_0, y_0)\)为域\( R \)中的任一点．由于\( f(x, y) \)在\( R \)内连续，故对任给的\( \varepsilon > 0 \)，存在\( \delta > 0 \)，使当
\[ |x - x_0| < \delta, \quad |y - y_0| < \delta \]
时，就有
\[ |f(x, y) - f(x_0, y_0)| < \varepsilon. \]
再由\( \varphi \)及\( \psi \)的连续性知，对于上述的\( \delta \)，存在\( \eta > 0 \)，使当\( |u - u_0| < \eta, \quad |v - v_0| < \eta \)时，就有
\[ |x - x_0| < \delta, \quad |y - y_0| < \delta, \]
其中\( x_0 = \varphi(u_0, v_0), \quad y_0 = \psi(u_0, v_0). \)
于是，对任给的\( \varepsilon > 0 \)，存在\( \eta > 0 \)，使当\( |u - u_0| \quad \eta, \quad |v - v_0| < \eta \)时，就有
\[ |f(\varphi(u, v), \psi(u, v)) - f(\varphi(u_0, v_0), \psi(u_0, v_0))| < \varepsilon, \]
即
\[ |F(u, v) - F(u_0, v_0)| < \varepsilon. \]
因此，\( F(u, v) \)在点\((u_0, v_0)\)连续，由\((u_0, v_0)\)的任意性知，函数\( F(u, v) \)于域\( R' \)内连续．

§2. 偏导函数．多变量函数的微分

1° 偏导函数 若所论及的多变数的函数的一切偏导函
数是连续的，则微分的结果与微分的次序无关。

2° 多变量函数的微分 若自变数 \( x, y, z \) 的函数 \( f(x, y, z) \) 的全增量可写为下形

\[
Af(x, y, z) = A\Delta x + B\Delta y + C\Delta z + o(\rho),
\]
式中 \( A, B, C \) 与 \( \Delta x, \Delta y, \Delta z \) 无关而 \( \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \)，则称函数 \( f(x, y, z) \) 可微分，而增量的线性主部 \( A\Delta x + B\Delta y + C\Delta z \) 等于

\[
df(x, y, z) = f'_x(x, y, z)dx + f'_y(x, y, z)dy + f'_z(x, y, z)dz,
\]
(其中 \( dx = \Delta x, dy = \Delta y, dz = \Delta z \)) 称为此函数的微分。

当变数 \( x, y, z \) 为其他自变数的可微分的函数时，公式(1)仍有其意义。

若 \( x, y, z \) 为自变数，则对于高阶的微分，有符号公式

\[
d^nf(x, y, z) = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^nf(x, y, z).
\]

3° 复合函数的导函数 若 \( w = f(x, y, z) \)，其中 \( x = \varphi(u, v) \)，\( y = \psi(u, v) \)，\( z = \chi(u, v) \) 且函数 \( \varphi, \psi, \chi \) 可微分，则

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u},
\]

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}.
\]

计算函数 \( w \) 的二阶导函数时最好用下列符号公式：

\[
\frac{\partial^2 w}{\partial u^2} = \left( P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z} \right)^2 w + \frac{\partial P_1}{\partial u} \frac{\partial w}{\partial x}
\]
\[ + \frac{\partial Q_1}{\partial u} \frac{\partial w}{\partial y} + \frac{\partial R_1}{\partial u} \frac{\partial w}{\partial z} \]

及 \[ \frac{\partial^2 w}{\partial u \partial v} = \left( P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z} \right) \left( P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} 
+ R_2 \frac{\partial}{\partial z} \right) w + \frac{\partial P_1}{\partial v} \frac{\partial w}{\partial x} + \frac{\partial Q_1}{\partial v} \frac{\partial w}{\partial y} + \frac{\partial R_1}{\partial v} \frac{\partial w}{\partial z}, \]

其中 \[ P_1 = \frac{\partial x}{\partial u}, \quad Q_1 = \frac{\partial y}{\partial u}, \quad R_1 = \frac{\partial z}{\partial u} \]
及 \[ R_2 = \frac{\partial x}{\partial v}, \quad Q_2 = \frac{\partial y}{\partial v}, \quad R_2 = \frac{\partial z}{\partial v}. \]

4° 在已知方向上的导函数 若用方向余弦 \( \{ \cos \alpha, \cos \beta, \cos \gamma \} \) 表示 \( Oxyz \) 空间内的方向 \( l \)，且函数 \( u = f(x, y, z) \) 可微分，则沿方向 \( l \) 的导函数按下式来计算：

\[ \frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma. \]

在已知点函数增加最迅速的速度之大小与方向用矢量——函数的梯度

\[ \text{grad } u = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} + \frac{\partial u}{\partial z} \mathbf{k} \]

来表示，它的大小等于

\[ |\text{grad } u| = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2}. \]

3211. 证明。
\[ f'_x(x, b) = \frac{d}{dx}[f(x, b)]. \]

证 令 \( \varphi(x) = f(x, b) \)，则
\[
\frac{d}{dx}[f(x, b)] = \varphi'(x) = \lim_{\Delta x \to 0} \frac{\varphi(x + \Delta x) - \varphi(x)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, b) - f(x, b)}{\Delta x} = f'_x(x, b).
\]

注 在求某一固定点的导数及微分时，用本题的结果常可减少运算量。在本节中，我们就多次利用本题的结果来简化运算。

3212. 设
\[
f(x, y) = x + (y - 1) \arcsin \sqrt{\frac{x}{y}},
\]
求 \( f'_x(x, 1) \)。
解 由于 \( f(x, 1) = x \)，故 \( f'_x(x, 1) = 1 \)。

3213. \( u = x^4 + y^4 - 4x^2y^2 \)。
解 \( \frac{\partial u}{\partial x} = 4x^3 - 8xy^2, \quad \frac{\partial u}{\partial y} = 4y^3 - 8x^2y, \)
\[
\frac{\partial^2 u}{\partial x^2} = 12x^2 - 8y^2, \quad \frac{\partial^2 u}{\partial y^2} = 12y^2 - 8x^2,
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -16xy.
\]
*）以下各题不再写 \( \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \)，而仅写 \( \frac{\partial^2 u}{\partial x \partial y} \)，
因为当它们连续时是相等的，并且在今后各题中均按
\[\frac{\partial^2 u}{\partial x \partial y} \text{ 理解为 } \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right).\]

3214. \( u = xy + \frac{x}{y} \).

解 \( \frac{\partial u}{\partial x} = y + \frac{1}{y}, \quad \frac{\partial u}{\partial y} = x - \frac{x}{y^2}, \)
\[\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 2y, \quad \frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{1}{y^2} \cdot\]

3215. \( u = \frac{x}{y^2} \).

解 \( \frac{\partial u}{\partial x} = \frac{1}{y^2}, \quad \frac{\partial u}{\partial y} = -\frac{2x}{y^3}, \)
\[\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{6x}{y^4}, \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{2}{y^3}.\]

3216. \( u = \frac{x}{\sqrt{x^2 + y^2}} \).

解 \( \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} - \frac{2x \cdot x}{2(x^2 + y^2)^{\frac{3}{2}}} = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}, \)
\[\frac{\partial u}{\partial y} = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}},\]
\[\frac{\partial^2 u}{\partial x^2} = -\frac{3}{2} \cdot \frac{y^2}{(x^2 + y^2)^{\frac{5}{2}}}, \quad \frac{2x}{(x^2 + y^2)^{\frac{5}{2}}} = -\frac{3xy^2}{(x^2 + y^2)^{\frac{5}{2}}}, \]
\[\frac{\partial^2 u}{\partial y^2} = -\frac{x}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{3}{2} xy \cdot \frac{2y}{(x^2 + y^2)^{\frac{6}{2}}} \]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y} \left[ \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}} \right] = 2\frac{y}{(x^2 + y^2)^{\frac{3}{2}}} - \frac{3y^3}{(x^2 + y^2)^{\frac{5}{2}}} = \frac{y(2x^2 - y^2)}{(x^2 + y^2)^{\frac{5}{2}}}
\]

3217. \( u = x \sin(x + y) \).

解

\[
\frac{\partial u}{\partial x} = \sin(x + y) + x \cos(x + y), \\
\frac{\partial u}{\partial y} = x \cos(x + y), \\
\frac{\partial^2 u}{\partial x^2} = \cos(x + y) + \cos(x + y) - x \sin(x + y) \\
= 2 \cos(x + y) - x \sin(x + y), \\
\frac{\partial^2 u}{\partial y^2} = -x \sin(x + y), \\
\frac{\partial^2 u}{\partial x \partial y} = \cos(x + y) - x \sin(x + y).
\]

3218. \( u = \frac{\cos x^2}{y} \).

解

\[
\frac{\partial u}{\partial x} = -\frac{2x \sin x^2}{y}, \quad \frac{\partial u}{\partial y} = -\frac{\cos x^2}{y^2}, \\
\frac{\partial^2 u}{\partial x^2} = -\frac{2 \sin x^2 + 4x^2 \cos x^2}{y},
\]
\[
\frac{\partial^2 u}{\partial y^2} = \frac{2\cos x^2}{y^3}, \\
\frac{\partial^2 u}{\partial x \partial y} = \frac{2x \sin x^2}{y^2}.
\]

3219. \( u = \tan \frac{x^2}{y} \).

解

\[
\frac{\partial u}{\partial x} = \frac{2x}{y} \sec^2 \frac{x^2}{y}, \quad \frac{\partial u}{\partial y} = \frac{x^2}{y^2} \sec \frac{x^2}{y},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{2}{y} \sec \frac{x^2}{y} + \frac{2x}{y} \sec^2 \frac{x^2}{y} \tan \frac{x^2}{y}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2}{y} \sec \frac{x^2}{y} + \frac{3x^2}{y^2} \sec \frac{x^2}{y} \sin \frac{x^2}{y},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{2x}{y^2} \sec \frac{x^2}{y} - \frac{4x^3}{y^3} \sec \frac{x^2}{y} \sin \frac{x^2}{y},
\]

3220. \( u = x^y \).

解 由 \( u = x^y = e^{y \ln x} \) 即得

\[
\frac{\partial u}{\partial x} = yx^{y-1}, \quad \frac{\partial u}{\partial y} = e^{y \ln x} \cdot \ln x = x^y \ln x,
\]

\[
\frac{\partial^2 u}{\partial x^2} = y(y-1)x^{y-2}, \quad \frac{\partial^2 u}{\partial y^2} = x^y \ln^2 x,
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + yx^{y-1} \ln x
\]
\[ u = \ln(x + y^2), \]

解
\[
\frac{\partial u}{\partial x} = \frac{1}{x + y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x + y^2},
\]
\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x + y^2)^2},
\]
\[
\frac{\partial^2 u}{\partial y^2} = -\frac{2y \cdot 2y}{(x + y^2)^2} = \frac{2(x - y^2)}{(x + y^2)^2},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{2y}{(x + y^2)^2}.
\]

3222. \[ u = \arctg \frac{y}{x}. \]

解
\[
\frac{\partial u}{\partial x} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( -\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2},
\]
\[
\frac{\partial u}{\partial y} = -\frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{1}{x} = -\frac{y}{x^2 + y^2},
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{x^2 + y^2} + \frac{y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.
\]

3223. \[ u = \arctg \frac{x + y}{1 - xy}. \]
解 由776题知

\[ \arctg \frac{x + y}{1 - xy} = \arctg x + \arctg y - e \pi, \]

其中 \( e = 0, 1 \) 或 \(-1\). 于是，

\[ \frac{\partial u}{\partial x} = \frac{1}{1 + x^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{1 + y^2}, \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{2x}{(1 + x^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{2y}{(1 + y^2)^2}, \]

\[ \frac{\partial^2 u}{\partial x \partial y} = 0. \]

本题如不用776题的结果，直接求导数也可获解。
例如，

\[ \frac{\partial u}{\partial x} = \frac{1}{1 + (\frac{x + y}{1 - xy})^2} \cdot \frac{1 - xy + y(x + y)}{(1 - xy)^2} \]

\[ = \frac{1}{1 + x^2}. \]

3224. \( u = \arcsin \frac{x}{\sqrt{x^2 + y^2}} \).

解 \( \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \cdot \frac{x}{\sqrt{x^2 + y^2}}, \)

\[ = \frac{\sqrt{x^2 + y^2}}{|y|} \cdot \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}. \]

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\[
\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2}} - \frac{x y}{(x^2 + y^2)^{3/2}}
\]

\[
= \sqrt{x^2 + y^2} \left[ - \frac{x y}{(x^2 + y^2)^{3/2}} \right]
\]

\[
= - \frac{x}{x^2 + y^2} \cdot \frac{y}{|y|} = - \frac{x \text{sgn} y}{x^2 + y^2},
\]

\[
\frac{\partial^2 u}{\partial x^2} = - \frac{2x |y|}{(x^2 + y^2)^{3/2}},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left[ - \frac{x y}{|y|(x^2 + y^2)} \right]
\]

\[
= - \frac{x |y|(x^2 + y^2) - x y \left( \frac{|y|}{y} \right) (x^2 + y^2) + 2 y |y|}{y^2 (x^2 + y^2)^2}
\]

\[
= \frac{2x |y|}{(x^2 + y^2)^{3/2}},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\frac{|y|}{y} (x^2 + y^2) - 2 y |y|}{(x^2 + y^2)^{3/2}}
\]

\[
= \frac{x^2 \text{sgn} y - y |y|}{(x^2 + y^2)^2} = \frac{(x^2 - y^2) \text{sgn} y}{(x^2 + y^2)^2} \quad (y \neq 0).
\]

\[
\]

\[
* \) 利用3216题的结果.
\]

3225. \( u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}. \)
解 \[
\frac{\partial u}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial u}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial u}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x^2 + y^2 + z^2)^{\frac{8}{3}}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{3}}}
\]
\[
= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{3xy}{(x^2 + y^2 + z^2)^{\frac{5}{3}}},
\]

利用对称性，即得

\[
\frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial^2 u}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{\frac{8}{3}}},
\]
\[
\frac{\partial^2 u}{\partial y \partial z} = \frac{3yz}{(x^2 + y^2 + z^2)^{\frac{5}{3}}},
\]
\[
\frac{\partial^2 u}{\partial z \partial x} = \frac{3xz}{(x^2 + y^2 + z^2)^{\frac{5}{3}}},
\]

3226. \( u = \left( \frac{x}{y} \right)^n \).

解 \( u = x^r y^{-r} \).
\[
\frac{\partial u}{\partial x} = z x^{z-1} y^{-z} = z \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial u}{\partial y} = -z x^z y^{-z-1} = -\frac{z}{y} \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial u}{\partial z} = \left( \frac{x}{y} \right)^z \ln \frac{x}{y} ,
\]
\[
\frac{\partial^2 u}{\partial x^2} = z(z-1) x^{z-2} y^{-z} = \frac{z(z-1)}{x^2} \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial^2 u}{\partial y^2} = (-z)(-z-1) x^z y^{-z-2} = \frac{z(z+1)}{y^2} \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial^2 u}{\partial z^2} = \left( \frac{x}{y} \right)^z \ln^2 \frac{x}{y} ,
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \left( \frac{z}{x} u \right)_z = \frac{z}{x} \left[ -\frac{z}{y} \left( \frac{x}{y} \right)^z \right] = -\frac{z^2}{xy} \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial^2 u}{\partial y \partial x} = (-z u)_x = -\frac{z}{y} \left( \frac{x}{y} \right)^z \ln \frac{x}{y} - \frac{1}{y} \left( \frac{x}{y} \right)^z ,
\]
\[
= -\frac{1 + z \ln \frac{x}{y}}{y} \left( \frac{x}{y} \right)^z ,
\]
\[
\frac{\partial^2 u}{\partial z \partial x} = \left( u \ln \frac{x}{y} \right)_x = \frac{z}{x} \left( \frac{x}{y} \right)^z \ln \frac{x}{y} + \frac{1}{x} \left( \frac{x}{y} \right)^z = \frac{1 + z \ln \frac{x}{y}}{x} \left( \frac{x}{y} \right)^z \left( \frac{x}{y} \gg 0 \right) .
\]
3227. \( u = x^y \).

解

\[
\frac{\partial u}{\partial x} = \frac{y x^{y-1}}{x^y} = yu, \\
\frac{\partial u}{\partial y} = \frac{1}{z} x^y \ln x = \frac{u \ln x}{z}, \\
\frac{\partial u}{\partial z} = -\frac{y}{z^2} x^y \ln x = -\frac{yu \ln x}{z^2},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{xy - yxu}{x^2 z^2} = \frac{y(y-z)u}{x^2 z^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \ln x \frac{\partial u}{\partial y} = \frac{u \ln x}{z^2},
\]

\[
\frac{\partial^2 u}{\partial z^2} = -yu \ln x \cdot \left( \frac{z^2}{z^4} - 2uz \right) = \frac{yu \ln x \cdot (2z + y \ln x)}{z^4},
\]

\[
\frac{\partial^3 u}{\partial x \partial y} = \frac{1}{xz} \left( u + y \frac{\partial u}{\partial y} \right) = \frac{u(z + y \ln x)}{xz^2},
\]

\[
\frac{\partial^2 u}{\partial y \partial z} = \ln x \left( \frac{1}{z} \frac{\partial u}{\partial z} - \frac{u}{z^2} \right) = -\frac{u \ln x \cdot (z + y \ln x)}{z^3},
\]

\[
\frac{\partial^2 u}{\partial z \partial x} = -\frac{y}{z^2} \left( \ln x \frac{\partial u}{\partial x} + \frac{u}{x} \right) = -\frac{yu(z + y \ln x)}{xy^3}.
\]
解

\[
\frac{\partial u}{\partial x} = y^z x^{y^z - 1} = \frac{uy^z}{x},
\]

\[
\frac{\partial u}{\partial y} = z y^{z-1} x^{y^z} \ln x = zu y^{z-1} \ln x,
\]

\[
\frac{\partial u}{\partial z} = x^{y^z} y^z \ln x \cdot \ln y = uy^z \ln x \cdot \ln y,
\]

\[
\frac{\partial^2 u}{\partial x^2} = y^z \left( -\frac{u}{x^2} + \frac{1}{x} \frac{\partial u}{\partial x} \right) = \frac{uy^z(y^z - 1)}{x^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = 2 \ln x \left[ y^{z-1} \frac{\partial u}{\partial y} + (z - 1) y^{z-2} u \right]
\]

\[
= uz y^{z-2} \ln x \cdot (z y^z \ln x - z - 1),
\]

\[
\frac{\partial^2 u}{\partial z^2} = \left( y^z \frac{\partial u}{\partial z} + uy^z \ln y \right) \ln x \cdot \ln y
\]

\[
= uy^z \ln x \cdot \ln^2 y \cdot (1 + y^z \ln x),
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{x} \left( y^z \frac{\partial u}{\partial y} + uz y^{z-1} \right)
\]

\[
= \frac{uz y^{z-1} (y^z \ln x + 1)}{x},
\]

\[
\frac{\partial^2 u}{\partial y \partial z} = \left( y^{z-1} u + uz y^{z-1} \ln y + z y^{z-1} \frac{\partial u}{\partial z} \right) \ln x
\]

\[
= uy^{z-1} \ln x \cdot (1 + uz y^z (1 + y^z \ln x)),
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = y^2 \ln y \cdot \left( \frac{\partial u}{\partial x} \ln x + \frac{u}{x} \right) \\
= \frac{u y^2 \ln y \cdot (y^2 \ln x + 1)}{x} \quad (x \gg 0, \ y \gg 0).
\]

3228. 设 (a) \( u = x^2 - 2xy - 3y^2 \); (b) \( u = x^2 \); (b) \( u = \arccos \sqrt{\frac{x}{y}} \)，验证等式

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
\]

证 (a) \( \frac{\partial u}{\partial x} = 2x - 2y \), \( \frac{\partial u}{\partial y} = -2x - 6y \),

\[
\frac{\partial^2 u}{\partial x \partial y} = -2, \quad \frac{\partial^2 u}{\partial y \partial x} = -2,
\]

于是，\( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \).

(b) \( \frac{\partial u}{\partial x} = y^2 x^{2-1} \), \( \frac{\partial u}{\partial y} = 2yx^{2} \ln x \) (\( x \gg 0 \)),

\[
\frac{\partial^2 u}{\partial x \partial y} = 2yx^{2} - 1 + 2y^2 x^{2-1} \ln x,
\]

\[
\frac{\partial^2 u}{\partial y \partial x} = 2y^3 x^{2-1} \ln x + 2yx^{2-1},
\]

于是，\( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \).
(b) 当 \( 0 \leq x \leq y \) 时，我们有

\[
\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x}{y}}} = \frac{1}{2 \sqrt{x} \sqrt{y}} = -\frac{1}{2 \sqrt{x(y-x)}},
\]

\[
\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x}{y}}} \left( -\frac{\sqrt{x}}{2y} \right) = -\frac{\sqrt{x}}{2\sqrt{y^2(y-x)}},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4 \sqrt{x} (y-x)^{\frac{3}{2}}},
\]

\[
\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{4 \sqrt{x} \sqrt{y^2(y-x)}} + \frac{\sqrt{x}}{4y(y-x)^{\frac{3}{2}}}
\]

\[
= \frac{1}{4 \sqrt{x} (y-x)^{\frac{3}{2}}},
\]

于是，当 \( 0 \leq x \leq y \) 时，有

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
\]

当 \( y \leq x \leq 0 \) 时，\( u = \arccos \frac{\sqrt{-x}}{\sqrt{-y}} \).

\[
\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{1 - \frac{x}{y}}} \left( -\frac{1}{2 \sqrt{-x} \sqrt{-y}} \right)
\]
\[
\frac{1}{2 \sqrt{-x \sqrt{x-y}}} \\
\frac{\partial u}{\partial y} = -\frac{1}{\sqrt{1-x/y}} \left( \frac{\sqrt{-x}}{2(-y)^2} \right) = -\frac{\sqrt{-x}}{2\sqrt{xy^2-y^3}} \\
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4\sqrt{-x(x-y)^3}} \\
\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{4\sqrt{-x} \sqrt{xy^2-y^3}} + \frac{\sqrt{-x}}{4\sqrt{y^2}(x+y)^3} \\
= \frac{1}{4\sqrt{-x(x-y)^3}}.
\]

于是，当 $y < x < 0$ 时，也有

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
\]

仔细观察可以想到，在不同的区域上，一阶偏导数相差一个符号，但二阶混合偏导数却是相等的。

3230. 设 $f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$，若 $x^2 + y^2 \neq 0$ 及 $f(0, 0) = 0$，证明

\[
f''_{xx}(0, 0) \neq f''_{yx}(0, 0).
\]

证 由于

\[
\lim_{x \to 0} \frac{f(x, y) - f(0, y)}{x} = \lim_{x \to 0} xy \frac{x^2 - y^2}{x^2 + y^2} = -y,
\]

$55$
故，$f'_x(0, y) = -y$，从而

$$f''_{xx}(0, 0) = \left. \frac{d}{dy} \left[ f'_x(0, y) \right] \right|_{y=0} = -1$$

同法可求得 $f'_y(x, 0) = x$，从而

$$f''_{yy}(0, 0) = \left. \frac{d}{dx} \left[ f'_y(x, 0) \right] \right|_{x=0} = 1.$$  于是，$f''_{xx}(0, 0) \neq f''_{yy}(0, 0)$.

3231. 设 $u = f(x, y, z)$ 为 $n$ 次齐次函数，就下列各题验证关于齐次函数的尤拉定理：

(a) $u = (x - 2y + 3z)^2$;  
(b) $u = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$;  
(c) $u = \left(\frac{x}{y}\right)^t$.

证 关于 $n$ 次齐次函数的尤拉定理如下：

设 $n$ 次齐次函数 $f(x, y, z)$ * 在域 $A$ 中关于所有变量均有连续偏导函数，则下述等式成立

$$xf'_x(x, y, z) + yf'_y(x, y, z) + zf'_z(x, y, z) = nf(x, y, z).$$

(a) 由于 $(tx - 2ty + 3tz)^2 = t^2u$，故 $u$ 为二次齐次函数。又因

* 为了书写的简便，在这里我们仅限于讨论三个变量的情形。
\[ \frac{\partial u}{\partial x} = 2(x - 2y + 3z), \quad \frac{\partial u}{\partial y} = -4(x - 2y + 3z), \]

\[ \frac{\partial u}{\partial z} = 6(x - 2y + 3z), \]

故得

\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = (x - 2y + 3z) (2x - 4y + 6z) = 2u, \]

即函数 \( u \) 满足尤拉定理。

（6）由于对任何的 \( t \geq 0 \),

\[ \frac{tx}{\sqrt{(tx)^2 + (ty)^2 + (tz)^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = t^0 \cdot u, \]

故 \( u \) 为零次齐次函数。又因

\[ \frac{\partial u}{\partial x} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \quad \frac{\partial u}{\partial y} = -\frac{xy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \]

\[ \frac{\partial u}{\partial z} = -\frac{xz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \]

故得

\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \left( xy^2 + xz^2 - xy^2 - xz^2 \right) = 0 \cdot u, \]

即函数 \( u \) 满足尤拉定理。

（7）由于
\[
\left(\frac{tx}{ty}\right)^\frac{x}{y} = \left(\frac{x}{y}\right)^{\frac{x}{y}} = t^0 \cdot u \quad (t > 0),
\]

故函数 \(u\) 为零次齐次函数。又因
\[
\frac{\partial u}{\partial x} = \frac{1}{y} \cdot \frac{y}{z} \left(\frac{x}{y}\right)^{\frac{x}{y}-1} = \frac{yu}{xz},
\]
\[
\frac{\partial u}{\partial y} = \left(\left(\frac{y}{x}\right)^\frac{x}{y}\right)' \cdot \left(\frac{y}{x}\right)^\frac{x}{y} \cdot \left[\frac{1}{y} \ln \frac{x}{z} - \frac{1}{z} \cdot \frac{1}{y}\right]
\]
\[
= \frac{u}{z} \left(\ln \frac{x}{y} - 1\right),
\]
\[
\frac{\partial u}{\partial z} = \left(\frac{x}{y}\right)^{\frac{x}{y}} \cdot \ln \frac{x}{y} \cdot \left(-\frac{y}{z^2}\right) = -\frac{yu}{z^2} \ln \frac{x}{y},
\]

故得
\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = x \cdot \frac{yu}{xz} + y \cdot \frac{u}{z} \left(\ln \frac{x}{y} - 1\right)
\]
\[
- z \cdot \frac{yu}{z^2} \ln \frac{x}{y} = 0 \cdot u,
\]

即函数 \(u\) 满足尤拉定理。

3232. 证明：若可微函数 \(u = f(x, y, z)\) 满足方程式
\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu,
\]

则它为 \(n\) 次齐次函数。

证：任意固定域中一点 \((x_0, y_0, z_0)\)，考察下面的 \(t\) 的函数 \((t > 0)\):
\[
F(t) = \frac{f(tx_0, ty_0, tz_0)}{t^n},
\]

它当 \( t > 0 \) 时有定义且是可微的。应用复合函数的求导法则，对 \( t \) 求导数即得

\[
F'(t) = \frac{1}{t^n} \left\{ x_0f_x'(tx_0, ty_0, tz_0) + y_0f_y'(tx_0, ty_0, tz_0) + z_0f_z'(tx_0, ty_0, tz_0) \right\}
\]

\[
- \frac{n}{t^{n+1}} f(tx_0, ty_0, tz_0)
\]

\[
= \frac{1}{t^{n+1}} \left\{ tx_0f_x'(tx_0, ty_0, tz_0) + ty_0f_y'(tx_0, ty_0, tz_0) + tz_0f_z'(tx_0, ty_0, tz_0) - nf(tx_0, ty_0, tz_0) \right\},
\]

由于 \( tx_0f_x'(tx_0, ty_0, tz_0) + ty_0f_y'(tx_0, ty_0, tz_0) + tz_0f_z'(tx_0, ty_0, tz_0) - nf(tx_0, ty_0, tz_0) = 0 \)

故

\[
F'(t) = 0.
\]

从而当 \( t > 0 \) 时，\( F(t) = c \)，其中 \( c \) 为常数。现在确定 \( c \)。为此，在定义 \( F(t) \) 的等式中令 \( t = 1 \)，则得

\[
c = f(x_0, y_0, z_0).
\]

于是，
\[ F(t) = \frac{f(tx_0, ty_0, tz_0)}{t^n} = f(x_0, y_0, z_0), \]

即

\[ f(tx_0, ty_0, tz_0) = t^n f(x_0, y_0, z_0). \]

上式说明函数 \( f(x, y, z) \) 为一个 \( n \) 次的齐次函数，这就是所要证明的。

3233. 证明：若 \( f(x, y, z) \) 是可微分的 \( n \) 次共齐次函数，则其偏导函数 \( f'_x(x, y, z), f'_y(x, y, z), f'_z(x, y, z) \) 是 \((n-1) \) 次的齐次函数。

证 由等式

\[ f(tx, ty, tz) = t^n f(x, y, z) \]

两端分别对 \( x, y, z \) 求偏导函数，则得

\[ tf'_1(tx, ty, tz) = t^n f'_1(x, y, z), \]

\[ tf'_2(tx, ty, tz) = t^n f'_2(x, y, z), \]

\[ tf'_3(tx, ty, tz) = t^n f'_3(x, y, z), \]

其中 \( f'_1(\cdot, \cdot, \cdot), f'_2(\cdot, \cdot, \cdot), f'_3(\cdot, \cdot, \cdot) \) 分别代表

\( f(\cdot, \cdot, \cdot) \) 对第一个，第二个，第三个变量的偏导数。

于是，

\[ f'_1(tx, ty, tz) = t^{n-1} f'_1(x, y, z), \]

\[ f'_2(tx, ty, tz) = t^{n-1} f'_2(x, y, z), \]

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\[ f'_y(tx, ty, tz) = t^{n-1} f'_y(x, y, z), \]

即偏导函数 \( f'_x(x, y, z), f'_y(x, y, z) \) 及 \( f'_z(x, y, z) \)

均为 \((n-1)\)次的齐次函数，

3234. 设 \( u = f(x, y, z) \) 是可微分两次的 \( n \) 次齐次函数。证明

\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z})^2 u = n(n-1)u. 
\]

证 由3233题知：\( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \) 均为 \((n-1)\)次齐次函数。应用尤拉定理，即得

\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) \frac{\partial u}{\partial x} = (n-1) \frac{\partial u}{\partial x}, \quad (1) 
\]

\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) \frac{\partial u}{\partial y} = (n-1) \frac{\partial u}{\partial y}, \quad (2) 
\]

\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) \frac{\partial u}{\partial z} = (n-1) \frac{\partial u}{\partial z}. \quad (3) 
\]

将(1)式两端乘以 \( x \)，(2)式两端乘以 \( y \)，(3)式两端乘以 \( z \)，然后相加，即得

\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z})^2 u = (n-1)(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}) = n(n-1)u,
\]

这就是所要证明的等式。
求下列函数的一阶和二阶微分（x, y, z 为自变量）：

3235. \( u = x^n y^m \).

解 \[ du = x^{n-1} y^{m-1} (m y dx + n x dy), \]
\[ d^2 u = m(m-1) x^{n-2} y^m d x^2 + 2mn x^{n-1} y^{m-1} d x d y \]
\[ + n(n-1) x^n y^{m-2} d y^2, \]
\[ = x^{n-2} y^{m-2} (m(m-1) y^2 d x^2 + 2mn y dx d y \]
\[ + n(n-1) x^2 d y^2). \]

3236. \( u = \frac{x}{y} \).

解 \[ du = \frac{y dx - x dy}{y^2}, \]
\[ d^2 u = \frac{y^2 (d x d y - d x d y) - 2 y dy (y dx - x dy)}{y^4} \]
\[ = -\frac{2}{y^3} (y dx - x dy) dy. \]

3237. \( u = \sqrt{x^2 + y^2} \).

解 \[ du = \frac{x dx + y dy}{\sqrt{x^2 + y^2}}, \]
\[ d^2 u = \frac{d}{\sqrt{x^2 + y^2}} \left( \frac{x dx + y dy}{\sqrt{x^2 + y^2}} + (xd x + y dy) \right) \]
\[ \cdot d \left( \frac{1}{\sqrt{x^2 + y^2}} \right) = d \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - \frac{x dx + y dy}{\sqrt{x^2 + y^2}^3} \]
\[ = \frac{(y dx - x dy)^2}{(x^2 + y^2)^{3/2}}. \]
3238. \( u = \ln \sqrt{x^2 + y^2} \).

解：
\[
d u = \frac{xdx + ydy}{x^2 + y^2},
\]
\[
d^2 u = \frac{d(xdx + ydy)}{x^2 + y^2} - \frac{2(xdx + ydy)^2}{(x^2 + y^2)^2} = \frac{dx^2 + dy^2}{x^2 + y^2} - \frac{2(xdx + ydy)^2}{(x^2 + y^2)^2} = \frac{(y^2 - x^2)(dx^2 - dy^2) - 4xdydx}{(x^2 + y^2)^2}.
\]

3239. \( u = e^{xy} \).

解：
\[
d u = e^{xy}(ydx + xdy),
\]
\[
d^2 u = e^{xy}[(ydx + xdy)^2 + 2dx dy] = e^{xy}(y^2 dx^2 + 2(1 + xy)dy dx + x^2 dy^2).
\]

3240. \( u = xy + yz + zx \).

解：
\[
d u = (y + z)dx + (z + x)dy + (x + y)dz,
\]
\[
d^2 u = 2(d dx + d ydz + d zdx).
\]

3241. \( u = \frac{z}{x^2 + y^2} \).

解：
\[
d u = -\frac{2z}{(x^2 + y^2)^2} (xdx + ydy) + \frac{dz}{x^2 + y^2} = -\frac{(x^2 + y^2)dz - 2z(xdx + ydy)}{(x^2 + y^2)^2},
\]
\[
d^2 u = \frac{1}{(x^2 + y^2)^4} \left[ (x^2 + y^2)^2(2(xdx + ydy)dz
\]
\[
- 2(xdx + ydy)dz - 2z(dx^2 + dy^2) \right].
\]
\[-4(x^2 + y^2)(xdx + ydy) + ((x^2 + y^2)dz
\]
\[-2z(xdx + ydy)\right)\right\}
\[
= \frac{1}{(x^2 + y^2)^3} \left\{ 2x(3x^2 - y^2)dx^2 + 8xyzdxdy
\right.
\]
\[+(3y^2 - x^2)dy^2 - 4(x^2 + y^2)(xdx + ydy)dz \right\}.
\]

3242. 设 $f(x, y, z) = \frac{x}{\sqrt{y}}$，求 $df(1, 1, 1)$ 及 $d^2 f(1, 1, 1)$。

解 本题将采用分别先求一阶及二阶偏导函数，然后再合成以求一阶及二阶微分的方法。由于

\[f_x(x, 1, 1) = 1, \quad f_x(1, 1, 1) = 1,
\]

\[f_y(1, y, 1) = -\frac{1}{y^{\frac{3}{2}}}, \quad f_y(1, 1, 1) = -1,
\]

\[f_z(1, 1, z) = 0, \quad f_z(1, 1, 1) = 0,
\]

故得

\[df(1, 1, 1) = f_x(1, 1, 1)dx + f_y(1, 1, 1)dy
\]

\[+ f_z(1, 1, 1)dz = dx - dy.
\]

又因

\[f_x(x, 1, 1) = 1, \quad f_{xx}(x, 1, 1) = 0, \quad f_{xx}(1, 1, 1) = 0.
\]
\[ f_z'(1, y, 1) = \frac{1}{y}, \quad f_{xx}'(1, y, 1) = -\frac{1}{y^2}, \]
\[ f_{yy}'(1, 1, 1) = -1, \]
\[ f_{zz}'(1, 1, 1) = -\frac{1}{z}, \quad f_{zz}'(1, 1, z) = -\frac{1}{z^2}, \]
\[ f_{xx}'(1, 1, 1) = -1, \]
\[ f_{yy}'(1, y, 1) = -\frac{1}{y^2}, \quad f_{yy}'(1, y, 1) = \frac{2}{y^2}, \]
\[ f_{yy}(1, 1, 1) = 2, \]
\[ f_{zz}'(1, 1, z) = -\frac{1}{z}, \quad f_{zz}'(1, 1, z) = \frac{1}{z^2}, \]
\[ f_{zz}'(1, 1, 1) = 1, \]
\[ f_{zz}'(1, 1, z) = 0, \quad f_{xx}'(1, 1, z) = 0, \quad f_{zz}'(1, 1, 1) = 0, \]

故得
\[
d^2 f(1, 1, 1) = f_{xx}'(1, 1, 1) dx^2 + f_{yy}'(1, 1, 1) dy^2 \]
\[+ f_{zz}'(1, 1, 1) dz^2 + 2 f_{xy}'(1, 1, 1) dx dy \]
\[+ 2 f_{yz}'(1, 1, 1) dy dz + 2 f_{zx}'(1, 1, 1) dx dz \]
\[= 2 dy^2 - 2 dx dy + 2 dy dz - 2 dx dz \]
\[= 2(dy - dx)(dy + dz). \]

3243. 证明：若
\[ u = \sqrt{x^2 + y^2 + z^2}, \]

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则
\[ d^2u \geq 0. \]
证 \[ du = \frac{xdx + ydy + zdz}{u}, \]
\[ d^2u = \frac{1}{u^2}(u(dx^2 + dy^2 + dz^2) - (xdx + ydy + zdz)du) \]
\[ = \frac{1}{u^3}[(xdy - ydx)^2 + (ydz - zdy)^2 + (zdz - xdx)^2]. \]
由于 \( u > 0 \)（在原点处 \( du \) 不存在），故 \( du \geq 0 \).
3244. 假定 \( x, y \) 的绝对值甚小，对下列各式推出近似公式：
(a) \((1 + x)^n(1 + y)^n\);  (b) \(\ln(1 + x) \cdot \ln(1 + y)\);
(b) \(\arctan \frac{x + y}{1 - xy}\).
解 (a) 设 \( f(x, y) = (1 + x)^n(1 + y)^n \)，则
\[ f_x(x, 0) = m(1 + x)^{n-1}, f'_x(0, 0) = m, \]
\[ f_y(0, y) = n(1 + y)^{n-1}, f'_y(0, 0) = n. \]
于是
\[ f(x, y) \approx f(0, 0) + f_x(0, 0)x + f'_y(0, 0)y \]
\[ = 1 + mx + ny, \]
即有近似公式

\[(1+x)^n(1+y)^n \approx 1+mx+ny,\]

(6) 设 \(f(x, y) = \ln(1+x) \cdot \ln(1+y)\)，则

\[f'_x(x, 0) = 0, \quad f'_y(0, 0) = 0,\]

\[f''_{xx}(x, 0) = 0, \quad f''_{xx}(0, 0) = 0,\]

\[f''_{xx}(0, y) = 0, \quad f''_{yy}(0, 0) = 0,\]

\[f''_{xy}(0, y) = \ln(1+y), \quad f''_{xy}(0, y) = \frac{1}{1+y}, \quad f''_{xy}(0, 0) = 1.\]

于是，

\[f(x, y) \approx f(0, 0) + f'_x(0, 0)x + f'_y(0, 0)y\]

\[+ \frac{1}{2!} \left[ f''_{xx}(0, 0)x^2 + 2f''_{xy}(0, 0)xy + f''_{yy}(0, 0)y^2 \right]\]

\[= xy,\]

即有近似公式

\[\ln(1+x) \cdot \ln(1+y) \approx xy.\]

本题如不用求偏导函数的方法，也可直接获解：

\[\ln(1+x) \cdot \ln(1+y) = (x + o(x)) \cdot (y + o(y)).\]

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\[ \approx xy. \]

设 \( f(x, y) = \arctg \frac{x + y}{1 + xy} \)，则

\[
\begin{align*}
    f'_x(x, 0) &= \frac{1}{1 + x^2}, \quad f'_x(0, 0) = 1, \\
    f'_y(0, y) &= \frac{1}{1 + y^2}, \quad f'_y(0, 0) = 1.
\end{align*}
\]

于是，

\[ f(x, y) \approx f(0, 0) + f'_x(0, 0)x + f'_y(0, 0)y = x + y, \]

即有近似公式

\[ \arctg \frac{x + y}{1 + xy} \approx x + y. \]

3245. 用微分来代替函数的增量，近似地计算：

(a) 1.002·2.003²·3.004³；(b) \( \frac{1.03^2}{\sqrt{0.98} \sqrt{1.05^3}} \);

(b) \( \sqrt[3]{1.02^9 + 1.97^6} \)；(r) \( \sin 29^\circ \tan 4\theta \);

(A) 0.971.05.

解 (a) 设 \( f(x, y, z) = (1 + x)^n(1 + y)^s(1 + z)^t \)；则当 \(|x|, |y|, |z| \) 甚小时，有近似公式（参阅 3244(a))

\[ f(x, y, z) \approx 1 + mx + ny + lz. \]

利用上式即得

\[ 1.002\cdot2.003^2\cdot3.004^3 = (1 + 0.002) \cdot 2^2 \cdot \left(1 + \frac{0.003}{2}\right)^2 \cdot 3^8 \cdot \left(1 + \frac{0.004}{3}\right)^8 \]

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\[ \approx 1 \cdot 2^2 \cdot 3^3 \left(1 + 0.002 + 2 \cdot \frac{0.003}{2} + 3 \cdot \frac{0.004}{3} \right) \]

\[ = 108.972; \]

\[ (6) \quad \frac{1.03^2}{3 \sqrt[3]{0.98} \cdot \sqrt[3]{1.05}} = (1 + 0.03)^{\frac{1}{3}} \cdot (1 - 0.02)^{-\frac{1}{4}} \]

\[ \approx 1 + 2 \cdot 0.03 + \left( -\frac{1}{3} \right) (-0.02) + \left( -\frac{1}{4} \right) 0.05 \]

\[ \approx 1.054; \]

\[ (v) \quad \sqrt{1.02^3 + 1.97^3} = (1.97)^\frac{3}{2} \left[ 1 + \left( \frac{1.02}{1.97} \right)^3 \right]^{\frac{1}{2}} \]

\[ = 2^{\frac{3}{2}} \left( 1 - \frac{0.03}{2} \right)^{\frac{3}{2}} \left[ 1 + \left( \frac{1.02}{1.97} \right)^3 \right]^{\frac{1}{2}} \]

\[ \approx 2^{\frac{3}{2}} \left[ 1 + \frac{3}{2} \left( -\frac{0.03}{2} \right) + \frac{1}{2} \left( \frac{1.02}{1.97} \right)^3 \right] \]

\[ \approx 2.95; \]

（1）设 \( f(x, y) = \sin x \cdot \tan y \)，则有近似公式

\[ f(x, y) \approx \sin x_0 \cdot \tan y_0 + \cos x_0 \cdot \tan y_0 \cdot (x - x_0) \]

\[ + \frac{\sin x_0}{\cos^2 y_0} \cdot (y - y_0). \]

在本题中，令 \( x_0 = \frac{\pi}{6}, \ y_0 = \frac{\pi}{4}, \ x - x_0 = -\frac{\pi}{180}, \)

\[ y - y_0 = \frac{\pi}{180}, \] 即得

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\[ \sin 29^\circ \tan 46^\circ \approx \sin \frac{\pi}{6} \tan \frac{\pi}{4} + \cos \frac{\pi}{6} \tan \frac{\pi}{4} \]

\[ \left( -\frac{\pi}{180} \right) + \frac{\sin \frac{\pi}{6}}{\cos^2 \frac{\pi}{4}} \left( \frac{\pi}{180} \right) \]

\approx 0.502;

（Ⅱ）设 \( f(x, y) = x^y \)，由于

\[ f_x'(1, 1) = \left. \frac{d}{dx} f(x, 1) \right|_{x=1} = 1, \]

\[ f_y'(1, 1) = \left. \frac{d}{dy} f(1, y) \right|_{y=1} = 0, \]

于是，\( x^y \approx x \)。所以，我们有

\[ 0.97105 \approx 0.97. \]

3246. 设矩形的边长为 6 米和 8 米，若第 1 个边增加 2 毫米，而第 2 个边减少 5 毫米，问矩形的对角线和面积变化多少？

解  面积 \( A = xy \)，对角线 \( l = \sqrt{x^2 + y^2} \)。于是，

\[ \Delta A \approx ydx + xdy, \quad \Delta l \approx \frac{x dx + y dy}{\sqrt{x^2 + y^2}}. \]

以 \( x = 6000, \ y = 8000, \ dx = 2, \ dy = -5 \) 代入上述二式，即得

\[ \Delta A \approx 8000 \cdot 2 + 6000 \cdot (-5) = -14000 \text{（平方毫米）} = -140 \text{（平方厘米）}, \]
\[ A \approx \frac{6000 \cdot 2 \pm 8000 \cdot (-5)}{\sqrt{6000^2 + 8000^2}} \approx -3 \text{ (毫米)} , \]

即对角线减少约3毫米，面积减少约140平方厘米。

3247. 扇形的中心角 \( \alpha = 60^\circ \) 增加 \( M \alpha = 1^\circ \)。为了使扇形的面积仍然不变，则应当把扇形的半径 \( R = 20 \) 厘米减少若干？

解 扇形的面积 \( A = \frac{1}{2} R^2 \alpha \)。于是，

\[ \Delta A = R \alpha dR + \frac{1}{2} R^2 d\alpha. \]

按题设，应有 \( \Delta A = 0 \)，即

\[ 20 \cdot \frac{\pi}{3} dR + \frac{1}{2} \cdot 20^2 \cdot \frac{\pi}{180} \approx 0. \]

解之，得

\[ dR \approx -\frac{1}{6} \text{（厘米）} \approx -1.7 \text{（毫米）}, \]

即应当使半径减少约1.7毫米。

3248. 证明乘积的相对误差近似地等于乘数的相对误差的和。

证 设 \( u = xy, \) 则 \( du = xdy + ydx, \) 从而

\[ \frac{du}{u} = \frac{dx}{x} + \frac{dy}{y} . \]

取绝对值，得

\[ \left| \frac{du}{u} \right| < \left| \frac{dx}{x} \right| + \left| \frac{dy}{y} \right| , \]

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3249. 当测量圆柱的底半径 $R$ 和高 $H$ 时所得的结果如下：

$$R = 2.5 \text{ 米 } \pm 0.1 \text{ 米}; \ H = 4.0 \text{ 米 } \pm 0.2 \text{ 米},$$

则所计算出圆柱的体积可有怎样的绝对误差 $\Delta V$ 和相对误差 $\delta V$ ?

解  体积 $V = \pi R^2 H$. 于是，

$$\Delta V \approx dV = 2\pi RdR + \pi R^2 dH.$$  

以 $R = 2.5$, $H = 4.0$, $dR = 0.1$, $dH = 0.2$ 代入上式，即得

$$\Delta V \approx 10.2 \text{ 立方米},$$

$$\delta V = \left| \frac{\Delta V}{V} \right| \approx 13\%.$$  

3250. 三角形的边 $a = 200 \text{ 米 } \pm 2 \text{ 米}, b = 300 \text{ 米 } \pm 5 \text{ 米}$，它们之间的角 $C = 60^\circ \pm 1^\circ$，则所计算出三角形的第三边 $c$ 可有怎样的绝对误差?

解  按余弦定律，有

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

微分之，得

$$dc = da + db - b \cos C da - a \cos C db + ab \sin C dc.$$  

以 $a = 200, b = 300, c = \sqrt{200^2 + 300^2 - 2 \cdot 200 \cdot 300 \cdot \cos 60^\circ},$

$$C = \frac{\pi}{3}, \ \ dc = 2, \ \ db = 5, \ \ dC = \frac{\pi}{180} \text{ 代入上式，即得}$$

$$dc \approx 7.6 \text{ 米},$$

故第三边 $c$ 之绝对误差约为 $7.6$ 米。

3251. 证明：在点 $(0,0)$ 连续的函数

$$f(x, y) = \sqrt{|xy|}$$
于点 \((0, 0)\)有两个偏导函数 \(f_x'(0, 0)\) 和 \(f_y'(0, 0)\)，但在点 \((0, 0)\) 并非可微分的。

说明导函数 \(f_x'(x, y)\) 和 \(f_y'(x, y)\) 在点 \((0, 0)\) 的邻域中的性质。

解
\[
f_x'(0, 0) = \frac{d}{dx} [f(x, 0)] \bigg|_{x=0} = 0,
\]
\[
f_y'(0, 0) = \frac{d}{dy} [f(0, y)] \bigg|_{y=0} = 0.
\]

考察极限
\[
\lim_{\rho \to 0} \frac{f(x, y) - f(0, 0) - f_x'(0, 0)x - f_y'(0, 0)y}{\rho}
\]
\[
= \lim_{\rho \to 0} \frac{\sqrt{|xy|}}{\rho}.
\]

当动点 \((x, y)\) 沿直线 \(y = kx\) 趋于点 \((0, 0)\) 时，显然对不同的 \(k\) 有不同的极限值 \(\frac{\sqrt{|k|}}{\sqrt{1 + k^2}}\)。因此，上述极限不存在，即在点 \((0, 0)\)，

\[
f(x, y) - f(0, 0) - f_x'(0, 0)x - f_y'(0, 0)y
\]

不能表成 \(o(\rho)\)，其中 \(\rho = \sqrt{x^2 + y^2}\)，故知 \(\sqrt{|xy|}\) 在点 \((0, 0)\) 不可微分。

不难得到
\[ f^1_x(x, y) = \begin{cases} \frac{\sqrt{|xy|}}{2x}, & x \neq 0, \\ 0, & x^2 + y^2 = 0, \\ \text{无意义}, & x = 0, y \neq 0. \end{cases} \]

因此，\( f^1_x(x, y) \) 在点 \((0, 0)\) 的任何邻域中均有无意义之点及无界，\( f^1_x(x, y) \) 的性质类似。

3252. 证明：函数

\[ f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, \text{若} \ x^2 + y^2 \neq 0 \text{及} f(0, 0) = 0, \]

于点 \((0, 0)\) 的邻域中连续且有有界的偏导函数 \( f^1_x(x, y) \) 和 \( f^1_y(x, y) \)，但此函数于点 \((0, 0)\) 不能微分。

证 函数 \( f(x, y) \) 在 \( x^2 + y^2 \neq 0 \) 的点显然是连续的。由不等式

\[
|f(x, y)| = \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq \frac{x^2 + y^2}{2\sqrt{x^2 + y^2}}
\]

\[
= \frac{\sqrt{x^2 + y^2}}{2}
\]

知 \( \lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = 0 = f(0, 0) \)，故 \( f(x, y) \) 在点 \((0, 0)\) 的邻域中连续。

\[ f^1_x(x, y) = \begin{cases} \frac{y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ \left(\frac{x^2 + y^2}{x^2 + y^2}\right)^2, & x^2 + y^2 = 0. \end{cases} \]
当 \( x^2 + y^2 \neq 0 \) 时，由于

\[
|f'_z(x, y)| \leq \frac{|y^2|}{(y^2)^{\frac{3}{2}}} = 1,
\]

故 \( f'_z(x, y) \) 在点(0,0)的邻域内有界。同法可以证明 \( f'_y(x, y) \) 在点(0,0)的邻域内有界。

由于 \( f'_z(0,0) = f'_y(0,0) = 0 \)，且极限

\[
\lim_{\rho \to 0} \frac{f(x, y) - f(0, 0) - xf'_z(0,0) - yf'_y(0,0)}{\rho}
\]

\[
= \lim_{\rho \to 0} \frac{xy}{x^2 + y^2}
\]

是不存在的，因此可知函数 \( f(x, y) \) 在点(0,0)不可微分。

3253. 证明：函数

\[
f(x, y) = (x^2 + y^2)\sin \frac{1}{x^2 + y^2}, \text{ 若 } x^2 + y^2 \neq 0
\]

和 \( f(0, 0) = 0 \)

于点(0,0)的邻域中有偏导函数 \( f'_z(x, y) \) 和 \( f'_y(x, y) \)，这些偏导函数于点(0,0)是不连续的且在此点的任何邻域中是无界的；然而此函数于点(0,0)可微分。

证 当 \( x^2 + y^2 \neq 0 \) 时，\( f'_z(x, y) \) 及\( f'_y(x, y) \) 均存在，且
\[ f'_x(x, y) = 2x \sin \frac{1}{x^2 + y^2} - \frac{2x}{x^2 + y^2} \cos \frac{1}{x^2 + y^2}, \]

\[ f'_y(x, y) = 2y \sin \frac{1}{x^2 + y^2} - \frac{2y}{x^2 + y^2} \cos \frac{1}{x^2 + y^2}, \]

又因

\[ f'_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} x \sin \frac{1}{x^2} = 0, \]

\[ f'_y(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} y \sin \frac{1}{y^2} = 0, \]

故知在点 (0, 0) 内有偏导函数 \( f'_x(x, y) \) 及 \( f'_y(x, y) \).

考虑在点 \( \left( \frac{1}{\sqrt{2n\pi}}, 0 \right) \) 的偏导函数 \( f'_x(x, y) \):

\[ f'_x \left( \frac{1}{\sqrt{2n\pi}}, 0 \right) = \frac{2}{\sqrt{2n\pi}} \sin 2n\pi - 2 \sqrt{\frac{1}{2n\pi}} \cos 2n\pi \]

\[ = -2 \sqrt{\frac{1}{2n\pi}} \to -\infty \quad (n \to \infty), \]

因此，\( f'_x(x, y) \) 在点 (0, 0) 的任何邻域内无界，由此又知 \( f'_y(x, y) \) 在点 (0, 0) 不连续。同法可证 \( f'_y(x, y) \) 在 (0, 0) 的任何邻域中也无界，从而 \( f'_y(x, y) \) 在点 (0, 0) 也不连续.
最后，我们证明 $f(x, y)$ 在点 $(0, 0)$ 可微分。事实上，

$$f'_x(0, 0) = f'_y(0, 0) = 0,$$

且

$$\lim_{\rho \to 0} \frac{f(x, y) - f(0, 0) - xf'_x(0, 0) - yf'_y(0, 0)}{\rho}$$

$$= \lim_{\rho \to 0} \frac{\sqrt{x^2 + y^2} \cdot \sin \frac{1}{x^2 + y^2}}{\rho} = 0,$$

故得

$$f(x, y) = f(0, 0) + xf'_x(0, 0) + yf'_y(0, 0) + o(\rho),$$

即函数 $f(x, y)$ 在点 $(0, 0)$ 可微分。

3254. 证明：于某凸形的域 $E$ 内有有界偏导函数 $f'_x(x, y)$

和 $f'_y(x, y)$ 的函数 $f(x, y)$ 于域 $E$ 内一致连续。

证  由于 $f'_x(x, y)$ 及 $f'_y(x, y)$ 在 $E$ 内有界，故存在 $L > 0$，使当 $(x, y) \in E$ 时，恒有

$$|f'_x(x, y)| \leq \frac{L}{2},$$

及

$$|f'_y(x, y)| \leq \frac{L}{2}.$$

在 $E$ 内取两点 $P_1(x_1, y_1)$ 及 $P_2(x_2, y_2)$。

（1）如果以 $|P_1P_2|$ 为直径的圆（包括圆周在内）

都属于 $E$（图 6-25），则点 $P_3(x_3, y_3)$ 及线段
$P_1P_3, P_2P_3$ 都在 $E$ 内。
于是，
$$|f(x_1, y_1) - f(x_2, y_2)| = |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|$$
$$= |f'\left(x_1, \xi\right)|$$
$$\cdot |y_1 - y_2| + |f'\left(\eta, y_2\right)| \cdot |x_1 - x_2|,$$
其中 $\xi$ 介于 $y_1, y_2$ 之间，$\eta$ 介于 $x_1, x_2$ 之间。由偏导函数的有界性，即得
$$|f(x_1, y_1) - f(x_2, y_2)|$$
$$\leq \frac{L}{2} |y_1 - y_2| + \frac{L}{2} |x_1 - x_2|$$
$$\leq \frac{L}{2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$+ \frac{L}{2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= L \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$
或
$$|f(P_1) - f(P_2)| \leq L \cdot |P_1P_2|.$$  
（2）如图 6.26 所示，$P_1 \in E, P_2 \in E$，但点 $(x_1, y_2)$ 和 $(x_2, y_1)$ 不一定属于 $E$。由于 $P_1$ 和 $P_2$ 均为 $E$ 的内点，故存在 $R > 0$，使得分别以 $P_1, P_2$ 为
圆心，$R$ 为半径的圆（包括圆周在内）都在 $E$ 内。作两圆的外公切线 $Q_1Q_4$ 及 $Q_2Q_3$，则由切点均在 $E$ 内知，矩形 $Q_1Q_2Q_3Q_4$ 整个落在 $E$ 内。

![图 6-26](image)

不难看出，在直线段 $P_1P_2$ 上可取足够多的点：$P_1 = M_0, M_1, M_2, \cdots, M_n = P_2$，使

$$|M_{k-1}M_k| < 2R \ (k = 1, 2, \cdots, n).$$

则以 $|M_{k-1}M_k|$ 为直径的圆全落在矩形内，从而也在 $E$ 内。于是，

$$|f(P_1) - f(P_2)| \leq \sum_{k=1}^{n} |f(M_k) - f(M_{k-1})|$$

$$\leq \sum_{k=1}^{n} L \cdot |M_kM_{k-1}| = L \cdot \sum_{k=1}^{n} |M_kM_{k-1}|$$

$$= L \cdot |P_1P_2|.$$

这就证明了对 $E$ 中任意两点，函数 $f(P)$ 满足里普什兹条件。

对于任给的 $\varepsilon > 0$，取 $\delta = \frac{\varepsilon}{L}$，则当 $P_1 \in E, P_2$
\[ f(P_1) - f(P_2) \leq L \cdot |P_1P_2| \leq L \delta = \varepsilon, \]
即函数 \( f(x, y) \) 在 \( E \) 中一致连续。

注：用 \( \partial E \) 表区域 \( E \) 的边界，\( \overline{E} \) 表 \( E \) 加上 \( \partial E \)所成的闭区域。在本题的假定下，还可证明 \( f(x, y) \)可开拓为 \( \overline{E} \) 上的一致连续函数。事实上，对 \( \partial E \) 上任一点 \( P_0 \)，由柯西收敛准则知当点 \( P \) 从 \( E \) 内趋于 \( P_0 \)时 \( f(P) \) 的极限 \( A \) 存在（根据 \( f(P) \) 在 \( E \) 的一致连续性易知它满足柯西收敛准则）。我们规定 \( f(P_0) = A \)。于是 \( f(P) \) 在整个 \( \overline{E} \) 上有定义。在不等式

\[ f(P_1) - f(P_2) \leq L \cdot |P_1P_2| \quad (P_1, P_2 \in E) \]
两端让 \( P_1 \to P_0 \) （\( P_0 \in \partial E \)）取极限，得

\[ f(P_0) - f(P_2) \leq L \cdot |P_0P_2| \]
\[ (P_0 \in \partial E, P_2 \in E), \]
再让 \( P_2 \to P_0' \) （\( P_0' \in \partial E \)）取极限，得

\[ f(P_0) - f(P_0') \leq L \cdot |P_0P_0'| \]
\[ (P_0, P_0' \in \partial E). \]
由此可知，\( f(P) \) 在 \( \overline{E} \) 上满足里普什兹条件，从而 \( f(P) \) 在 \( \overline{E} \) 上一致连续。

3255. 证明：若函数 \( f(x, y) \) 对变数 \( x \) 是连续的（对每一个固定的值 \( y \)）且有对变数 \( y \) 的有界的导函数 \( f'_y(x, y) \)。
则此函数对变数 \( x \) 和 \( y \) 的总体是连续的。
证 设 \( P_0(x_0, y_0) \) 是所论的开域 \( E \) 中任一点。取以 \( P_0 \)
为中心的一个充分小的开球 $G_0$，使 $G_0$ 完全含于 $E$ 内。设在 $G_0$ 内，有 $|f'(x, y)| \leq L$。于是，当 $(x, y'), (x, y'')$ 属于 $G_0$ 时，有

$$|f(x, y') - f(x, y'')| = |f'(x, \xi)| \cdot |y' - y''| \leq L |y' - y''|,$$

其中 $\xi$ 为介于 $y'$, $y''$ 之间的一数，故 $f(x, y)$ 在 $G_0$ 中满足里普什兹条件。因此，根据 3206 题结果知 $f(x, y)$ 在 $G_0$ 中连续，特别是在 $P_0$ 点连续。由 $P_0$ 点的任意性，即知 $f(x, y)$ 在 $E$ 内连续，证毕。

注。从证明过程中很明显，本题只要假定 $f'(x, y)$ 在 $E$ 中每一点的某邻域中有界即可。

在下列问题中求所指的偏导函数。

6. \[ \frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial x^3 \partial y}, \frac{\partial^4 u}{\partial x^2 \partial y^2}, \quad \text{若} \]

\[ u = x - y + x^2 + 2xy + y^2 + x^3 - 3x^2y - y^3 + x^4 - 4x^2y^2 + y^4. \]

解

\[ \frac{\partial^2 u}{\partial x^2} = 2 + 6x - 6y + 12x^2 - 8y^2, \]

\[ \frac{\partial^3 u}{\partial x^3} = 6 + 24x. \]

于是，

\[ \frac{\partial^4 u}{\partial x^4} = 24, \quad \frac{\partial^4 u}{\partial x^3 \partial y} = 0, \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} = -16. \]
3257. \( \frac{\partial^3 u}{\partial x^2 \partial y} \), 若 \( u = \ln(xy) \).

解 \( \frac{\partial u}{\partial x} = \ln(xy) + 1 \), \( \frac{\partial^2 u}{\partial x^2} = \frac{1}{x} \).

于是,

\[ \frac{\partial^3 u}{\partial x^2 \partial y} = 0. \]

3258. \( \frac{\partial^6 u}{\partial x^3 \partial y^3} \), 若 \( u = x^3 \sin y + y^3 \sin x \).

解 \( \frac{\partial^8 u}{\partial x^3 \partial y^3} = 6 \sin y + y^3 \sin (x + \frac{3\pi}{2}) \)

\[ = 6 \sin y - y^3 \cos x. \]

于是,

\[ \frac{\partial^6 u}{\partial x^3 \partial y^3} = 6 \sin \left( y + \frac{3\pi}{2} \right) - 6 \cos x \]

\[ = -6(\cos y + \cos x). \]

3259. \( \frac{\partial^8 u}{\partial x \partial y \partial z} \), 若 \( u = \arctg \frac{x + y + z - xyz}{1 - xy - xz - yz} \).

解 注意到

\[ u = \arctg x + \arctg y + \arctg z + \varepsilon \pi \quad (\varepsilon = 0, \pm 1), \]

即得

\[ \frac{\partial^3 u}{\partial x \partial y \partial z} = 0. \]

3260. \( \frac{\partial^3 u}{\partial x \partial y \partial z} \), 若 \( u = e^{xy} \).
解 \( \frac{\partial u}{\partial x} = yze^{xyz} \), \( \frac{\partial^2 u}{\partial x \partial y} = ze^{xyz} + x yz e^{xyz} \).

于是，
\[
\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + x yz e^{xyz} + 2 x y z e^{xyz} + x^2 y^2 z^2 e^{xyz} = e^{xyz}(1 + 3 x y z + x^2 y^2 z^2).
\]

3261. \( \frac{\partial^4 u}{\partial x \partial y \partial z \partial \eta} \) 若 \( u = \ln \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \).

解 设 \( r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \)，则 \( u = -\ln r \).

\[
\frac{\partial u}{\partial x} = -\frac{1}{r} \frac{\partial r}{\partial x} = -\frac{x-\xi}{r^2},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{2(x-\xi)(y-\eta)}{r^4},
\]
\[
\frac{\partial^3 u}{\partial x \partial y \partial \xi} = -\frac{2(y-\eta) + 8(x-\xi)^2(y-\eta)}{r^4} + \frac{8(x-\xi)^2(y-\eta)}{r^6}.
\]

于是，
\[
\frac{\partial^4 u}{\partial x \partial y \partial \xi \partial \eta} = \frac{2}{r^4} - \frac{8(y-\eta)^2}{r^6} - \frac{8(x-\xi)^2}{r^4} + \frac{48(x-\xi)^2(y-\eta)^2}{r^8} = \frac{-6}{r^4} + \frac{48(x-\xi)^2(y-\eta)^2}{r^8}.
\]

3262. \( \frac{\partial^6 u}{\partial x^6 \partial y^6} \)，若 \( u = (x-x_0)^6 (y-y_0)^6 \).
解  \( \frac{\partial^q u}{\partial x^q} = p \cdot (y - y_0)^q \cdot \).

于是，

\[ \frac{\partial^{q+a} u}{\partial x^q \partial y^a} = p \cdot q \cdot (\tilde{p}, \tilde{q} 均为自然数) \]

3263. \( \frac{\partial^{m+n} u}{\partial x^m \partial y^n} \)，若 \( u = \frac{x + y}{x - y} \).

解  \( u = 1 + \frac{2y}{x - y} \cdot \left( \frac{\partial^m u}{\partial x^m} = (-1)^m \frac{2y}{(x - y)^{m+1}} \right) \)，利用求高阶导数的莱布尼茨公式，即得

\[
\frac{\partial^{m+n} u}{\partial x^m \partial y^n} = (-1)^m \cdot 2(\frac{m!}{}) \cdot \left\{ y \frac{\partial^n}{\partial y^n} \left( \frac{1}{(x - y)^{m+1}} \right) \right\} \\
+ C_n^1 \frac{\partial}{\partial y} \left( y \right) \cdot \frac{\partial^{n-1}}{\partial y^{n-1}} \left( \frac{1}{(x - y)^{m+1}} \right) \\
= 2 \cdot (-1)^m m! \cdot \left\{ \frac{(m+1)(m+2) \cdots (m+n)y}{(x - y)^{m+n+1}} + \frac{n(m+1)(m+2) \cdots (m+n-1)}{(x - y)^{n+1}} \right\} \\
= \frac{2 \cdot (-1)^m (m+n-1) y}{(x - y)^{m+n+1}}.
\]

3264. \( \frac{\partial^{m+n} u}{\partial x^m \partial y^n} \)，若 \( u = (x^2 + y^2) e^{x+y} \).

解  \( u = (x^2 + y^2) e^{x+y} = x^2 e^x e^y + y^2 e^y e^x = u_1 + u_2 \).

显然 \( \frac{\partial^n u_2}{\partial x^n} = e^x, y^2 e^y \)，利用求高阶导数的莱布尼兹公式
式，即得
\[ \frac{\partial^{n+n}u_2}{\partial x^n \partial y^n} = \frac{\partial^n}{\partial y^n} \left( \frac{\partial^{n}u_2}{\partial x^n} \right) = \frac{\partial^n}{\partial y^n} (e^x y^2 e^y) \]
\[ = e^x \frac{\partial^n}{\partial y^n} (y^2 e^y) = e^x \left\{ y^2 \frac{\partial^n}{\partial y^n} (e^y) \right\} \]
\[ + C_1^n \frac{\partial}{\partial y} (y^2) \frac{\partial^{n-1}}{\partial y^{n-1}} (e^y) \]
\[ + C_2^n \frac{\partial^2}{\partial y^2} (y^2) \frac{\partial^{n-2}}{\partial y^{n-2}} (e^y) \]}
\[ = e^{x+y} \{ y^2 + 2ny + n(n-1) \}. \]

同法可求得
\[ \frac{\partial^{n+n+1}u_1}{\partial x^n \partial y^{n+1}} = e^{x+y} \{ x^2 + 2mx + m(m-1) \}. \]

于是，
\[ \frac{\partial^{n+n}u}{\partial x^n \partial y^n} = \frac{\partial^{n+n}u_1}{\partial x^n \partial y^n} + \frac{\partial^{n+n}u_2}{\partial x^n \partial y^n} \]
\[ = e^{x+y} (x^2 + y^2 + 2mx + 2ny + m(m-1) + n(n-1)). \]

3285*. \[ \frac{\partial^{n+n+n+}u}{\partial x^n \partial y^n \partial z^r}, \text{若} \ u = xye^{x+y+z}. \]

解
\[ \frac{\partial^{n+n+n+}u}{\partial x^n \partial y^n \partial z^r} = \frac{\partial^{n+n+n}}{\partial x^n \partial y^n \partial z^r} (xe^x \cdot ye^y \cdot ze^z) \]
\[ = \frac{\partial^n}{\partial x^n} (xe^x) \cdot \frac{\partial^2}{\partial y^n} (ye^y) \cdot \frac{\partial^n}{\partial z^n} (ze^z) \]

85.
\[
\begin{align*}
&= e^x(x+p) \cdot e^y(y+q) \cdot e^z(z+r) \\
&= e^{x+y+z}(x+p)(y+q)(z+r).
\end{align*}
\]

3266. 若 \( f(x, y) = e^x \sin y \), 求 \( f_{x, y}^{(m+n)} (0,0) \).

解 \[
\begin{align*}
&f_{x, y}^{(m+n)} (0,0) = e^x \sin \left( y + \frac{\pi}{2} \right) \bigg|_{\substack{x=0 \\ y=0}} = \sin \frac{\pi}{2}.
\end{align*}
\]

3267. 证明：若

\[
u = f(xyz),
\]

则

\[
\frac{\partial^3 u}{\partial x \partial y \partial z} = F(t),
\]

式中 \( t = xyz \), 并求函数 \( F \).

解 \[
\begin{align*}
&\frac{\partial u}{\partial x} = yz f'(t), \\
&\frac{\partial^2 u}{\partial x \partial y} = yz f''(t) + xz + zf'(t).
\end{align*}
\]

于是，

\[
\begin{align*}
&\frac{\partial^3 u}{\partial x \partial y \partial z} = x^2 y^2 z^2 f''(t) + 2xyz f''(t) \\
&+ f'(t) + xz f'(t) \\
&= x^2 y^2 z^2 f''(t) + 3xyz f''(t) + f'(t) \\
&= t^2 f''(t) + 3t f''(t) + f'(t) = F(t).
\end{align*}
\]

3268. 设 \( u = x^4 - 2x^3 y - 2xy^3 + y^4 + x^3 - 3x^2 y - 3xy^2 + y^3 + 2x^2 - xy + 2y^2 + x + y + 1 \), 求 \( d^4 u \).
导函数 $\frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial x^2 \partial y}, \frac{\partial^4 u}{\partial x^2 \partial y^2}, \frac{\partial^4 u}{\partial x \partial y^3}$ 和 $\frac{\partial^4 u}{\partial y^4}$

等于什么?

解 $d^4 u = 24 \ dx^4 - 2C_4 \ dx^3 (x^3) \ dy$

$-2C_4 \ dx \ dx^3 (y^3) + 24 \ dy^4$

$= 24(dx^4 - 2dx^3 \ dy - 2dx \ dy^3 + dy^4)$。

由 $d^4 u = (dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y})^4 u$，得

$$
\frac{\partial^4 u}{\partial x^4} = 24, \quad \frac{\partial^4 u}{\partial x^2 \partial y} = -12, \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} = 0, \quad \frac{\partial^4 u}{\partial x \partial y^3} = -12, \quad \frac{\partial^4 u}{\partial y^4} = 24.
$$

在下列各题中求所指出的阶的全微分；

3269. $d^3 u$，若 $u = x^3 + y^3 - 3xy(x-y)$。

解 $d^3 u = 6(dx^3 + dy^3 - 3dx^2 \ dy + 3dx \ dy^2)$。

3270. $d^3 u$，若 $u = \sin(x^2 + y^2)$。

解 $d^3 u = 2x \cos(x^2 + y^2) \ dx + 2y \cos(x^2 + y^2) \ dy$

$= 2(dx \ dx + y \ dy) \ cos(x^2 + y^2)$

$$
d^3 u = -4 \sin(x^2 + y^2) \cdot (dx \ dx + y \ dy)^2
+ 2 \cos(x^2 + y^2) \cdot (dx^2 + dy^2).
$$

于是，

$$
d^3 u = -8 \cos(x^2 + y^2) \cdot (dx \ dx + y \ dy)^3
$$

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\[-8\sin(x^2+y^2) \cdot (xdx+ydy) \cdot (dx^2+dy^2) \]
\[-4\sin(x^2+y^2) \cdot (xdx+ydy) \cdot (dx^2+dy^2) \]
\[= -8(xdx+ydy)^8 \cos(x^2+y^2) \]
\[-12(xdx+ydy)(dx^2+dy^2) \sin(x^2+y^2). \]

3271. \(d^{10}u, \text{ 若 } u = \ln(x+y).\)

解  \(du = \frac{dx+dy}{x+y} \). 于是，

\[d^{10}u = -\frac{91(dx+dy)^{10}}{(x+y)^{10}}.\]

3272. \(d^6u, \text{ 若 } u = \cos x \cosh y.\)

解  \(d^6u = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^6 u \)

\[= -\cos x \cosh y dx^6 - 6 \sin x \sinh y dx^5 dy \]
\[+ 15 \cos x \cosh y dx^4 dy^2 \]
\[+ 20 \sin x \sinh y dx^3 dy^3 - 15 \cos x \cosh y dx^2 dy^4 \]
\[- 6 \sin x \sinh y dx dy^5 + \cos x \cosh y dy^6 \]
\[= - (dx^6 - 15 dx^4 dy^2 + 15 dx^2 dy^4 \]
\[- dy^6) \cos x \cosh y - 2 dx dy(3 dx^4 \]
\[- 10 dx^2 dy^2 + 3 dy^4) \sin x \sinh y. \]

3273. \(d^3u, \text{ 若 } u = xyz.\)

解  注意到 \(d^2x = d^2y = d^2z = 0\), 即得

\[d^3u = d^3(xyz) = C_{1}^1 dx d^2(yz) = 3 dx \cdot (C_{1}^1 dy dz) \]
\[= 6 dx dy dz.\]

3274. \(d^4u, \text{ 若 } u = \ln(x^y y^z).\)

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解 由于 \( u = x \ln x + y \ln y + z \ln z \)，故
\[
d^4 u = (x \ln x)^{(4)} d x^4 + (y \ln y)^{(4)} d y^4 + (z \ln z)^{(4)} d z^4
\]
\[
= 2 \left( \frac{d x^4}{x^3} + \frac{d y^4}{y^3} + \frac{d z^4}{z^3} \right).
\]

3275. \( d^n u \)，若 \( u = e^{ax+by} \)。

解 注意到 \( d^2 (ax + by) = 0 \)，即得
\[
d^n u = d^n (e^{ax+by}) = e^{ax+by} (d (ax + by))^n
\]
\[
= e^{ax+by} (a dx + b dy)^n.
\]

3276. \( d^n u \)，若 \( u = X(x)Y(y) \)。

解
\[
d^n u = \sum_{k=0}^{n} C_n^k d^{n-k} X(x) \cdot d^k Y(y)
\]
\[
= \sum_{k=0}^{n} C_n^k X^{(n-k)}(x) Y^{(k)}(y) d x^{n-k} d y^k.
\]

3277. \( d^n u \)，若 \( u = f(x+y+z) \)。

解 注意到 \( d^2 (x+y+z) = 0 \)，即得
\[
d^n u = f^{(n)}(x+y+z) \cdot (d x + d y + d z)^n.
\]

3278. \( d^n u \)，若 \( u = e^{ax+by+cz} \)。

解 注意到 \( d^2 (ax + by + cz) = 0 \)，即得
\[
d^n u = e^{ax+by+cz} (a dx + b dy + c dz)^n.
\]

3279. \( P_n(x,y,z) \) 为 \( n \) 次齐次多项式，证明
\[
d^n P_n(x,y,z) = n! P_n(d x, d y, d z).
\]

证 \( P_n(x,y,z) \) 可表示为形如
\[
A x^p y^q z^r
\]
的单项式之和，其中 \( A \) 为常数，\( p,q,r \) 为非负整数。
且 $p+q+r=n$。

由于微分运算对加法及乘以常数是线性的（可交换的），因此要证

$$d^n P_n(x, y, z) = n! P_n(dx, dy, dz),$$

只要证明

$$d^n(x^p y^q z^r) = n! dx^p dy^q dz^r$$

就可以了。事实上，

$$d^n(x^p y^q z^r) = C_{p+q+r}^{p+q} d^p(x^p) d^q(y^q) d^r(z^r)$$

$$= \frac{n!}{r! (p+q)!} \left[ C_{p+q+r}^{p+q} d^p(x^p) d^q(y^q) d^r(z^r) \right]$$

$$= \frac{n!}{r! (p+q)!} \cdot \frac{(p+q)!}{p!q!r!} dx^p dy^q dz^r$$

$$= n! dx^p dy^q dz^r.$$

3280. 设：

$$Au = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}.$$ 

求 $Au$ 和 $A^2 u = A(Au)$，若

(a) $u = \frac{x}{x^2 + y^2}$;  
(b) $u = \ln \sqrt{x^2 + y^2}$. 

解 (a) $\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$,  
$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}$.

于是，

$$Au = \frac{x(y^2 - x^2)}{(x^2 + y^2)^2} - \frac{2xy^2}{(x^2 + y^2)^2} = -\frac{x}{x^2 + y^2} = -u,$$

$$A^2 u = A(Au) = A(-u) = -Au = u.$$
\[ \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}. \] 于是，

\[ Au = \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} = 1, \]

\[ A^2 u = A(Au) = 0. \]

3281. 设:

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \]

求 \( \Delta u \)，若

(a) \( u = \sin x \cosh y \);  (b) \( u = \ln \sqrt{x^2 + y^2} \).

解 (a) \[ \frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y, \quad \frac{\partial^2 u}{\partial y^2} = \sin x \cosh y. \] 于是，

\[ \Delta u = -\sin x \cosh y + \sin x \cosh y = 0. \]

(b) \[ \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \] 由对称

性质 \[ \frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}. \] 于是，

\[ \Delta u = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0. \]

3282. 设:

\[ \Delta_1 u = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \]

及
\[ \Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \]

求 \( \Delta_1 u \) 和 \( \Delta_2 u \)，若
(a) \( u = x^3 + y^3 + z^3 - 3xyz \);
(b) \( u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \).

解 (a) \( \Delta_1 u = 9 \left[ (x^2 - yz)^2 + (y^2 - zx)^2 \right. \\
+ (z^2 - xy)^2 \left. \right] \), \\
\( \Delta_2 u = 6(x + y + z) \).
(b) 令 \( r = \sqrt{x^2 + y^2 + z^2} \)，则 \( u = \frac{1}{r} \).

\[ \frac{\partial u}{\partial x} = -\frac{1}{r^2} \frac{\partial r}{\partial x} = -\frac{x}{r^3}, \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^3} + \frac{3x}{r^4} \frac{\partial r}{\partial x} = -\frac{1}{r^3} + \frac{3x^2}{r^5}. \]

由对称性即知

\[ \Delta_1 u = \frac{x^2 + y^2 + z^2}{r^6} = \frac{1}{r^4} = \frac{1}{(x^2 + y^2 + z^2)^2}, \]

\[ \Delta_2 u = \left( -\frac{1}{r^3} + \frac{3x^2}{r^6} \right) + \left( -\frac{1}{r^3} + \frac{3y^2}{r^6} \right) \\
+ \left( -\frac{1}{r^3} + \frac{3z^2}{r^6} \right) = 0. \]

求下列复合函数的一阶和二阶导函数:

3283. \( u = f(x^2 + y^2 + z^2) \).
解 \[ \frac{\partial u}{\partial x} = 2xf'(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial x^2} = 2f'(x^2 + y^2 + z^2) \]

\[ + 4x^2f''(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial x \partial y} = 4xyf''(x^2 + y^2 + z^2). \]

由对称性即知

\[ \frac{\partial u}{\partial y} = 2yf'(x^2 + y^2 + z^2), \]

\[ \frac{\partial u}{\partial z} = 2zf'(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial y^2} = 2f'(x^2 + y^2 + z^2) \]

\[ + 4y^2f''(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial z^2} = 2f'(x^2 + y^2 + z^2) \]

\[ + 4z^2f''(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial y \partial z} = 4yxf''(x^2 + y^2 + z^2), \]

\[ \frac{\partial^2 u}{\partial z \partial x} = 4zxf''(x^2 + y^2 + z^2). \]

3284. \( u = f(x, \frac{x}{y}). \)
解  \( \frac{\partial u}{\partial x} = f'_{1}(x, \frac{x}{y}) + \frac{1}{y} f'_{2}(x, \frac{x}{y}), \)

\[ \frac{\partial u}{\partial y} = -\frac{x}{y^2} f'_{2}(x, \frac{x}{y}), \]

\[ \frac{\partial^2 u}{\partial x^2} = f''_{11}(x, \frac{x}{y}) + \frac{2}{y} f''_{12}(x, \frac{x}{y}) + \frac{1}{y^2} f''_{22}(x, \frac{x}{y}), \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{2x}{y^3} f'_{2}(x, \frac{x}{y}) + \frac{x^2}{y^4} f''_{22}(x, \frac{x}{y}), \]

\[ \frac{\partial^2 u}{\partial x \partial y} = -\frac{x}{y^2} f''_{12}(x, \frac{x}{y}) - \frac{1}{y^2} f'_{2}(x, \frac{x}{y}) - \frac{x}{y^3} f''_{22}(x, \frac{x}{y}). \]

*)  \( f'_{1}, f'_{2}, f''_{11}, f''_{12}, f''_{22} \) 均系按其下标的次序分别对第一、第二个中间变量求导函数，以下各题均同，不再说明。

3285.  \( u = f(x, xy, xyz). \)

解  \( \frac{\partial u}{\partial x} = f'_{1}(x, xy, xyz) + y f'_{2}(x, xy, xyz) \)

\[ + yz f'_{3}(x, xy, xyz). \]

将 \( f'_{1}(x, xy, xyz), f'_{2}(x, xy, xyz), f'_{3}(x, xy, xyz) \)
简记为 $f'_1$, $f'_2$, $f'_3$, 以后不再说明。于是，

$$\frac{\partial u}{\partial x} = f'_1 + yf'_2 + yzf'_3, \quad \frac{\partial u}{\partial y} = xf'_2 + xzf'_3,$$

$$\frac{\partial u}{\partial z} = yf'_1,$$

$$\frac{\partial^2 u}{\partial x^2} = f''_{11} + yf''_{12} + yzf''_{13} + y\left(f''_{21} + yf''_{22}\right) + yzf''_{23} + yzf''_{33}.$$

由于 $f''_{12} = f''_{21}$, $f''_{13} = f''_{31}$, $f''_{23} = f''_{32}$ (以下各题均同)，故

$$\frac{\partial^2 u}{\partial x^2} = f''_{11} + y^2f''_{12} + y^2zf''_{13} + 2yf''_{12} + 2yzf''_{13} + 2y^2zf''_{23}.$$

同法可求得

$$\frac{\partial^2 u}{\partial y^2} = x^2f''_{22} + x^2zf''_{23} + x^2zf''_{32} + x^2z^2f''_{33},$$

$$= x^2f''_{22} + 2xz^2f''_{23} + x^2z^2f''_{33},$$

$$\frac{\partial^2 u}{\partial z^2} = x^2yf''_{33},$$
\[
\frac{\partial^2 u}{\partial x \partial y} = xf''_{12} + xz f''_{18} + f'_2 + xy f''_{22} + xyz f''_{23} + zf''_3 + xyz f'_{32} + xyz^2 f'_{33} = xy f''_{22} + xyz^2 f'_{33} + zf''_3 + xz f'_8 + 2xyz f'_8 + z f'_8.
\]

\[
\frac{\partial^2 u}{\partial x \partial z} = xy f''_{13} + x y^2 f''_{23} + x y z f'_{33} + y f'_3,
\]

\[
\frac{\partial^2 u}{\partial y \partial z} = x^2 y f''_{23} + x^2 y z f'_{33} + x f'_8.
\]

3286. 设 \( u = f(x+y, xy) \)，求 \( \frac{\partial^2 u}{\partial x \partial y} \).

解

\[
\frac{\partial u}{\partial x} = f'_1 + y f'_2.
\]

于是，

\[
\frac{\partial^2 u}{\partial x \partial y} = f''_1 + xf''_{12} + f'_2 + y f''_{21} + xy f''_{22} = f''_1 + (x+y) f''_{12} + xy f''_{22} + f'_2.
\]

3287. 设 \( u = f(x+y+z, x^2 - y^2 + z^2) \)，求 \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \).
解 \( \frac{\partial u}{\partial x} = f'_1 + 2xf'_2, \)

\( \frac{\partial^2 u}{\partial x^2} = f''_1 + 2xf''_2 + 2f'_2 + 2xf'_2 + 4x^2f''_2 \)

\( = f''_1 + 4xf''_2 + 4x^2f''_2 + 2f'_2. \)

由对称性即得

\( \frac{\partial^2 u}{\partial y^2} = f''_1 + 4yf''_2 + 4y^2f''_2 + 2f'_2, \)

\( \frac{\partial^2 u}{\partial z^2} = f''_1 + 4zf''_2 + 4z^2f''_2 + 2f'_2. \)

于是，

\[ \Delta u = 3f''_1 + 4(x + y + z)f''_2 \]

\[ + 4(x^2 + y^2 + z^2)f''_2 + 6f'_2. \]

求下列复合函数的一阶和二阶全微分（\( x, y \)及 \( z \)为自变量）：

3288. \( u = f(t) \)，其中 \( t = x + y. \)

解  \( du = f'(t)(dx + dy), \quad d^2u = f''(t)(dx + dy)^2. \)

3289. \( u = f(t) \)，其中 \( t = \frac{y}{x}. \)

解  \( du = f'(t), \quad \frac{x dy - y dx}{x^2}, \)

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\[ d^2u = f''(t) \cdot \frac{(xd y - yd x)^2}{x^4} - 2f'(t) \cdot \frac{d(xd y - yd x)}{x^3}. \]

3290. \( u = f(\sqrt{x^2 + y^2}) \).

解  \[ du = f'(t) \cdot \frac{xd x + yd y}{\sqrt{x^2 + y^2}}, \]
\[ d^2u = f''(t) \cdot \frac{(xd x + yd y)^2}{x^2 + y^2} + f'(t) \cdot \frac{(yd x - xd y)^2}{(x^2 + y^2)^{\frac{3}{2}}}. \]

3291. \( u = f(t), \) 其中 \( t = xyz. \)

解  \[ du = f'(t)(yzd x + xzd y + xyd z), \]
\[ d^2u = f''(t)(yzd x + xzd y + xyd z)^2 + 2f'(t)(zd x d y + yd x d z + xd y d z). \]

3292. \( u = f(x^2 + y^2 + z^2). \)

解  \[ du = 2f'(t)(xd x + yd y + zd z), \]
\[ d^2u = 4f''(t)(xd x + yd y + zd z)^2 + 2f'(t)(d x^2 + d y^2 + d z^2). \]

3293. \( u = f(\xi, \eta), \) 其中 \( \xi = ax, \eta = by. \)

解  \[ du = af'_1 d x + bf'_2 d y, \]
\[ d^2u = a^2 f''_{11} d x^2 + 2ab f''_{12} d x d y + b^2 f''_{22} d y^2. \]

3294. \( u = f(\xi, \eta), \) 其中 \( \xi = x + y, \eta = x - y. \)
解 \( du = f'_1 \cdot (dx + dy) + f'_2 \cdot (dx - dy) \),

\[ d^2 u = f''_{11} \cdot (dx + dy)^2 + 2f''_{12} \cdot (dx^2 - dy^2) + f''_{22} \cdot (dx - dy)^2. \]

3295. \( u = f(\xi, \eta) \), 其中 \( \xi = xy \), \( \eta = \frac{x}{y} \).

解 \( du = f'_1 \cdot (yd + xdy) + f'_2 \cdot \frac{ydx - xdy}{y^2} \),

\[ d^2 u = f''_{11} \cdot (yd + xdy)^2 + f''_{22} \cdot \frac{(ydx - xdy)^2}{y^4} \]

\[ + 2f''_{12} \cdot \frac{y^2dx^2 - x^2dy^2}{y^2} + 2f'_1 \cdot dxdy - 2f'_2 \cdot \frac{(ydx - xdy)dy}{y^3}. \]

3296. \( u = f(x + y, z) \).

解 \( du = f'_1 \cdot (dx + dy) + f'_2 \cdot dz \),

\[ d^2 u = f''_{11} \cdot (dx + dy)^2 + 2f''_{12} \cdot (dx + dy)dz + f''_{22}dz^2. \]

3297. \( u = f(x + y + z, x^2 + y^2 + z^2) \).

解 \( du = f'_1 \cdot (dx + dy + dz) + 2f'_2 \cdot (xdx + ydy + zdz) \).
\( + y dy + zdz \),
\[
d^2u = f''_{11} \cdot (dx + dy + dz)^2 + 4f''_{12} \cdot (dx + dy + dz) (xdx + ydy + zdz) \\
+ 4f''_{22} \cdot (xdx + ydy + zdz)^2 + 2f''_2 \cdot (dx^2 + dy^2 + dz^2).
\]

3298. \( u = f\left(\frac{x}{y}, \frac{y}{z}\right) \).

解
\[
du = f'_1 \cdot \frac{ydxdy - xdy}{y^2} + f'_2 \cdot \frac{zdy - ydz}{z^2},
\]
\[
d^2u = f''_{11} \cdot \frac{(ydxdy - xdy)^2}{y^4} + f''_{22} \cdot \frac{(zdy - ydz)^2}{z^4} \\
+ 2f''_{12} \cdot \frac{(ydxdy - xdy)(zdy - ydz)}{y^2 z^2} \\
- 2f'_1 \cdot \frac{(ydxdy - xdy)dy}{y^3} - 2f'_2 \cdot \frac{(zdy - ydz)dz}{z^3}.
\]

3299. \( u = f(x, y, z) \)，其中 \( x = t, y = t^2, z = t^4 \).

解
\[
du = (f'_1 + 2tf'_2 + 3t^2 f'_3) dt,
\]
\[
d^2u = (f''_{11} + 4t^2 f''_{12} + 9t^4 f''_{13}) + 4t f''_{22} + 4t f''_{12} + 6t^2 f''_{13} \\
+ 12t^3 f''_{23} + 2f'_2 + 6tf'_3) dt^2.
\]

3300. \( u = f(\xi, \eta, \zeta) \)，其中 \( \xi = ax, \eta = by, \zeta = cz \).
解  \( du = af'_1 \cdot dx + bf'_2 \cdot dy + cf'_3 \cdot dz, \)

\[ d^2 u = a^2 f''_{11} \cdot dx^2 + b^2 f''_{22} \cdot dy^2 + c^2 f''_{33} \cdot dz^2 \]

\[ + 2ab f''_{12} \cdot dx \cdot dy + 2ac f''_{13} \cdot dx \cdot dz + 2bc f''_{23} \cdot dy \cdot dz. \]

3301. \( u = f(\xi, \eta, \zeta), \) 其中 \( \xi = x^2 + y^2, \) \( \eta = x^2 - y^2, \)

\( \zeta = 2xy. \)

解  \( du = 2f'_1 \cdot (xdx + ydy) + 2f'_2 \cdot (xdx - ydy) \)

\[ + 2f'_3 \cdot (ydx + xdy), \]

\[ d^2 u = 4f''_{11} \cdot (xdx + ydy)^2 + 4f''_{22} \cdot (xdx - ydy)^2 \]

\[ + 4f''_{33} \cdot (ydx + xdy)^2 + 8f''_{12} \cdot (x^2dx^2 - y^2dy^2) \]

\[ + 8f''_{13} \cdot (xdx + ydy)(ydx + xdy) \]

\[ + 8f''_{23} \cdot (xdx - ydy)(ydx + xdy) + 2f'_1 \cdot (dx^2 \]

\[ + dy^2) + 2f'_2 \cdot (dx^2 - dy^2) + 4f'_3 \cdot dxdy. \]

求 \( d^nu, \) 设;

3302. \( u = f(ax + by + cz). \)

解  \( d^nu = f^{(n)}(ax + by + cz) \cdot (adx + bdy + cdz)^n. \)

3303. \( u = f(ax, by, cz). \)
解  \( d'u = \left( a_d x \frac{d}{d\xi} + b_d y \frac{d}{d\eta} + c_d z \frac{d}{d\zeta} \right)^n f(\xi, \eta, \zeta), \)

其中 \( \xi = a x, \eta = b y, \zeta = cz. \)

3304. \( u = f(\xi, \eta, \zeta), \) 其中 \( \xi = a_1 x + b_1 y + c_1 z, \)
\( \eta = a_2 x + b_2 y + c_2 z, \) \( \zeta = a_3 x + b_3 y + c_3 z. \)

解  \( d'u = \left[ \left( a_1 d x + b_1 d y + c_1 d z \right) \frac{d}{d\xi} + \left( a_2 d x + b_2 d y + c_2 d z \right) \frac{d}{d\eta} + \left( a_3 d x + b_3 d y + c_3 d z \right) \frac{d}{d\zeta} \right]^n f(\xi, \eta, \zeta) \)

\( = \left[ d x \left( a_1 \frac{\partial}{\partial\xi} + a_2 \frac{\partial}{\partial\eta} + a_3 \frac{\partial}{\partial\zeta} \right) \right.
\left. + d y \left( b_1 \frac{\partial}{\partial\xi} + b_2 \frac{\partial}{\partial\eta} + b_3 \frac{\partial}{\partial\zeta} \right) \right.
\left. + d z \left( c_1 \frac{\partial}{\partial\xi} + c_2 \frac{\partial}{\partial\eta} + c_3 \frac{\partial}{\partial\zeta} \right) \right]^n f(\xi, \eta, \zeta). \)

3305. 设 \( u = f(r), \) 其中 \( r = \sqrt{x^2 + y^2 + z^2} \) 和 \( f \) 为可微分两次的函数。证明，

\( \Delta u = F(r), \)

其中 \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \) \( \Delta \) 为拉普拉斯算子，

并求函数 \( F. \)

解  \( \frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}, \)

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\[ \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x^2}{r^2} + f'(r) \cdot \frac{r^2 - x^2}{r^3}. \]

由对称性即得

\[ \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{y^2}{r^2} + f'(r) \cdot \frac{r^2 - y^2}{r^3}, \]

\[ \frac{\partial^2 u}{\partial z^2} = f''(r) \cdot \frac{z^2}{r^2} + f'(r) \cdot \frac{r^2 - z^2}{r^3}. \]

于是，

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) \]

\[ + 2f'(r) \cdot \frac{1}{r} = F(r). \]

3306．设 \( u \) 和 \( v \) 为可微分两次的函数而 \( \Delta \) 为拉普拉斯算子（参阅 3305 题）。证明：

\[ \Delta (uv) = u \Delta v + v \Delta u + 2 \Delta (u, v), \]

其中 \( \Delta (u, v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z}. \)

证  \[ \Delta (uv) = \frac{\partial^2 (uv)}{\partial x^2} + \frac{\partial^2 (uv)}{\partial y^2} + \frac{\partial^2 (uv)}{\partial z^2} \]

\[ = \left( u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \right) \]

\[ + \left( u \frac{\partial^2 v}{\partial y^2} + v \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \]

[103]
\[ + \left( u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) \]

\[ = u \Delta v + v \Delta u + 2 \Delta (u, v), \]

这就是所要证明的。

3307. 证明：函数

\[ u = \ln \sqrt{(x-a)^2 + (y-b)^2} \]

（a和b为常数）满足拉普拉斯方程

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]

证

\[ \frac{\partial u}{\partial x} = \frac{x-a}{(x-a)^2 + (y-b)^2}, \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{(y-b)^2 - (x-a)^2}{((x-a)^2 + (y-b)^2)^2}. \]

由对称性即得

\[ \frac{\partial^2 u}{\partial y^2} = \frac{(x-a)^2 - (y-b)^2}{((x-a)^2 + (y-b)^2)^2}. \]

于是，

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]

3308. 证明：若函数 \( u = u(x, y) \) 满足拉普拉斯方程（参阅3307题），则函数

\[ v = u \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) \]

也满足这方程。

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证 设 \( \xi = \frac{x}{x^2 + y^2} \), \( \eta = \frac{y}{x^2 + y^2} \), 则 \( u(x, y) = u(\xi, \eta) \)。从而

\[
v''_{xx} = u_{\xi\xi} \cdot (\xi_x^2)^2 + u_{\eta\eta} \cdot (\eta_x^2)^2 + 2u_{\xi\eta} \cdot \xi_x \cdot \eta_x
\]

\[
+ u_{\xi} \cdot \xi_{xx} + u_{\eta} \cdot \eta_{xx},
\]

\[
v''_{yy} = u_{\xi\xi} \cdot (\xi_y^2)^2 + u_{\eta\eta} \cdot (\eta_y^2)^2 + 2u_{\xi\eta} \cdot \xi_y \cdot \eta_y
\]

\[
+ u_{\xi} \cdot \xi_{yy} + u_{\eta} \cdot \eta_{yy}.
\]

由于

\[
\xi_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} = -\eta_y, \quad \xi_y = \frac{2xy}{(x^2 + y^2)^2} = \eta_x,
\]

\[
\xi_{xx} = (\xi_x)_x = (\eta_y)_y = (\eta_x)_x = -\xi_{yy},
\]

\[
\eta_{xx} = (\eta_x)_y = (\eta_y)_y = (\xi_x)_x = -\eta_{yy},
\]

及

\[
u''_{\xi\xi}(\xi, \eta) + u''_{\eta\eta}(\xi, \eta) = 0,
\]

故

\[
\Delta u = u''_{xx} + u''_{yy} = u''_{\xi\xi}(\xi_x^2)^2 + u''_{\eta\eta}(\eta_x^2)^2
\]

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\[ + 2u_1^\prime \cdot \xi^1 \eta^2 + u_1^\prime \cdot \xi^2 \eta^2 \\
+ u_1^\prime \cdot \eta^2 x + u_1^\prime \cdot \eta^2 \cdot (\eta^2)^2 + u_1^\prime \cdot (\eta^2)^2 \\
+ 2u_1^\prime \cdot \eta^2 \cdot (\eta^2)^2 + u_1^\prime \cdot (\eta^2)^2 + u_1^\prime \cdot (\eta^2)^2 \\
= \left[ u_1^\prime + u_1^\prime \right] \left[ (\frac{\xi^2}{\xi})^2 + (\frac{\eta^2}{\eta})^2 \right] = 0, \]

即函数 \( u \) 也满足拉普拉斯方程。

3309. 证明：函数

\[
u = \frac{1}{2a \sqrt{\pi t}} e^{-\frac{(x-b)^2}{4a^2 t}}
\]

（\( a \) 和 \( b \) 为常数）满足热传导方程

\[
\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.
\]

证

\[
\frac{\partial u}{\partial t} = \frac{1}{8a^3 t^2 \sqrt{\pi t}} e^{-\frac{(x-b)^2}{4a^2 t}} \cdot \left[ (x-b)^2 - 2a^2 t \right],
\]

\[
\frac{\partial u}{\partial x} = -\frac{x-b}{4a^2 t \sqrt{\pi t}} e^{-\frac{(x-b)^2}{4a^2 t}},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{8a^5 t^2 \sqrt{\pi t}} e^{-\frac{(x-b)^2}{4a^2 t}} \cdot \left[ (x-b)^2 - 2a^2 t \right].
\]

将 \( \frac{\partial u}{\partial t} \) 与 \( \frac{\partial^2 u}{\partial x^2} \) 比较即得
\[
\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.
\]

即函数 \(u\) 满足热传导方程。

3310. 证明：若函数 \(u = u(x, t)\) 满足热传导方程（参阅 3309 题），则函数

\[
v = \frac{1}{a \sqrt{t}} e^{-\frac{x^2}{4a^2 t}} \left( -\frac{x}{a^3 t} - \frac{1}{a^4 t} \right) (t \gg 0)
\]

也满足该方程。

证 设 \(w = w(x, t) = \frac{1}{a \sqrt{t}} e^{-\frac{x^2}{4a^2 t}}\)，此函数即 3309 题中的函数 \(u\) 乘以 \(2 \sqrt{t}\)，并令 \(b = 0\) 后得到。因此，它满足热传导方程

\[
\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}.
\]

显然有

\[
\frac{\partial w}{\partial x} = -\frac{2x}{4a^2 t} w = -\frac{xw}{2a^2 t}.
\]

令 \(\xi = \xi(x, t) = \frac{x}{a^2 t}\)， \(\eta = \eta(t) = -\frac{1}{a^4 t}\)，则

\[
\xi_t = \frac{1}{a^2 t}, \xi_{tt} = 0, \quad \xi_t = -\frac{x^2}{a^2 t^2}, \quad \eta_t = \frac{1}{a^4 t^2}.
\]

由于 \(v = w(x, t) \cdot u(\xi, \eta)\) 及 \(u_t = a^2 u_{tt}\)，故

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\[ v_i^1 = w_i^1 \cdot u + w \cdot (u_i^1 \cdot \xi_i^1 + u_i^1 \cdot \eta_i^1) \]

\[ = a^2 w_{xx}^1 \cdot u + w \left[ u_x^1 \cdot \left( -\frac{x^2}{a^2 t^2} \right) + a^2 u_{xx}^1 \cdot \left( -\frac{1}{a^4 t^2} \right) \right] \]

\[ v_x^1 = w_x^1 \cdot u + w u_x^1 \cdot \xi_x^1, \]

\[ v_{xx}^1 = w_{xx}^1 \cdot u + 2 w_x^1 \cdot u_x^1 \xi_x^1 + w u_{xx}^1 \cdot (\xi_x^1)^2 + w u_x^1 \cdot \xi_x^1 \]

\[ = w_{xx}^1 \cdot u + 2 \left( -\frac{xw}{2a^2 t} \right) u_x^1 \cdot \left( \frac{x}{a^2 t} \right) + w u_{xx}^1 \cdot \left( -\frac{1}{a^4 t^2} \right)^2 \]

\[ = w_{xx}^1 \cdot u - \frac{x^2 w}{a^4 t^2} u_x^1 + \frac{w}{a^4 t^2} u_{xx}^1. \]

将 \( v_i^1 \) 与 \( v_{xx}^1 \) 比较即得

\[ v_i^1 = a^2 v_{xx}^1, \]

即函数 \( v \) 也满足热传导方程。

3311. 证明：函数

\[ u = \frac{1}{r} \]

（式中 \( r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \) 当 \( r \neq 0 \)

时，满足拉普拉斯方程

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \]
证  本题证法与3282 题(6)的证法完全类似，只要将该题中的 \(x, y, z\) 换成 \(x-a, y-b, z-b\) 即可。事实上，

\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^3} + \frac{3(x-a)^2}{r^5},
\]

\[
\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^3} + \frac{3(y-b)^2}{r^5},
\]

\[
\frac{\partial^2 u}{\partial z^2} = -\frac{1}{r^3} + \frac{3(z-c)^2}{r^5}.
\]

将上述三式相加，即证得

\[\Delta \left( \frac{1}{r} \right) = 0.\]

3312. 证明：若函数 \(u=u(x, y, z)\) 满足拉普拉斯方程（参阅3311题），则函数

\[v = \frac{1}{r} u \left( \frac{k^2 x}{r^2}, \frac{k^2 y}{r^2}, \frac{k^2 z}{r^2} \right)\]

（式中 \(k\)为常数及 \(r = \sqrt{x^2 + y^2 + z^2}\) 也 满足该方程。

证  证法一

设 \(S=S(x, y, z)=\frac{1}{r}\)，则由3282题(6)知

\[\Delta S = S_{xx} + S_{yy} + S_{zz} = 0.\]
\[(S_1')^2 + (S_2')^2 + (S_3')^2 = \frac{1}{r^2} = S^4.\]

\[S_1' = -\frac{x}{r^2} = -S^3 x, \quad S_2' = -S^3 y, \quad S_3' = -S^3 z.\]

记 \[v = \frac{1}{r} a \left( \frac{h^2 x}{r^2}, \frac{h^2 y}{r^2}, \frac{h^2 z}{r^2} \right)\]

\[= Su \left( h^2 S^2 x, h^2 S^2 y, h^2 S^2 z \right)\]

\[= Su(x, y, z, S) = F(x, y, z, S).\]

于是，

\[v_x' = F'_x + F'_y \cdot S_x'.\]

注意到 \(F'_x\) 和 \(F'_y\) 也是自变量 \(x, y, z\) 和中间变量 \(S\) 的函数，即得

\[v_{xx}'' = F''_{xx} + 2F''_{xs} \cdot S_x' + F''_{ss} \cdot (S_x')^2 + F'_x \cdot S_{xx}'.\]

由对称性得

\[v_{yy}'' = F''_{yy} + 2F''_{ys} \cdot S_y' + F''_{ss} \cdot (S_y')^2 + F'_y \cdot S_{yy}'.\]

\[v_{zz}'' = F''_{zz} + 2F''_{zs} \cdot S_z' + F''_{ss} \cdot (S_z')^2 + F'_z \cdot S_{zz}'.\]

于是，

\[\Delta v = (F''_{xx} + F''_{yy} + F''_{zz}) + F'_x \cdot (S''_{xx} + S''_{yy} + S''_{zz}).\]

110.
\[
+ \left\{ 2(F_{iS}^{i'} \cdot S^i_x + F_{iS}^{i'} \cdot S^i_y + F_{iS}^{i'} \cdot S^i_z) + F_{SS}^{i'} \left[ (S_x^i)^2 + (S_y^i)^2 + (S_z^i)^2 \right] \right\},
\]

显然第二个括弧为零，也不难验证第一个括弧为零。事实上，

\[
F_{xx}^{ii'} + F_{xy}^{ii'} + F_{xz}^{ii'} = k^4 S^5 \cdot (u_{i1}^{ii'} + u_{i2}^{ii'} + u_{i3}^{ii'}) = 0.
\]

现在来计算最后一个括弧。注意到

\[
S w'_s = 2k^2 S^2 x u'_1 + 2k^2 S^2 y u'_2 + 2k^2 S^2 z u'_3
\]

\[
= 2x w'_x + 2y w'_y + 2z w'_z,
\]

即得

\[
F_{SS}^{ii'} \cdot \left[ (S_x^i)^2 + (S_y^i)^2 + (S_z^i)^2 \right] = (S w)'_{SS} \cdot S^4
\]

\[
= (w + S w'_S)^{ii'} \cdot S^4
\]

\[
= (w + 2x w'_x + 2y w'_y + 2z w'_z)\cdot S^4
\]

\[
= S^4 w'_s + 2x S^4 w'_{sS} + 2y S^4 w'_{sS} + 2z S^4 w'_{sS} + 2z S^4 w'_{is}. \quad (1)
\]

而

\[
2(F_{iS}^{i'} \cdot S^i_x + F_{iS}^{i'} \cdot S^i_y + F_{iS}^{i'} \cdot S^i_z)
\]

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\[= 2(S w)_{x_3}^{x_3} (-S^x x) + 2(S w)_{y_3}^{y_3} (-S^y y) + 2(S w)_{z_3}^{z_3} (-S^z z)\]

\[=-2S^x x \cdot (S w_x^x)_3 - 2S^y y \cdot (S w_y^x)_3 - 2S^z z \cdot (S w_z^x)_3\]

\[=-2S^x x \cdot (w_x^x + Sw_{x_3}^x) - 2S^y y \cdot (w_y^x + Sw_{y_3}^x)\]

\[= -S^x (2xw^x + 2yw^x + 2zw^x) - 2S^x w_{x_3}^x\]

\[-2yS^4 w_{y_3}^x - 2zS^4 w_{z_3}^x\]

\[= -S^4 w_x^x - 2xS^4 w_{x_3}^x - 2yS^4 w_{y_3}^x - 2zS^4 w_{z_3}^x. \quad (2)\]

比较(1)式和(2)式即知第三个括弧也为零。于是，最后证得

\[\Delta v = 0\]

证法二

本题也可直接求出\(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}\)，进而证得

\[\Delta v = 0.\] 事实上，设

\[\frac{k^2 x}{r^2} = t_1, \quad \frac{k^2 y}{r^2} = t_2, \quad \frac{k^2 z}{r^2} = t_3, \]

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利用3306题的结果即可得

$$\Delta v = \frac{1}{r} \left[ \frac{\partial^2 u(t_1, t_2, t_3)}{\partial x^2} + \frac{\partial^2 u(t_1, t_2, t_3)}{\partial y^2} + \frac{\partial^2 u(t_1, t_2, t_3)}{\partial z^2} \right] + u(t_1, t_2, t_3) \Delta \left( \frac{1}{r} \right)$$

$$+ 2 \left[ \frac{\partial u(t_1, t_2, t_3)}{\partial x} \frac{\partial \left( \frac{1}{r} \right)}{\partial x} + \frac{\partial u(t_1, t_2, t_3)}{\partial y} \frac{\partial \left( \frac{1}{r} \right)}{\partial y} + \frac{\partial u(t_1, t_2, t_3)}{\partial z} \frac{\partial \left( \frac{1}{r} \right)}{\partial z} \right]. \tag{1}$$

为书写简便起见，记 $u(t_1, t_2, t_3) = u$。分别求 $u$ 及 $\frac{1}{r}$ 对 $x, y, z$ 的一阶偏导函数：

$$\frac{\partial u}{\partial x} = k^2 \cdot \left[ \frac{\partial u}{\partial t_1} \cdot \left( \frac{r^2 - 2x^2}{r^4} \right) + \frac{\partial u}{\partial t_2} \cdot \left( -\frac{2xy}{r^4} \right) \right]$$

$$+ \frac{\partial u}{\partial t_3} \cdot \left( -\frac{2xz}{r^4} \right),$$

$$\frac{\partial u}{\partial y} = k^2 \cdot \left[ \frac{\partial u}{\partial t_1} \cdot \left( -\frac{2xy}{r^4} \right) + \frac{\partial u}{\partial t_2} \cdot \left( \frac{r^2 - 2y^2}{r^4} \right) + \frac{\partial u}{\partial t_3} \cdot \left( -\frac{2yz}{r^4} \right) \right],$$

$$\frac{\partial u}{\partial z} = k^2 \cdot \left[ \frac{\partial u}{\partial t_1} \cdot \left( -\frac{2xz}{r^4} \right) + \frac{\partial u}{\partial t_2} \cdot \left( -\frac{2yz}{r^4} \right) + \frac{\partial u}{\partial t_3} \cdot \left( \frac{r^2 - 2z^2}{r^4} \right) \right];$$
\[
\begin{align*}
\frac{\partial (\frac{1}{r})}{\partial x} &= -\frac{x}{r^3}, \quad \frac{\partial (\frac{1}{r})}{\partial y} = -\frac{y}{r^3}, \\
\frac{\partial (\frac{1}{r})}{\partial z} &= -\frac{z}{r^3}.
\end{align*}
\]

从而得

\[
\frac{\partial^2 u}{\partial x^2} = k^4 \cdot \left[ \frac{\partial^2 u}{\partial t_1^2} \cdot \left( \frac{r^2 - 2x^2}{r^4} \right) + \frac{\partial^2 u}{\partial t_1 \partial t_2} \cdot \left( - \frac{2xy}{r^4} \right) \right] + k^2 \frac{\partial u}{\partial t_1} \cdot \left( \frac{-2xr^4 - 4xr^2(r^2 - 2x^2)}{r^8} \right)
\]

\[
+ k^4 \cdot \left[ \frac{\partial^2 u}{\partial t_2 \partial t_1} \cdot \left( \frac{r^2 - 2x^2}{r^4} \right) + \frac{\partial^2 u}{\partial t_2^2} \cdot \left( - \frac{2xy}{r^4} \right) \right] + \frac{\partial^2 u}{\partial t_2 \partial t_3} \cdot \left( - \frac{2xz}{r^4} \right) \left( - \frac{2xy}{r^4} \right)
\]

\[
+ k^4 \cdot \left[ \frac{\partial^2 u}{\partial t_3 \partial t_1} \cdot \left( \frac{r^2 - 2x^2}{r^4} \right) + \frac{\partial^2 u}{\partial t_3 \partial t_2} \cdot \left( - \frac{2xy}{r^4} \right) \right] + \frac{\partial^2 u}{\partial t_3^2} \cdot \left( - \frac{2xz}{r^4} \right) \left( - \frac{2xz}{r^4} \right) + k^3 \frac{\partial u}{\partial t_3} \cdot \left( \frac{-2xr^4 - 4xr^2(-2xz)}{r^8} \right).
\]
\[
\frac{\partial^4 u}{\partial y^2} = k^4 \left[ \frac{\partial^2 u}{\partial t_1^2} \cdot \left( -\frac{2xy}{r^4} \right) + \frac{\partial^2 u}{\partial t_1 \partial t_3} \cdot \left( -\frac{2yz}{r^4} \right) \right] \left( -\frac{2xy}{r^4} \right) \\
+ k^2 \frac{\partial u}{\partial t_1} \cdot \left[ -\frac{2xr^4 - 4yr^2}{r^8} (\frac{2xy}{r^4}) \right] \\
+ k^4 \left[ \frac{\partial^2 u}{\partial t_2 \partial t_1} \cdot \left( -\frac{2xy}{r^4} \right) + \frac{\partial^2 u}{\partial t_2^2} \cdot \left( -\frac{2y^2}{r^4} \right) \right] \left( -\frac{2y^2}{r^4} \right) \\
+ k^2 \frac{\partial u}{\partial t_2} \cdot \left[ -\frac{2yr^4 - 4yr^2(r^2 - 2y^2)}{r^8} \right] \\
+ k^4 \left[ \frac{\partial^2 u}{\partial t_3 \partial t_1} \cdot \left( -\frac{2xy}{r^4} \right) + \frac{\partial^2 u}{\partial t_3 \partial t_2} \cdot \left( -\frac{2yz}{r^4} \right) \right] \left( -\frac{2yz}{r^4} \right) \\
+ k^2 \frac{\partial u}{\partial t_3} \cdot \left[ -\frac{2xr^4 - 4yr^2(-2yz)}{r^8} \right] \\
\frac{\partial^2 u}{\partial z^2} = k^4 \left[ \frac{\partial^2 u}{\partial t_1^2} \cdot \left( -\frac{2xz}{r^4} \right) + \frac{\partial^2 u}{\partial t_1 \partial t_2} \cdot \left( -\frac{2x^2}{r^4} \right) \right] \left( -\frac{2xz}{r^4} \right) \\
+ k^2 \frac{\partial u}{\partial t_1} \cdot \left[ -\frac{2xr^4 - 4xz^2}{r^8} (-2xz) \right] \\
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\]
\[
+ h^4 \left[ \frac{\partial^2 u}{\partial t_1 \partial t_2} \left( -\frac{2xz}{r^4} \right) + \frac{\partial^2 u}{\partial t_1^2} \left( -\frac{2yz}{r^4} \right) \right] \\
+ \frac{\partial^2 u}{\partial t_2 \partial t_3} \left( \frac{r^2 - 2z^2}{r^4} \right) \left( -\frac{2yz}{r^4} \right) \\
+ h^2 \frac{\partial u}{\partial t_2} \left[ -\frac{2yz}{r^4} \left( -\frac{2yz}{r^4} \right) \right] \\
+ h^4 \left[ \frac{\partial^2 u}{\partial t_3 \partial t_1} \left( -\frac{2xz}{r^4} \right) + \frac{\partial^2 u}{\partial t_3 \partial t_2} \left( -\frac{2yz}{r^4} \right) \right] \\
+ \frac{\partial^2 u}{\partial t_3^2} \left( \frac{r^2 - 2z^2}{r^4} \right) \left( \frac{r^2 - 2z^2}{r^4} \right) \right] \\
+ h^2 \frac{\partial u}{\partial t_3} \left[ -\frac{2yz}{r^4} \left( -\frac{2yz}{r^4} \right) \right].
\]

将 \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial (\frac{1}{r})}{\partial x}, \frac{\partial (\frac{1}{r})}{\partial y}, \frac{\partial (\frac{1}{r})}{\partial z} \) 及 \( \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2} \) 代入 (1) 式，合并整理，并注意到

\( \Delta \left( \frac{1}{r} \right) = 0 \) 及 \( \frac{\partial^2 u}{\partial t_1^2} + \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial^2 u}{\partial t_3^2} = 0 \),

即得

\[
\Delta u = \frac{1}{r} \left[ \frac{h^4}{r^4} \left( \frac{\partial^2 u}{\partial t_1^2} + \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial^2 u}{\partial t_3^2} \right) \right] \\
- \frac{2h^2}{r^4} \left( \frac{\partial^2 u}{\partial t_1 \partial t_2} + \frac{\partial^2 u}{\partial t_2 \partial t_3} + \frac{\partial^2 u}{\partial t_3 \partial t_1} \right).
\]
\begin{align*}
+ 0 & \cdot \sum_{i=1}^{3} \frac{\partial^2 u}{\partial t_i \partial t_i} \bigg] + u \cdot 0 + \frac{2b^2}{r^6} \left( x \frac{\partial u}{\partial f_1} \\
+ y \frac{\partial u}{\partial f_2} + z \frac{\partial u}{\partial f_3} \right) = 0 ,
\end{align*}

上述说明函数 \( v = v(x, y, z) \) 也满足拉普拉斯方程。

3313. 证明：函数

\[ u = \frac{C_1 e^{-x}}{r} + C_2 e^{r} \]

（式中 \( r = \sqrt{x^2 + y^2 + z^2} \) 及 \( C_1, C_2 \) 为常数）满足

爱尔兰戈尔兹方程

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 u . \]

证 设

\[ v = \frac{1}{r} e^{-xr}, \quad w = \frac{1}{r} e^{xr}, \]

则有

\[ u = C_1 v + C_2 w . \]

\[ v_x = v_x^* \cdot r_x = e^{-xr} \cdot \left( -\frac{1}{r^2} - \frac{a}{r} \right) \cdot \frac{x}{r} \]

\[ = -xv \cdot \left( \frac{1}{r^2} + \frac{a}{r} \right) , \]

\[ v_{xx} = -v_x^* \cdot \left( \frac{1}{r^2} + \frac{a}{r} \right) x - v \cdot \left( -\frac{2}{r^3} - \frac{a}{r^2} \right) \]

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\[
\frac{x}{r} \cdot x - \nu \cdot \left( \frac{1}{r^2} + \frac{a}{r} \right) \\
= x^2 \nu \cdot \left( \frac{1}{r^2} + \frac{a}{r} \right) \cdot \frac{1}{r} \\
= \nu \cdot \left[ \left( \frac{3}{r^4} + \frac{3a}{r^2} + \frac{a^2}{r^2} \right) \cdot (x^2 + y^2) \\
+ r^2 - \frac{3a}{r} \right] = a^2 \nu.
\]

利用对称性，即得

\[
\Delta u = \nu \cdot \left[ \left( \frac{3}{r^4} + \frac{3a}{r^2} + \frac{a^2}{r^2} \right) \cdot (x^2 + y^2) \\
+ r^2 - \frac{3a}{r} \right] = a^2 \nu.
\]

记 \( b = -a \)，则 \( w = \frac{1}{r} e^{-br} \)。仿上述证明，有

\[
\Delta w = b^2 w = a^2 w.
\]

于是，

\[
\Delta u = \Delta (C_1 u + C_2 w) = C_1 \Delta u + C_2 \Delta w
\]

\[= C_1 a^2 v + C_2 a^2 w = a^2 u,
\]

即

\[\Delta u - a^2 u.
\]

3314. 设函数 \( u_1 = u_1(x, y, z) \) 及 \( u_2 = u_2(x, y, z) \) 满足拉普拉
\[ \Delta (\Delta v) = 0. \]

证  利用 3306 题的结果，即得
\[ \Delta v = \Delta u_1 + (x^2 + y^2 + z^2) \Delta u_2 + u_2 \cdot \Delta (x^2 + y^2 + z^2) + 2 \left( 2x \frac{\partial u_2}{\partial x} + 2y \frac{\partial u_2}{\partial y} + 2z \frac{\partial u_2}{\partial z} \right) \]
\[ = 6u_2 + 4 \left( x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} + z \frac{\partial u_2}{\partial z} \right). \]

重复应用同一结果于 \( \Delta v \)，得
\[ \Delta (\Delta v) = 6\Delta u_2 + 4 \left\{ x \Delta \left( \frac{\partial u_2}{\partial x} \right) + y \Delta \left( \frac{\partial u_2}{\partial y} \right) + z \Delta \left( \frac{\partial u_2}{\partial z} \right) + \frac{\partial u_2}{\partial x} \Delta x + \frac{\partial u_2}{\partial y} \Delta y + \frac{\partial u_2}{\partial z} \Delta z + 2 \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} \right) \right\}. \]

由于
\[ \Delta \left( \frac{\partial u_2}{\partial x} \right) = \frac{\partial^2}{\partial x^2} \left( \frac{\partial u_2}{\partial x} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial u_2}{\partial x} \right) + \frac{\partial^2}{\partial z^2} \left( \frac{\partial u_2}{\partial x} \right) \]
\[ = \frac{\partial}{\partial x} \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} \right) = \frac{\partial}{\partial x} (\Delta u_2) = 0, \]
\[ \Delta \left( \frac{\partial u_2}{\partial y} \right) = 0, \Delta \left( \frac{\partial u_2}{\partial z} \right) = 0, \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} \right) = \frac{\partial}{\partial x} (\Delta u_2) = 0, \]
故最后证得
\[ A(Av) = 0. \]

3315. 设 \( f(x, y, z) \) 是可微分 \( m \) 次的 \( n \) 次齐次函数。证明

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^m f(x, y, z)
= n(n-1) \cdots (n-m+1) f(x, y, z).
\]

证 证法一
根据齐次函数的定义知，函数 \( f(x, y, z) \) 满足

\[ f(ix, iy, iz) = t^m f(x, y, z). \]  (1)

在(1)式两端分别对 \( t \) 求 \( m \) 次导数，首先考察 \( \frac{d^m f}{dt^m} \)。由求全导数的公式知

\[
\frac{df}{dt} = x \frac{\partial f}{\partial (xt)} + y \frac{\partial f}{\partial (yt)} + z \frac{\partial f}{\partial (zt)}
= t^{n-1} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) f(x, y, z),
\]

\[
\frac{d^2 f}{dt^2} = \frac{d}{dt} \left( \frac{df}{dt} \right)
= x \left\{ x \frac{\partial^2 f}{\partial (xt)^2} \right. \\
+ y \frac{\partial^2 f}{\partial (xt) \partial (yt)} + z \frac{\partial^2 f}{\partial (xt) \partial (zt)} \left\} \\
+ y \left\{ x \frac{\partial^2 f}{\partial (yt) \partial (xt)} + y \frac{\partial^2 f}{\partial (yt)^2} + z \frac{\partial^2 f}{\partial (yt) \partial (zt)} \right\} \\
+ z \left\{ x \frac{\partial^2 f}{\partial (zt) \partial (xt)} + y \frac{\partial^2 f}{\partial (zt) \partial (yt)} + z \frac{\partial^2 f}{\partial (zt)^2} \right\}
\]
\[\begin{align*}
&= x^2 \frac{\partial^2 f}{\partial (xt)^2} + y^2 \frac{\partial^2 f}{\partial (yt)^2} + z^2 \frac{\partial^2 f}{\partial (zt)^2} \\
&\quad + 2xy \frac{\partial^2 f}{\partial (xt) \partial (yt)} + 2yz \frac{\partial^2 f}{\partial (yt) \partial (zt)} \\
&\quad + 2zx \frac{\partial^2 f}{\partial (zt) \partial (xt)} \\
&= t^{n-2} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^2 f(x, y, z).
\end{align*}\]

一般地，由数学归纳法可得

\[
\frac{d^mf}{dt^m} = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = m} C_{\alpha_1, \alpha_2, \alpha_3} \frac{\partial^n f}{\partial (xt)^{\alpha_1} \partial (yt)^{\alpha_2} \partial (zt)^{\alpha_3}} \cdot x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}
\]

\[
= t^{n-m} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^m f(x, y, z), \quad (2)
\]

其中总和是关于 \( \alpha_1 + \alpha_2 + \alpha_3 = m \) 的非负整数 \( \alpha_1, \alpha_2, \alpha_3 \) 的一切可能组合而取的，且

\[
C_{\alpha_1, \alpha_2, \alpha_3} = \frac{m!}{\alpha_1! \alpha_2! \alpha_3!},
\]

而 (1) 式右端对 \( t \) 求 \( m \) 次导数，得

\[
\left[ t^m f(x, y, z) \right]^{(n)} = n(n-1) \cdots (n-m+1) t^{n-m} f(x, y, z). \quad (3)
\]

比较 (2) 式和 (3) 式，令 \( t = 1 \)，即证得

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^n f(x, y, z)
\]
\[ = n(n-1) \cdots (n-m+1) f(x, y, z). \]

证法二
当 \( m = 1 \) 时，则由
\[ f(tx, ty, tz) = t^n f(x, y, z) \]
两端对 \( t \) 求导，可得
\[
\frac{x \frac{\partial f(tx, ty, tz)}{\partial (tx)}}{\partial (tx)} + \frac{y \frac{\partial f(tx, ty, tz)}{\partial (ty)}}{\partial (ty)} + \frac{z \frac{\partial f(tx, ty, tz)}{\partial (tz)}}{\partial (tz)}
\]
\[ = n t^{n-1} f(x, y, z) \quad (t > 0). \]
令 \( t = 1 \)，即有
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^1 f = nf. \]

当 \( m = 2 \) 时，由 3234 题的结果知
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^2 f = n(n-1)f. \]

在 3233 题中已证得 \( f'_x(x, y, z), f'_y(x, y, z), f'_z(x, y, z) \),
\[ f'_x(x, y, z) \text{ 为} (n-1) \text{ 次的齐次函数}. \]

今设 \( m = k - 1 \) 时命题为真，对 \( f'_x, f'_y, f'_z \) 用数学归纳法的假设，即
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^{k-1} f'_x. \]
\[ = (n - 1)(n - 2) \cdots (n - k + 1) f'_x, \quad (4) \]
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^{k-1} f'_x
\]
\[ = (n - 1)(n - 2) \cdots (n - k + 1) f'_z. \quad (5) \]
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^{k-1} f'_z
\]
\[ = (n - 1)(n - 2) \cdots (n - k + 1) f'_z. \quad (6) \]

将(4)两端乘以x，(5)式两端乘以y，(6)式两端乘以z，然后相加，即得
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^{k} f(x, y, z)
\]
\[ = (n - 1)(n - 2) \cdots (n - k + 1) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) f(x, y, z)
\]
\[ = n(n-1)(n-2) \cdots (n-k+1) f(x, y, z). \]

即当 \( m = k \) 时命题也为真。

于是，命题对于一切自然数 \( m \) 为真，即
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^{n} f
\]
\[ = n(n-1) \cdots (n-k+1) f. \]

3316. 若
\[ z = \sin y + f(\sin x - \sin y), \]
其中 \( f \) 为可微分的函数。简化式子

\[ \sec x \frac{\partial z}{\partial x} + \sec y \frac{\partial z}{\partial y}, \]

解

\[ \sec x \frac{\partial z}{\partial x} + \sec y \frac{\partial z}{\partial y} = \sec x \cos x \cdot f' \]

\[ + \sec y \cdot (\cos y - \cos y \cdot f') \]

\[ = f' + 1 - f' = 1, \]
即

\[ \sec x \frac{\partial z}{\partial x} + \sec y \frac{\partial z}{\partial y} = 1. \]

3317. 证明：函数

\[ z = x^n f\left(\frac{y}{x^2}\right) \]

（其中 \( f \) 为任意的可微分函数）满足方程

\[ x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = nz. \]

证

\[ x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = x \left\{ nx^{n-1} f\left(\frac{y}{x^2}\right) ight. \]

\[ - \frac{2x^n y}{x^3} f'\left(\frac{y}{x^2}\right) \right\} + 2y \frac{x^n}{x^2} f'\left(\frac{y}{x^2}\right) \]

\[ = nx^n f\left(\frac{y}{x^2}\right) = nz, \]
即

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\[ x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = nz. \]

3318. 证明：

\[ z = yf(x^2 - y^2) \]

（其中 \( f \) 为任意的可微分函数）满足方程

\[ y^2 \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = xz. \]

证  \[ y^2 \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = y^2 \cdot 2yf' + xy \cdot (f - 2y^2f') = xyf = xz, \]

即

\[ y^2 \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = xz. \]

3319. 若

\[ u = \frac{1}{12}x^4 - \frac{1}{6}x^3(y + z) + \frac{1}{2}x^2yz + f(y - x, z - x), \]

式中 \( f \) 为可微分的函数。简化式子

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}.
\]

解  \[ \frac{\partial u}{\partial x} = \frac{1}{3}x^3 - \frac{1}{2}x^2(y + z) + xyz - f'_1 - f'_2, \]

\[ \frac{\partial u}{\partial y} = -\frac{1}{6}x^3 + \frac{1}{2}x^2z + f'_1, \]
\[
\frac{\partial u}{\partial x} = -\frac{1}{6}x^3 + \frac{1}{2}x^2y + f_x^2.
\]

将上述三式相加，即得

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz.
\]

3320. 设：

\[x^2 = uv, \quad y^2 = uw, \quad z^2 = uv\]

及

\[f(x, y, z) = F(u, v, w)\]

证明：

\[xf_x^2 + yf_y^2 + zf_z^2 = uF_x^2 + vF_y^2 + wF_z^2.\]

证 把 \(u, v, w\) 当作自变量*1)，故

\[uF_x^2 = uf_x^2 \cdot x^2 + uf_y^2 \cdot y^2 + uf_z^2 \cdot z^2,\]

\[vF_y^2 = vf_x^2 \cdot x^2 + vf_y^2 \cdot y^2 + vf_z^2 \cdot z^2,\]

\[wF_z^2 = wf_x^2 \cdot x^2 + wf_y^2 \cdot y^2 + wf_z^2 \cdot z^2.\]

将上述三式相加，得

\[uF_x^2 + vF_y^2 + wF_z^2 = (ux_x^2 + vx_x^2 + wx_x^2) f_x^2 + (uy_y^2 + vy_y^2 + wy_y^2) f_y^2 + (uz_z^2 + vz_x^2 + wz_w^2) f_z^2 + (ux_x^2 + vx_x^2 + wx_x^2) f_x^2\]
\[ +uz'_w + wz'_w \) f'_z. \]  

(1)

由题设得 \( 2x \frac{\partial x}{\partial u} = 0 \)。因为 \( x \) 不恒等于零，所以 \( \frac{\partial x}{\partial u} = 0 \)。同法可得 \( \frac{\partial y}{\partial v} = 0, \frac{\partial z}{\partial w} = 0 \)。

再由题设，得

\[ 2x \frac{\partial x}{\partial w} = v, 2x \frac{\partial x}{\partial v} = w, 2y \frac{\partial y}{\partial u} = w, \]
\[ 2y \frac{\partial y}{\partial w} = u, 2x \frac{\partial z}{\partial u} = v, 2x \frac{\partial z}{\partial v} = u. \]

将上述结果代入 (1) 式，得

\[ uF'_v + vF'_u + wF'_w = \left( \frac{uv}{2x} + \frac{wu}{2x} \right) f'_z \]
\[ + \left( \frac{uw}{2y} + \frac{wu}{2y} \right) f'_z + \left( \frac{uv}{2z} + \frac{uw}{2z} \right) f'_z \]
\[ = xf'_z + yf'_z + zf'_z. \]

即

\[ uF'_v + vF'_u + wF'_w = xf'_z + yf'_z + zf'_z. \]

*) 如果把 \( x, y, z \) 当作自变量，也可以证明本题的结果。

假定任意函数 \( \psi, \phi \) 等等为可微分足够多次的函数，
验证下列等式：

3321. \( y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0 \)，若 \( z = \varphi(x^2 + y^2) \)。

证 由于

\[ y \frac{\partial z}{\partial x} = y \cdot 2x \varphi'(x^2 + y^2), \]

\[ x \frac{\partial z}{\partial y} = x \cdot 2y \varphi'(x^2 + y^2), \]

所以

\[ y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0. \]

3322. \( x^2 \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial y} + y^2 = 0 \)，若 \( z = \frac{y^2}{3x} + \varphi(xy) \)。

证 \( x^2 \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial y} + y^2 = x^2 \left[ -\frac{y^2}{3x^2} + y \varphi'(xy) \right] \]

\[ -xy \left[ \frac{2y}{3x} + x \varphi'(xy) \right] + y^2 = 0. \]

3323. \( (x^2 - y^2) \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = xyz \)，若 \( z = e^y \varphi\left( ye^{\frac{x^2}{y^2}} \right) \)。

证 \( (x^2 - y^2) \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = (x^2 - y^2) e^y \cdot \frac{xy}{y^2} ye^{\frac{x^2}{y^2}} \]

\[ + xy \left\{ e^y \cdot \varphi + e^y \varphi' \left[ \frac{e^\frac{x^2}{y^2} - \frac{x^2}{y^2} ye^{\frac{x^2}{y^2}}}{x^2} \right] \right\} \]

\[ = xyz e^y \varphi = xyz. \]

3324. \( x \frac{\partial u}{\partial x} + \alpha y \frac{\partial u}{\partial y} + \beta z \frac{\partial u}{\partial z} = nu \)，若 \( u = x^a \varphi\left( \frac{y}{x^a}, \frac{z}{x^b} \right) \)。
证 \( x \frac{\partial u}{\partial x} + ay \frac{\partial u}{\partial y} + \beta z \frac{\partial u}{\partial z} = nx^n \varphi - ax^n - y \varphi_1' \)
\( - \beta x^{n-2} z \varphi_2' + ay x^{n-2} \varphi_1' + \beta z x^{n-2} \varphi_2' \)
\( = nx^n \varphi = nu. \)

3325. \( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u + \frac{xy}{z}, \) 若

\( u = \frac{xy}{z} \ln x + x \varphi (\frac{y}{x}, \frac{z}{x}). \)

证 \( x \frac{\partial u}{\partial x} = x \cdot \frac{y}{z} \ln x + \frac{x y}{z} \cdot x \varphi - y \varphi_1' - z \varphi_2', \)
\( y \frac{\partial u}{\partial y} = \frac{x y}{z} \ln x + y \varphi_1', z \frac{\partial u}{\partial z} = -\frac{x y}{z} \ln x + z \varphi_2'. \)

将上述三式相加，即得

\( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u + \frac{xy}{z}. \)

3326. \( \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \) 若 \( u = \varphi(x - at) + \psi(x + at). \)

证 \( \frac{\partial^2 u}{\partial t^2} = a^2 \varphi'' + a^2 \psi'' , \frac{\partial^2 u}{\partial x^2} = \varphi'' + \psi''. \)

将上述二式比较，即得

\( \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \)

3327. \( \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \) 若
\[ u = x\varphi(x + y) + y\psi(x + y). \]

证 \[
\frac{\partial u}{\partial x} = \varphi + y\psi' + x\varphi', \quad \frac{\partial u}{\partial y} = x\varphi' + \psi + y\psi',
\]
\[
\frac{\partial^2 u}{\partial x^2} = 2\varphi' + y\psi'' + x\varphi'', \quad (1)
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \varphi' + \psi' + y\psi'' + x\varphi'', \quad (2)
\]
\[
\frac{\partial^2 u}{\partial y^2} = x\varphi'' + 2\psi' + y\psi''. \quad (3)
\]

(1) $- 2 \times (2) + (3)$，即得
\[
\frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

3328. \[ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0, \] 若
\[ u = \varphi\left(\frac{y}{x}\right) + x\psi\left(\frac{y}{x}\right). \]

证  $u_1 = \varphi\left(\frac{y}{x}\right)$ 为零次齐次函数， $u_2 = x\psi\left(\frac{y}{x}\right)$ 为一
次齐次函数。由 3234 题的结果（对于二元更成立）知
\[
(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^2 u_1 = 0, \quad (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^2 u_2 = 0.
\]

于是，
\[ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \]
\[\begin{align*}
&= \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 (u_1 + u_2) \\
&= \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 u_1 + \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 u_2 \\
&= 0 + 0 = 0.
\end{align*}\]

注：也可不引用3234题的结果，求出偏导数直接验证。

3329. \(x \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n - 1)u\)，若

\[u = x^n \varphi \left( \frac{y}{x} \right) + x^{1-n} \psi \left( \frac{y}{x} \right).\]

证 \(u_1 = x^n \varphi \left( \frac{y}{x} \right)\) 为 \(n\) 次齐次函数，\(u_2 = x^{1-n} \psi \left( \frac{y}{x} \right)\) 为 \(1-n\) 次齐次函数。由 3234 题的结果知

\[\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 u_1 = n(n - 1)u_1,\]

\[\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 u_2 = (1-n)(1-n-1)u_2\]

\[= n(n - 1)u_2.\]

于是，

\[\begin{align*}
x \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}
&= \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 (u_1 + u_2).
\end{align*}\]
\[ n(n-1)(u_1 + u_2) = n(n-1)u. \]

值得注意的是，3328题即为本题的特殊情况。

\[ n = 0. \]

3330. \[ \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x^2}, \quad \text{若} \quad u = \varphi(x + \psi(y)). \]

证 \[ \frac{\partial u}{\partial x} = \varphi', \quad \frac{\partial^2 u}{\partial x \partial y} = \varphi'' \psi', \]

\[ \frac{\partial u}{\partial y} = \varphi' \psi, \quad \frac{\partial^2 u}{\partial x^2} = \varphi''. \]

于是，

\[ \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x^2}. \]

用逐次微分的方法消去任意函数 \( \varphi \) 和 \( \psi \)。

3331. \[ z = x + \varphi(xy). \]

解 \[ \frac{\partial z}{\partial x} = 1 + y \varphi', \quad \frac{\partial z}{\partial y} = x \varphi'. \]

于是，

\[ x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = x. \]

3332. \[ z = x \varphi\left(\frac{x}{y^2}\right). \]

解 \[ \frac{\partial z}{\partial x} = \varphi + \frac{x}{y^2} \varphi', \quad \frac{\partial z}{\partial y} = -\frac{2x^2}{y^3} \varphi'. \]

于是，
\[2x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2x \varphi + \frac{2x^2}{y^2} \varphi' - \frac{2x^2}{y^2} \varphi'
\]

即

\[2x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z.
\]

3333. \( z = \varphi \left( \sqrt{x^2 + y^2} \right) \).

解

\[\frac{\partial z}{\partial x} = \frac{x \varphi'}{\sqrt{x^2 + y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y \varphi'}{\sqrt{x^2 + y^2}}.
\]

于是，

\[y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0.
\]

3334. \( u = \varphi(x-y, y-z) \).

解

\[\frac{\partial u}{\partial x} = \varphi', \quad \frac{\partial u}{\partial y} = -\varphi' + \varphi', \quad \frac{\partial u}{\partial z} = -\varphi'.
\]

于是，

\[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.
\]

3335. \( u = \varphi \left( \frac{x}{y}, \frac{y}{z} \right) \).

解

\[\frac{\partial u}{\partial x} = \frac{1}{y} \varphi' \frac{x}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2} \varphi' + \frac{1}{z} \varphi',
\]

\[\frac{\partial u}{\partial z} = -\frac{y}{z^2} \varphi'.
\]
于是，
\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0. \]

*) 注意到 \( \varphi \left( \frac{x}{y}, \frac{y}{z} \right) \) 为零次齐次函数，本题即3315题的特殊情形：\( n = 0. \)

3336. \( z = \varphi (x) + \psi (y). \)

解 \( \frac{\partial z}{\partial x} = \varphi' (x) \). 于是，

\[ \frac{\partial^2 z}{\partial x \partial y} = 0. \]

3337. \( z = \varphi (x) \psi (y). \)

解 \( \frac{\partial z}{\partial x} = \varphi' \psi, \ \frac{\partial z}{\partial y} = \varphi \psi', \ \frac{\partial^2 z}{\partial x \partial y} = \varphi' \psi'. \)

于是，

\[ z \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}. \]

3338. \( z = \varphi (x + y) + \psi (x - y). \)

解 \( \frac{\partial z}{\partial x} = \varphi' + \psi', \ \frac{\partial z}{\partial y} = \varphi' - \psi', \)

\[ \frac{\partial^2 z}{\partial x^2} = \varphi'' + \psi'', \ \frac{\partial^2 z}{\partial y^2} = \varphi'' + \psi''. \]

于是，
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}.
\]

3339. \( z = x\varphi\left(\frac{x}{y}\right) + y\psi\left(\frac{x}{y}\right) \).

解 注意到函数 \( z \) 为一次齐次函数，由3315题知
\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z.
\]

3340. \( z = \varphi(xy) + \psi\left(\frac{x}{y}\right) \).

解 设 \( z_1 = \varphi(xy) \)，则由3331题知
\[
x \frac{\partial z_1}{\partial x} - y \frac{\partial z_1}{\partial y} = 0.
\]

又 \( z_2 = \psi\left(\frac{x}{y}\right) \) 为零次齐次函数，且函数
\[
x \frac{\partial z_2}{\partial x} - y \frac{\partial z_2}{\partial y} = \frac{2x}{y} \psi',
\]

也为零次齐次函数。从而，函数
\[
u = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = \left( x \frac{\partial z_1}{\partial x} - y \frac{\partial z_1}{\partial y} \right)
\]
\[
+ \left( x \frac{\partial z_2}{\partial x} - y \frac{\partial z_2}{\partial y} \right)
\]

是零次齐次函数。于是，由3315题知
\[
x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 0.
\]

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但是，
\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \frac{\\partial}{\\partial x} \left( x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} \right) \\
+ y \frac{\partial}{\partial y} \left( x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} \right) \\
= x^2 \frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial x} - xy \frac{\partial^2 z}{\partial x \partial y} + xy \frac{\partial^2 z}{\partial x \partial y} \\
- y \frac{\partial z}{\partial y} - y^2 \frac{\partial^2 z}{\partial y^2} \\
= x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y},
\]
故得
\[
x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0.
\]

3341. 求函数
\[
z = x^2 - y^2
\]
在点 \( M(1,1) \) 沿与 Ox 轴的正向组成角 \( \alpha = 60^\circ \) 的方向 \( l \) 上的导函数。

解 \( \frac{\partial z}{\partial x} \bigg|_{(x,y) = 1} = 2, \quad \frac{\partial z}{\partial y} \bigg|_{(x,y) = 1} = -2. \)

\[
\cos \alpha = \cos 60^\circ = \frac{1}{2}, \quad \cos \beta = \cos 30^\circ = \frac{\sqrt{3}}{2}.
\]

于是，
\[
\frac{\partial z}{\partial l} \bigg|_{(x,y) = 1} = 2 \cdot \frac{1}{2} + (-2) \cdot \frac{\sqrt{3}}{2} = 1 - \sqrt{3}.
\]
3342. 求函数

\[ z = x^2 - xy + y^2 \]

在点 \( M(1, 1) \) 沿与 \( Ox \) 轴的正向组成 \( \alpha \) 角的方向上导函数，求怎样的方向上此导函数有：(a) 最大的值；(b) 最小的值；(c) 等于 0。

解  \( \frac{\partial z}{\partial x} \bigg|_{y=1} = 1, \quad \frac{\partial z}{\partial y} \bigg|_{x=1} = 1 \)。于是，

\[ \frac{\partial z}{\partial l} \bigg|_{y=1} = \cos \alpha + \cos(90^\circ - \alpha) = \cos \alpha + \sin \alpha \]

\[ = \sqrt{2} \sin \left( \alpha + \frac{\pi}{4} \right). \]

(a) 当 \( \sin \left( \alpha + \frac{\pi}{4} \right) = 1 \)，即 \( \alpha = \frac{\pi}{4} \) 时，\( \frac{\partial z}{\partial l} \) 最大；

(b) 当 \( \sin \left( \alpha + \frac{\pi}{4} \right) = -1 \)，即 \( \alpha = \frac{5\pi}{4} \) 时，\( \frac{\partial z}{\partial l} \) 最小；

(c) 当 \( \sin \left( \alpha + \frac{\pi}{4} \right) = 0 \)，即 \( \alpha = \frac{3\pi}{4} \) 或 \( \alpha = \frac{7\pi}{4} \)

时，\( \frac{\partial z}{\partial l} = 0 \)。

3343. 求函数

\[ z = \ln(x^2 + y^2) \]

在点 \( M_0(x_0, y_0) \) 沿与过此点的等位线成垂直的方向上的导数。

解  与等位线垂直的方向即梯度的方向或与梯度相反。
的方向。于是，

\[
\left. \frac{\partial z}{\partial t} \right|_{z = z_0} = \pm |\nabla z| \left|_{z = z_0}ight.
\]

\[
= \pm \sqrt{\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \left|_{z = z_0}\right.
\]

\[
= \pm \sqrt{\left(\frac{2x_0}{x_0^2 + y_0^2} \right)^2 + \left(\frac{2y_0}{x_0^2 + y_0^2} \right)^2} = \pm \frac{2}{\sqrt{x_0^2 + y_0^2}}.
\]

3344. 求函数

\[z = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\]

在点 \(M\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)\) 沿曲线 \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) 在此点的内法线方向上的导数。

解 曲线 \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) 是函数 \(z\) 的一条等位线。随着 \(x, y\) 的绝对值增大，\(z\) 是减少的，因此，曲线的内法线方向即梯度方向。于是，

\[
\left. \frac{\partial z}{\partial t} \right|_{z = \frac{a}{\sqrt{2}}} = |\nabla z| \left|_{z = \frac{a}{\sqrt{2}}}\right.
\]

\[
= \sqrt{\frac{4x^2}{a^4} + \frac{4y^2}{b^4}} \left|_{z = \frac{a}{\sqrt{2}}}\right.
\]

\[
= \frac{\sqrt{2(a^2 + b^2)}}{ab} \quad (a > 0, b > 0).
\]

3345. 求函数

\[u = xyz\]

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在点 \( M(1, 1, 1) \)沿方向 \( \{ \cos \alpha, \cos \beta, \cos \gamma \} \) 上的导数。函数在该点的梯度的大小等于什么？

解

\[
\frac{\partial u}{\partial i} \bigg|_{z=1} = \cos \alpha + \cos \beta + \cos \gamma,
\]

\[
|\text{grad} u| \bigg|_{z=1} = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2} \bigg|_{z=1}
\]

\[
= \sqrt{3}.
\]

3345. 求函数

\[ u = \frac{1}{r} \]

（式中 \( r = \sqrt{x^2 + y^2 + z^2} \)）在点 \( M_0(x_0, y_0, z_0) \) 处梯度的大小和方向。

解

\[
\frac{\partial u}{\partial x} = -\frac{x}{r^3}, \quad \frac{\partial u}{\partial y} = -\frac{y}{r^3}, \quad \frac{\partial u}{\partial z} = -\frac{z}{r^3}.
\]

于是，

\[
\text{grad} u = -\frac{1}{r^3} (x \vec{i} + y \vec{j} + z \vec{k})
\]

或简记成

\[
\text{grad} u = \left\{ -\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3} \right\}
\]

在点 \( M_0 \) 处的梯度为

\[
\text{grad} u = \left\{ -\frac{x_0}{r_0^3}, -\frac{y_0}{r_0^3}, -\frac{z_0}{r_0^3} \right\}
\]

其中 \( r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2} \)。从而得

\[
|\text{grad} u| = \sqrt{\left( -\frac{x_0}{r_0^3} \right)^2 + \left( -\frac{y_0}{r_0^3} \right)^2 + \left( -\frac{z_0}{r_0^3} \right)^2}
\]

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= \frac{1}{r_0^2},

\begin{align*}
\cos(\text{grad } u \wedge x) &= -\frac{x_0}{r_0^2} = -\frac{x_0}{r_0}, \\
\cos(\text{grad } u \wedge y) &= -\frac{y_0}{r_0^2} = -\frac{y_0}{r_0}, \\
\cos(\text{grad } u \wedge z) &= -\frac{z_0}{r_0^2} = -\frac{z_0}{r_0}.
\end{align*}

3347. 求函数

\[ u = x^2 + y^2 - z^2 \]

在点 \( A(\varepsilon, 0, 0) \) 及 \( B(0, \varepsilon, 0) \)二点的梯度之间的角度。

解 \( \text{grad } u = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\} = \left\{ 2x, 2y, -2z \right\}. \) 若

以 \( \text{grad } u_A \) 及 \( \text{grad } u_B \) 分别表示在 \( A \) 点及 \( B \) 点的梯度，则有

\[ \text{grad } u_A = \{ 2\varepsilon, 0, 0 \}, \quad \text{grad } u_B = \{ 0, 2\varepsilon, 0 \}. \]

由于

\[ \text{grad } u_A \cdot \text{grad } u_B = 2\varepsilon \cdot 0 + 0 \cdot 2\varepsilon + 0 \cdot 0 = 0, \]

故知

\[ \text{grad } u_A \perp \text{grad } u_B, \]

即在点 \( A \) 及点 \( B \) 二点的梯度之间的夹角为
\[(\text{grad } u_A, \text{grad } u_B) = \frac{\pi}{2}.\]

3348+ 在点 \(M(1, 2, 2)\) 处，函数
\[u = x + y + z\]
的梯度之大小与函数
\[v = x + y + z + 0.001 \sin (10^6 \pi \sqrt{x^2 + y^2 + z^2})\]
的梯度之大小相差若干？
解  \(\text{grad } u = \{1, 1, 1\}\)， \(|\text{grad } u| = \sqrt{3}\).
令 \(r = \sqrt{x^2 + y^2 + z^2}\)，则
\[
\frac{\partial v}{\partial x} = 1 + 1000\pi \frac{x}{r} \cos (10^6 \pi r),
\]
\[
\frac{\partial v}{\partial y} = 1 + 1000\pi \frac{y}{r} \cos (10^6 \pi r),
\]
\[
\frac{\partial v}{\partial z} = 1 + 1000\pi \frac{z}{r} \cos (10^6 \pi r).
\]
在点 \(M(1, 2, 2)\) 处,
\[
\frac{\partial v}{\partial x} = \frac{1000\pi}{3} + 1 \approx \frac{1000\pi}{3},
\]
\[
\frac{\partial v}{\partial y} = \frac{2000\pi}{3} + 1 \approx \frac{2000\pi}{3},
\]
\[
\frac{\partial v}{\partial z} = \frac{2000\pi}{3} + 1 \approx \frac{2000\pi}{3},
\]
\[|\text{grad } v| \approx 1000\pi \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}\]
\[= 1000\pi.\]
于是，两梯度之大小相差为
\[ |\text{grad } u| - |\text{grad } v| \approx 1000\pi - \sqrt{3} \approx 3140. \]

3349. 证明：在点 \(M_0(x_0, y_0, z_0)\) 处函数
\[ u = ax^2 + by^2 + cz^2 \]
及
\[ v = ax^2 + by^2 + cz^2 + 2mx + 2ny + 2pz \]
\((a, b, c, m, n, p\) 为常数且 \(a^2 + b^2 + c^2 \neq 0\) 二者之梯度之间的角度当点 \(M_0\) 无限远移时趋于零。
证 本题的题设条件“点 \(M_0(x_0, y_0, z_0)\) 无限远移”应理解为“\(x_0 \to \infty, y_0 \to \infty, z_0 \to \infty\) 同时成立”（此时 \(\sqrt{(ax_0)^2 + (by_0)^2 + (cz_0)^2} \to +\infty\)），否则，本题的结论不成立。

显见有
\[ \text{grad } u = \{2ax_0, 2by_0, 2cz_0\}, \]
\[ \text{grad } v = \{2ax_0 + 2m, 2by_0 + 2n, 2cz_0 + 2p\}. \]
令 \(a = ax_0, \beta = by_0, \gamma = cz_0; \)
\[ a_1 = ax_0 + m = a + m, \beta_1 = by_0 + n = b + n, \gamma_1 = cz_0 \]
\[ + p = \gamma + p. \]
于是，\(\text{grad } u\) 与 \(\text{grad } v\) 的夹角 \(\theta\) 满足
\[ \cos \theta = \frac{aa_1 + \beta \beta_1 + \gamma \gamma_1}{\sqrt{a^2 + \beta^2 + \gamma^2} \cdot \sqrt{a_1^2 + \beta_1^2 + \gamma_1^2}} \]
或
\[ \sin^2 \theta = 1 - \cos^2 \theta \]
\[ = \frac{(a^2 + \beta^2 + \gamma^2)(a_1^2 + \beta_1^2 + \gamma_1^2) - (aa_2 + \beta \beta_1 + \gamma \gamma_1)^2}{(a^2 + \beta^2 + \gamma^2)(a_1^2 + \beta_1^2 + \gamma_1^2)} \]

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\[ (\alpha\beta_1 - \alpha_1\beta)^2 + (\alpha\gamma_1 - \alpha_1\gamma)^2 + (\beta\gamma_1 - \beta_1\gamma)^2 \]
\[ = \frac{(m\alpha - m\beta)^2 + (m\alpha - m\gamma)^2 + (m\beta - m\gamma)^2}{(\alpha^2 + \beta^2 + \gamma^2)(\alpha^2 + \beta^2 + \gamma^2)} \]

令 \( \delta = \max(|ax_0|, |by_0|, |cz_0|) \)
\[ = \max(|a|, |b|, |c|) \]
\[ \delta \leq \sqrt{a^2 + b^2 + c^2} \leq \sqrt{3}\delta. \]
于是，当 \( \sqrt{a^2 + b^2 + c^2} \to +\infty \) 时，\( \delta \to +\infty \).
再令 \( q = \max(|m|, |n|, |p|) \)，则下述不等式显然成立:
\[ 0 \leq \sin^2 \theta = \frac{(m\alpha - m\beta)^2 + (m\alpha - m\gamma)^2 + (m\beta - m\gamma)^2}{(\alpha^2 + \beta^2 + \gamma^2)(\alpha^2 + \beta^2 + \gamma^2)} \]
\[ \leq \frac{(2q\delta)^2 + (2q\delta)^2 + (2q\delta)^2}{\delta^2(\delta^2 - 6\delta q - 3q^2)} \]
\[ = \frac{12q^2}{\delta^2 - 6\delta q - 3q^2} \to 0 \quad (\text{当} \delta \to +\infty \text{时}) . \]

于是，当 \( \sqrt{a^2 + b^2 + c^2} \to +\infty \) 时，\( \sin^2 \theta \to 0 \)，即当
\( \sqrt{a^2 + b^2 + c^2} \to +\infty \)，\( \theta \to 0 \)。证毕。
3350. 设 \( u = f(x, y, z) \) 为可微分两次的函数。若 \( \cos \alpha \)，\( \cos \beta \)，
\( \cos \gamma \) 为方向 \( l \) 的方向余弦，求 \( \frac{\partial^2 u}{\partial l^2} = \frac{\partial}{\partial l} \left( \frac{\partial u}{\partial l} \right) \).

解
\[ \frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma , \]
\[ \frac{\partial^2 u}{\partial l^2} = \left( \frac{\partial^2 u}{\partial x^2} \cos \alpha + \frac{\partial^2 u}{\partial y \partial x} \cos \beta + \right. \]

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\[
\frac{\partial^2 u}{\partial x \partial x} \cos \gamma \cos \alpha \\
+ \left( \frac{\partial^2 u}{\partial x \partial y} \cos \alpha + \frac{\partial^2 u}{\partial y \partial y} \cos \beta + \frac{\partial^2 u}{\partial z \partial y} \cos \gamma \right) \cos \beta \\
+ \left( \frac{\partial^2 u}{\partial x \partial z} \cos \alpha + \frac{\partial^2 u}{\partial y \partial z} \cos \beta + \frac{\partial^2 u}{\partial z^2} \cos \gamma \right) \cos \gamma \\
= -\frac{\partial^2 u}{\partial x^2} \cos^2 \alpha + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta + \frac{\partial^2 u}{\partial z^2} \cos^2 \gamma \\
+ 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha \cos \beta \\
+ 2 \frac{\partial^2 u}{\partial y \partial z} \cos \beta \cos \gamma + 2 \frac{\partial^2 u}{\partial z \partial x} \cos \gamma \cos \alpha.
\]

3351. 设 \( u = f(x, y, z) \) 为可微分两次的函数及

\[ l_1 \{ \cos \alpha_1, \cos \beta_1, \cos \gamma_1 \}, l_2 \{ \cos \alpha_2, \cos \beta_2, \cos \gamma_2 \}, \]

\[ l_3 \{ \cos \alpha_3, \cos \beta_3, \cos \gamma_3 \} \]

为三个互相垂直的方向。证明:

(a) \( \left( \frac{\partial u}{\partial l_1} \right)^2 + \left( \frac{\partial u}{\partial l_2} \right)^2 + \left( \frac{\partial u}{\partial l_3} \right)^2 \)

= \( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \); 

(b) \( \frac{\partial^2 u}{\partial l_1^2} + \frac{\partial^2 u}{\partial l_2^2} + \frac{\partial^2 u}{\partial l_3^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \)

证 (a) \( \left( \frac{\partial u}{\partial l_1} \right)^2 + \left( \frac{\partial u}{\partial l_2} \right)^2 + \left( \frac{\partial u}{\partial l_3} \right)^2 \)
\[
\sum_{i=1}^{3} \left( \frac{\partial u}{\partial x} \cos \alpha_i + \frac{\partial u}{\partial y} \cos \beta_i + \frac{\partial u}{\partial z} \cos \gamma_i \right)^2 \\
= \left( \frac{\partial u}{\partial x} \right)^2 \cdot \sum_{i=1}^{3} \cos^2 \alpha_i + \left( \frac{\partial u}{\partial y} \right)^2 \cdot \sum_{i=1}^{3} \cos^2 \beta_i \\
+ \left( \frac{\partial u}{\partial z} \right)^2 \cdot \sum_{i=1}^{3} \cos^2 \gamma_i \\
+ 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \cdot \sum_{i=1}^{3} \cos \alpha_i \cos \beta_i \\
+ 2 \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} \cdot \sum_{i=1}^{3} \cos \beta_i \cos \gamma_i \\
+ 2 \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{3} \cos \gamma_i \cos \alpha_i.
\]

由于 \( l_1, l_2, l_3 \) 是互相垂直的三个单位矢量，故

\[
\sum_{i=1}^{3} \cos \alpha_i \cos \beta_i = 0, \quad \sum_{i=1}^{3} \cos \beta_i \cos \gamma_i = 0, \\
\sum_{i=1}^{3} \cos \gamma_i \cos \alpha_i = 0, \\
\sum_{i=1}^{3} \cos^2 \alpha_i = 1, \quad \sum_{i=1}^{3} \cos^2 \beta_i = 1, \\
\sum_{i=1}^{3} \cos^2 \gamma_i = 1.
\]

将上述诸等式 (2) 代入 (1) 式，即得
\[
\left( \frac{\partial u}{\partial l_1} \right)^2 + \left( \frac{\partial u}{\partial l_2} \right)^2 + \left( \frac{\partial u}{\partial l_3} \right)^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2.
\]

（6）利用3350题的结果，得

\[
\sum_{i=1}^{3} \frac{\partial^2 u}{\partial l_i^2} = \frac{\partial^2 u}{\partial x^2} \cdot \sum_{i=1}^{3} \cos^2 \alpha_i
\]

\[
+ \frac{\partial^2 u}{\partial y^2} \cdot \sum_{i=1}^{3} \cos^2 \beta_i + \frac{\partial^2 u}{\partial z^2} \cdot \sum_{i=1}^{3} \cos^2 \gamma_i.
\]

\[
+ 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \cdot \sum_{i=1}^{3} \cos \alpha_i \cos \beta_i
\]

\[
+ 2 \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} \cdot \sum_{i=1}^{3} \cos \beta_i \cos \gamma_i
\]

\[
+ 2 \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} \cdot \sum_{i=1}^{3} \cos \gamma_i \cos \alpha_i.
\]

（3）

将诸等式（2）代入（3）式，即得

\[
\frac{\partial^2 u}{\partial l_1^2} + \frac{\partial^2 u}{\partial l_2^2} + \frac{\partial^2 u}{\partial l_3^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.
\]

3352. 设 \( u = u(x, y) \) 为可微分的函数且当 \( y = x^2 \) 时有，

\[
u(x, y) = 1
\]

及

\[
\frac{\partial u}{\partial x} = x.
\]

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求当 \( y = x^2 \) 时的 \( \frac{\partial u}{\partial y} \)。

解  \( \frac{d}{dx} u(x, x^2) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} \)。

当 \( y = x^2 \)，\( u(x, y) = u(x, x^2) = 1 \)，故 \( \frac{du(x, x^2)}{dx} = 0 \)。

且有 \( \frac{\partial u}{\partial x} = x \)，\( \frac{dy}{dx} = 2x \)。将这些结果代入上式，即得

\[
x + 2x \frac{\partial u}{\partial y} = 0.
\]

于是，\( \frac{\partial u}{\partial y} = -\frac{1}{2} \) (\( x \neq 0 \)）。

3353. 设函数 \( u = u(x, y) \) 满足方程

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0
\]

以及下列条件：

\[
u(x, 2x) = x, \quad u_x(x, 2x) = x^2.
\]

求：

\[
u''(x, 2x), \quad u''_{xx}(x, 2x), \quad u''_{yy}(x, 2x).
\]

解  由于 \( u(x, 2x) = x \)，故

\[
u'_x(x, 2x) + 2u'_x(x, 2x) = 1 \quad \text{(1)}.
\]

又因 \( u'_x(x, 2x) = x^2 \)，故由 (1) 式即得
\[ u_x^1(x, 2x) = \frac{1 - x^2}{2}. \quad (2) \]

将 (2) 式两端对 x 求导数，有

\[ u_{xx}^1(x, 2x) + 2u_{xy}^1(x, 2x) = -x; \quad (3) \]

由 \( u_x(x, 2x) = x^2 \) 两端对 x 求导数，有

\[ u_{xx}^1(x, 2x) + 2u_{xy}^1(x, 2x) = 2x. \quad (4) \]

联立 (3) 式和 (4) 式并利用题设条件 \( u_{xx}^1 = u_{yy}^1 \)，解之即得

\[ u_{xx}^1(x, 2x) = u_{yy}^1(x, 2x) = -\frac{4}{3} x, \]

\[ u_{xy}^1(x, 2x) = \frac{5}{3} x. \]

假定 \( z = z(x, y) \)，解下列方程：

3354. \( \frac{\partial^2 z}{\partial x^2} = 0. \)

解 \( \frac{\partial z}{\partial x} = \phi(y), \; z = x\phi(y) + \psi(y). \)

3355. \( \frac{\partial^2 z}{\partial x \partial y} = 0. \)

解 \( \frac{\partial z}{\partial x} = \phi_1(x), \)

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\[ z = \int_0^x \varphi_1(t) \, dt + \psi(y) = \varphi(x) + \psi(y). \]

3356. \[ \frac{\partial^2 z}{\partial y^2} = 0. \]

解
\[ \frac{\partial^{n-1} z}{\partial y^{n-1}} = \varphi_{n-1}(x), \]
\[ \frac{\partial^{n-2} z}{\partial y^{n-2}} = y \varphi_{n-1}(x) + \varphi_{n-2}(x), \]

累次积分 n 次，最后得
\[ x = y^{n-1} \varphi_{n-1}(x) + y^{n-2} \varphi_{n-2}(x) + \cdots + y \varphi_1(x) + \varphi_0(x). \]

3357. 假定 \( u = u(x, y, z) \) 解方程
\[ \frac{\partial^3 u}{\partial x \partial y \partial z} = 0. \]

解
\[ \frac{\partial^2 u}{\partial x \partial y} = \varphi_1(x, y), \]
\[ \frac{\partial u}{\partial x} = \varphi_2(x, y) + \psi_1(x, z), \]
\[ u = \varphi(x, y) + \psi(x, z) + x(y, z). \]

3358. 方程
\[ \frac{\partial z}{\partial y} = x^2 + 2y \]

的满足条件 \( z(x, x^2) = 1 \) 的解 \( z = z(x, y) \).

解 由 \[ \frac{\partial z}{\partial y} = x^2 + 2y \]
得
\[ z = x^2 y + y^2 + \varphi(x). \]

又因 \( z(x; x^2) = 1 \)，故

\[ 1 = x^4 + x^4 + \varphi(x), \]

从而有

\[ \varphi(x) = 1 - 2x^4. \]

最后得

\[ z = 1 + x^2 y + y^2 = 2x^4. \]

3359. 求方程

\[ \frac{\partial^2 z}{\partial y^2} = 2 \]

的满足条件 \( z(x, 0) = 1, z'_y(x, 0) = x \) 的解

\[ z = z(x, y). \]

解 由 \( \frac{\partial^2 z}{\partial y^2} = 2 \) 得

\[ \frac{\partial z}{\partial y} = 2y + \varphi(x). \]

又因 \( z'_y(x, 0) = x \)，所以

\[ x = 0 + \varphi(x) \] 或 \[ x = \varphi(x). \]

从而有

\[ \frac{\partial z}{\partial y} = 2y + x. \]

由此得

\[ z = y^2 + xy + \varphi_1(x). \]

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又因 $z(x,0) = 1$，故

$$1 = 0 + 0 + \varphi_1(x) \text{ 或 } 1 = \varphi_1(x).$$

最后得

$$z = 1 + xy + y^2.$$  

3360. 求方程

$$\frac{\partial^2 z}{\partial x \partial y} = x + y$$

的满足条件 $z(x,0) = x, z(0,y) = y^2$ 的解 $z = z(x,y)$.  

解 由 $\frac{\partial^2 z}{\partial x \partial y} = x + y$ 得

$$\frac{\partial z}{\partial x} = xy + \frac{1}{2} y^2 + \varphi_1(x),$$

$$z = \frac{1}{2} x^2 y + \frac{1}{2} xy^2 + \varphi(x) + \psi(y).$$

现确定 $\varphi(x)$ 及 $\psi(y)$. 由于 $z(x,0) = x, z(0,y) = y^2$，故有

$$x = \varphi(x) + \psi(0),$$

$$y^2 = \varphi(0) + \psi(y),$$

于是

$$z = x + y^2 + \frac{1}{2} x^2 y + \frac{1}{2} xy^2 - (\varphi(0) + \psi(0)).$$

又因 $z(0,0) = 0$，故 $\varphi(0) + \psi(0) = 0$. 最后得

$$z = x + y^2 + \frac{1}{2} xy(x + y).$$
§3. 隐函数的微分法

1° 存在定理  设：1）函数$F(x, y, z)$在某点$A_0(x_0, y_0, z_0)$等于零；2）$F(x, y, z)$和$F'(x, y, z)$在点$A_0$的邻域内有定义并且是连续的；3）$F'(x_0, y_0, z_0) \neq 0$，则在点$A_0(x_0, y_0)$的某充分小的邻域内存在唯一的连续函数

$$z = f(x, y)$$

满足方程

$$F(x, y, z) = 0$$

而且是$z_0 = f(x_0, y_0)$。

2° 隐函数的可微分性  设除了上面的条件外，4）如果函数$F(x, y, z)$在点$A_0(x_0, y_0, z_0)$的邻域内可微分，则函数

(1) 在点$A_0(x_0, y_0)$的邻域内也可微分并且它的导数$rac{\partial z}{\partial x}$和$rac{\partial z}{\partial y}$可从方程

$$
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0, \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0
$$

求得。若函数$F(x, y, z)$可微分任意多次，则用逐次微分方法（2）的方法也可计算函数$z$的高阶导函数。

3° 由方程组定义的隐函数  设函数$F_i(x_1, \cdots, x_m; y_1, \cdots, y_n)(i = 1, 2, \cdots, n)$满足下列条件：

(1) 于点$A_0(x_{10}, \cdots, x_{m0}; y_{10}, \cdots, y_{n0})$变成为零；

(2) 在点$A_0$的邻域内可微分；

(3) 在点$A_0$函数行列式$\frac{\partial (F_1, \cdots, F_n)}{\partial (y_1, \cdots, y_n)} \neq 0$。
在这种情况下，方程组
\[ F_i(x_1, \ldots, x_n; y_1, \ldots, y_n) = 0 \quad (i = 1, 2, \ldots, n) \]  （3）
在点 \( A_0 (x_{i0}, \ldots, x_{n0}) \) 的邻域内唯一地确定出一组可微分的函数:
\[ y_i = f_i(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n) , \]
这些方程满足方程（3）及原始条件
\[ f_i(x_{i0}, \ldots, x_{n0}) = y_{i0} \quad (i = 1, 2, \ldots, n) . \]
这些隐函数的微分可由方程组
\[ \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} \, dx_i + \sum_{i=1}^{n} \frac{\partial F_i}{\partial y_i} \, dy_i = 0 \]
\((i = 1, 2, \ldots, n)^*\) 求得。

3361. 证明：在每一点都不连续的迪里黑里函数
\[ y = \begin{cases} 1, & \text{若}\, x \text{为有理数} ; \\ 0, & \text{若}\, x \text{为无理数} \end{cases} \]
满足方程
\[ y^2 - y = 0 . \]
证 当 \( x \) 为有理数时，\( y^2 - y = 1 - 1 = 0 ; \) 当 \( x \) 为无理数时，\( y^2 - y = 0 - 0 = 0 . \) 因此，不论 \( x \) 为任何实数 \( x \)，均有
\[ y^2 - y = 0 . \]

3362. 设函数 \( f(x) \) 定义于区间 \( (a, b) \) 内。问在怎样的情况下方程
\[ f(x) y = 0 \]

* 这一段在简明陈述大多数的问题时无条件地假设隐函数和它们的对应导函数存在的条件满足。
当 $a < x < b$ 时才有唯一连续的解 $y = 0$ ?

解 函数 $f(x)$ 的非零点的集合在区间 $(a, b)$ 内是处处稠密的，即 $f(x)$ 的零点的集合不能充满区间 $(a, b)$ 的任意一个子区间 $(\alpha, \beta) \subset (a, b)$。此时，方程 $f(x) y = 0$ 有唯一连续的解 $y = 0$。事实上，设 $y = y(x)$ 为方程 $f(x) y = 0$ 的一个连续解，$x_0 \in (a, b)$，则

1. 当 $f(x_0) \neq 0$ 时，显然有 $y(x_0) = 0$；
2. 当 $f(x_0) = 0$ 时，由 $f(x)$ 的非零点的稠密性知，存在数列 $\{x_n\}$，满足 $x_n \to x_0$ 及 $f(x_n) \neq 0$ $(n = 1, 2, \ldots)$。于是，$y(x_n) = 0$。由 $y(x)$ 的连续性即得 $y(x_0) = y(\lim x_n) = \lim y(x_n) = 0$。

于是，当 $a < x < b$ 时，$y = 0$。

反之，若方程 $f(x) y = 0$ 在 $(a, b)$ 只有唯一的连续解 $y = 0$，则 $f(x)$ 的零点集必不能充满 $(a, b)$ 的任何子区间。事实上，设在 $(a, b)$ 的某子区间 $(\alpha, \beta)$ 上 $f(x) = 0$。定义 $(a, b)$ 上的函数 $y_0(x)$ 如下：

$$
\begin{align*}
0, & \quad \text{当 } a < x < a + \frac{\beta - \alpha}{4} \text{ 时,} \\
-\frac{4}{\beta - \alpha} \left(x - \alpha - \frac{\beta - \alpha}{4}\right), & \quad \text{当 } a + \frac{\beta - \alpha}{4} \leq x < a + \frac{\beta - \alpha}{2} \text{ 时,} \\
-\frac{4}{\beta - \alpha} \left(x - \alpha - \frac{3(\beta - \alpha)}{4}\right), & \quad \text{当 } a + \frac{\beta - \alpha}{2} \leq x < a + \frac{3}{4} (\beta - \alpha) \text{ 时;} \\
0, & \quad \text{当 } a + \frac{3}{4} (\beta - \alpha) \leq x < b \text{ 时。}
\end{align*}
$$
如图6.27所示，图中 $c_1 = a + \frac{\beta - \alpha}{4}$, $c_0 = a + \frac{\beta - \alpha}{2}$, $c_2 = a + \frac{3(\beta - \alpha)}{4}$.

显然 $y_0(x) \neq 0$, 但 $y = y_0(x)$ 是方程 $f(x)y = 0$ 在 $(a, b)$ 上的一个连续解。

图 6.27

3363. 设函数 $f(x)$ 和 $g(x)$ 于区间 $(a, b)$ 内有定义且连续。问在怎样的情况下，方程

$$f(x)y = g(x)$$

于区间 $(a, b)$ 内才有唯一连续的解。

解下面三个条件显然是必要的：

（1）$f(x)$ 的零点必须是 $g(x)$ 的零点，否则 $y$ 无解；

（2）$f(x)$ 的非零点集合必须在 $(a, b)$ 内稠密。否则，存在 $(\alpha, \beta) \subset (a, b)$，当 $x \in (\alpha, \beta)$ 时，恒有 $f(x) = g(x) = 0$。从而当 $x \in (\alpha, \beta)$ 时，任意改变原方程
一个连续解 \( y(x) \) 的函数值 (但保持连续性) 就得出原方程的另一个连续解 (参看3362题的图)，此与原方程连续解的唯一性矛盾。

(3) 如果 \( f(x_0) = 0 \)，则对任一列 \( x_n \to x_0 \)，\( f(x_n) \neq 0 \ (n = 1, 2, \cdots) \)，均有

\[
\lim_{n \to \infty} \frac{g(x_n)}{f(x_n)} = y_0 \quad (y_0 \text{ 是有限数且只与 } x_0 \text{ 有关}) .
\]

显然，如果上述极限不存在或对不同的序列取不同的值均导致 \( y \) 不连续。

反之，若上述三个条件满足，我们证明原方程的连续解唯一。事实上，这时令

\[
y_0(x) = \begin{cases} \frac{g(x)}{f(x)} , & \text{在 } f(x) \neq 0 \text{ 的点;} \\ \lim_{n \to \infty} \frac{g(x_n)}{f(x_n)} , & \text{在 } f(x) = 0 \text{ 的点，这里任意} \\ x_n \to x, \quad f(x_n) \neq 0 \quad (n = 1, 2, \cdots) . \end{cases}
\]

易知 \( y_0(x) \) 是 \((a, b)\) 内的连续函数且满足原方程，即是原方程的一个连续解。现若方程在 \((a, b)\) 内还有一连续解 \( y = y_1(x) \)，则

\[
f(x)y_1(x) = g(x), \quad f(x)y_0(x) = g(x) (a < x < b).
\]

对任何 \( x_0 \in (a, b) \)，若 \( f(x_0) \neq 0 \)，则 \( y_1(x_0) = \frac{g(x_0)}{f(x_0)} = y_0(x_0) \)；若 \( f(x_0) = 0 \)，取 \( x_n \to x_0, f(x_n) \neq 0 \ (n = 1, 2, \cdots) \)，则根据 \( y_1(x) \) 的连续性，得

\[
y_1(x_0) = \lim_{n \to \infty} y_1(x_n) = \lim_{n \to \infty} \frac{g(x_n)}{f(x_n)} = y_0(x_0) .
\]

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于是，\( y_1(x) = y_0(x) \) (\( a \leq x \leq b \))，唯一性获证。

3364. 设已知方程
\[ x^2 + y^2 = 1 \]
及
\[ y = y(x) \quad (-1 \leq x \leq 1) \]
为满足方程（1）的单调函数。
1）问有多少单调函数（2）满足方程（1）？
2）问有多少单调连续函数（2）满足方程（1）？
3）设：（a）\( y(0) = 1 \)；（b）\( y(1) = 0 \)，问有多少单调连续函数（2）满足方程（1）？
解 1）有限个。例如，令
\[ y_n(x) = \begin{cases} 
\sqrt{1-x^2}, & \text{当} -1 \leq x \leq 1 \text{且} x \neq \frac{1}{n} \text{时}, \\
-\sqrt{1-x^2}, & \text{当} x = \frac{1}{n} \text{时} 
\end{cases} 
\]
\( (n = 1, 2, 3, \cdots) \),
则显然 \( y = y_n(x) \) （\( n = 1, 2, 3, \cdots \)）都是满足方程（1）的单调函数。
2）二个：\( y = -\sqrt{1-x^2} \) 及 \( y = \sqrt{1-x^2} \)。
3）(a) 满足条件 \( y(0) = 1 \) 的仅 \( y = \sqrt{1-x^2} \) 这一个连续函数；(b) 满足条件 \( y(1) = 0 \) 的有 \( y = -\sqrt{1-x^2} \) 及 \( y = \sqrt{1-x^2} \) 这二个连续函数。

3365. 设已知方程
\[ x^2 = y^2 \]
及
\[ y = y(x) \quad (-\infty \leq x \leq +\infty) \]

157.
是满足方程（1）的单值函数。
1) 问有多少单值函数 (2) 满足方程 (1)?
2) 问有多少单值连续函数 (2) 满足方程 (1)?
3) 问有多少单值可微分的函数 (2) 满足方程 (1)?
4) 设；(a) \( y(1) = 1 \); (b) \( y(0) = 0 \), 问有多少单值连续函数 (2) 满足方程 (1)?
5) 设 \( y(1) = 1 \) 及 \( \delta \) 为充分小的数，问有多少单值连续函数 \( y = y(x) \) \((1 - \delta < x < 1 + \delta)\) 满足方程 (1)?

解 1) 无限个，例如，\( y_n(x) = \begin{cases} 
|x|, & x \neq \frac{1}{n}; \\
-|x|, & x = \frac{1}{n}; 
\end{cases} 
\)

\((n = 1, 2, \cdots)\) 都是。

2) 四个：\( y = -x, y = x, y = |x| \) 和 \( y = -|x| \).

3) 二个：\( y = -x \) 和 \( y = x \).

4) (a) 二个；\( y = x \) 和 \( y = |x| \); (b) 四个；即

2) 中之四个。

5) 一个；\( y = x \).

3366. 方程

\[ x^2 + y^2 = x^4 + y^4 \]

是定义 \( y \) 为 \( x \) 的多值函数。问这个函数在怎样的域内，1) 单值，2) 有二个值，3) 有三个值，4) 有四个值？求此函数的各枝点及它的单值连续的各枝。

解 由 \( x^2 + y^2 = x^4 + y^4 \) 得

\[ y^4 - y^2 + (x^4 - x^2) = 0 \]

解之，得

\[ y^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2 - x^4} \]
的六支，其中当 \( \frac{1}{4} + x^2 - x^4 \geq 0 \) 即 \( |x| \leq \sqrt{\frac{1 + \sqrt{2}}{2}} \) 时有二支：

\[
y_1 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad |x| \leq \sqrt{\frac{1 + \sqrt{2}}{2}},
\]

\[
y_2 = -\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad |x| \leq \sqrt{\frac{1 + \sqrt{2}}{2}}.
\]

而当 \( 0 \leq \frac{1}{4} + x^2 - x^4 \leq \left(\frac{1}{2}\right)^2 \) 即 \( 1 \leq x^2 \leq \frac{1 + \sqrt{2}}{2} \) 时有四支：

\[
y_3 = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad 1 \leq x \leq \sqrt{\frac{1 + \sqrt{2}}{2}};
\]

\[
y_4 = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad -\sqrt{\frac{1 + \sqrt{2}}{2}} \leq x \leq -1;
\]

\[
y_5 = -\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad 1 \leq x \leq \sqrt{\frac{1 + \sqrt{2}}{2}};
\]

\[
y_6 = -\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} + x^2 - x^4}}, \quad -\sqrt{\frac{1 + \sqrt{2}}{2}} \leq x \leq -1.
\]

此外还有一个孤立点 \((0, 0)\)（参看 1542 题的图形）。

考虑上述六支的公共定义域知：

1）没有单值区域。

2）双值区域为 \( 0 \leq |x| \leq 1 \) 及 \( x = \pm \sqrt{\frac{1 + \sqrt{2}}{2}} \)。
3) 枝状区域为 $x = 0$ 及 $x = \pm 1$。

4) 枝状区域为 $1 \leq |x| \leq \sqrt{\frac{1 + \sqrt{2}}{2}}$。

枝点的必要条件为

$$[(y^4 - y^2 + (\sqrt{2} - x^2)]_y = 0,$$

即

$$4y^3 - 2y = 0.$$

于是，

$$y = 0 \text{ 及 } y = \pm \frac{1}{\sqrt{2}}.$$

由 $y = 0$ 解得 $x = 0$ 及 $x = \pm 1$，而由 $y = \pm \frac{1}{\sqrt{2}}$ 解得

$$x = \pm \sqrt{\frac{1 + \sqrt{2}}{2}}.$$ 经验证，得六个枝点:

$$(-1, 0), (1, 0), (\sqrt{\frac{1 + \sqrt{2}}{2}}, \frac{1}{\sqrt{2}}),$$

$$\left(\sqrt{\frac{1 + \sqrt{2}}{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\sqrt{\frac{1 + \sqrt{2}}{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\left(-\sqrt{\frac{1 + \sqrt{2}}{2}}, -\frac{1}{\sqrt{2}}\right).$$

3367. 求由方程

$$(x^2 + y^2)^2 = x^2 - y^2$$

所定义的多值函数 $y$ 的各枝点和单值连续的各枝 $y = y(x)$ ($-1 \leq x \leq 1$)。

解 由 $(x^2 + y^2)^2 = x^2 - y^2$ 得

$$y^2 = \frac{-1 + 2x^2 \pm \sqrt{8x^2 + 1}}{2}.$$
因为当$|x| \leq 1$时，$\sqrt{3x^2 + 1} \geq 1 + 2x^2$，故单值连续的各枝为（共有四枝）

$$ y = \varepsilon(x) \sqrt[\frac{3x^2 + 1}{2} - (1 + 2x^2)} (-1 \leq x \leq 1), $$

其中$\varepsilon(x)$分别为$1, -1, sgnx, -sgnx$。

下面再求枝点，

$$ \left( (x^2 + y^2)^2 - x^2 + y^2 \right) = 2(x^2 + y^2) \cdot 2y + 2y = 0, $$

解之得$y = 0$，从而得$x = 0$及$x = \pm 1$。经验证得枝点为

$(0, 0), (1, 0)$及$(-1, 0)$。

3.5.8. 设函数$f(x)$当$a \leq x \leq b$时连续，并且函数$\varphi(y)$当$c \leq y \leq d$时单调增加而且连续。问在怎样的条件下方程

$$ \varphi(y) = f(x) $$

定义出单值函数

$$ y = \varphi^{-1}[f(x)]. $$

研究例子：(a)$\sin y + \sin y = x$；(6)$e^{-x} = -\sin^2 x$。

解，根据$\varphi(y)$的严格增加性以及$\varphi(y), f(x)$的连续性可知，若存在$(x_0, y_0)$满足$\varphi(y_0) = f(x_0)$，则在$x_0$附近有方程$\varphi(y) = f(x)$可唯一地确定$y$为$x$的单值连续函数

$$ y = \varphi^{-1}[f(x)] \quad (满足 y_0 = \varphi^{-1}[f(x_0)]); \quad (1) $$

若更设满足不等式

$$ \lim_{y \to +0} \varphi(y) = f(x) = \lim_{y \to -0} \varphi(y) \quad (a \leq x \leq b), \quad (2) $$

则显然函数$(1)$是整个$a \leq x \leq b$上定义的连续函数。

(a) 设$\varphi(y) = \sin y + \sin y \quad (-\infty \leq y \leq +\infty)$,
设：

$$x = y + \varphi(y),$$

（1）

其中 $\varphi(0) = 0$ 且当 $-\alpha < y < \alpha$ 时 $\varphi'(y)$ 连续并满足 $|\varphi'(y)| \leq k < 1$。证明：当 $-\epsilon < x < \epsilon$ 时存在唯一的可微分函数 $y = y(x)$ 满足方程（1）且 $y(0) = 0$。证 设 $F(x, y) = x - y - \varphi(y)$，则

1）由于 $\varphi(0) = 0$，故 $F(0, 0) = 0$；

2）当 $-\infty < x < +\infty$，$-\alpha < y < \alpha$ 时，$F(x, y)$$F'_x(x, y)$ 及 $F'_x(x, y) = -1 - \varphi'(y)$ 均连续；

3）$F'_x(0, 0) = -1 - \varphi'(0) < 0$，当然 $F'_x(0, 0) \neq 0$。

于是，由隐函数的存在及可微性定理知：存在 $\epsilon > 0$，使当 $-\epsilon < x < \epsilon$ 时，存在唯一的可微分函数 $y = y(x)$ 满足方程 $x = y + \varphi(y)$ 及 $y(0) = 0$。
\[ x = ky + \phi(y) \]

所定义的隐函数，其中常数 \( k \neq 0 \) 且 \( \phi(y) \) 为以 \( \omega \) 为周期的可微周期函数，且 \( |\phi'(y)| \leq |k| \)。证明

\[ y = \frac{x}{k} + \psi(x), \]

其中 \( \psi(x) \) 为以 \( |k|\omega \) 为周期的周期函数。

证 由于 \( x = ky + \phi(y) \)，故 \( \frac{dx}{dy} = k + \phi'(y) \)。又因

\( |\phi'(y)| \leq |k| \)，故 \( \frac{dx}{dy} \) 与 \( k \) 同号，即 \( x \) 为 \( y \) 的严格单调函数，且为连续的。由于 \( \phi(y) \) 是连续的以 \( \omega \) 为周期的函数，故有界，从而当 \( k \geq 0 \) 时，

\[ \lim_{y \to -\infty} x = -\infty, \quad \lim_{y \to +\infty} x = +\infty; \]

当 \( k < 0 \) 时，

\[ \lim_{y \to -\infty} x = +\infty, \quad \lim_{y \to +\infty} x = -\infty; \]

由此可知，其反函数 \( y = y(x) \) 存在唯一，且是 \( -\infty \leq x \leq +\infty \) 上有定义的严格单调可微函数。令

\[ y(x) - \frac{x}{k} = \psi(x) \quad (-\infty \leq x \leq +\infty), \quad (1) \]

则由 \( x = ky(x) + \phi(y(x)) \)，\( \phi(y(x)+\omega) = \phi(y(x)) \)

知 \( x + k\omega = ky(x) + \phi(y(x)) + k\omega = k \left[ y(x) + \omega \right] + \phi(y(x) + \omega) \)，从而，根据反函数的唯一性，得

\[ y(x + k\omega) = y(x) + \omega \quad (-\infty \leq x \leq +\infty). \quad (2) \]

由 (1) 式与 (2) 式，得

\[ \psi(x + k\omega) = y(x + k\psi) - \frac{x + k\omega}{k} = y(x) - \frac{x}{k} \]
\[ = \psi(x) \quad (-\infty < x < +\infty). \]

同理可证
\[ \psi(x - k\omega) = \psi(x) \quad (-\infty < x < +\infty), \]
故 \( \psi(x) \) 是以 \( |k| \omega \) 为周期的可微周期函数。由 (1) 得
\[ y = y(x) = \frac{1}{k} x + \psi(x). \]

证毕。

对于由下列各方程式所定义的函数 \( y \)，求出 \( y' \) 和 \( y'' \)。

3371. \( x^2 + 2xy - y^2 = a^2 \)。

解 用求导数及微分两种方法解之。

解法一
等式两端分别对 \( x \) 求导数，得
\[ 2x + 2y + 2xy' - 2yy' = 0, \]
故有
\[ y' = \frac{y + x}{y - x}. \]

再对上式求导数，得
\[ y'' = \frac{(y - x)(y' + 1) - (y + x)(y' - 1)}{(y - x)^2} \]
\[ = \frac{2y - 2xy'}{(y - x)^2} = \frac{2y(y - x) - 2x(y + x)}{(y - x)^3} \]
\[ = \frac{2(y^2 - 2xy - x^2)}{(y - x)^3} = - \frac{2a^2}{(y - x)^3} = \frac{2a^2}{(x - y)^3}. \]

解法二
等式两端分别微分，得
\[ 2x dx + 2yd y + 2yd x - 2yd y = 0, \]  \( \text{(1)} \)

故有
\[ \frac{dy}{dx} = \frac{y + x}{y - x}. \]

对(1)式两端再微分一次，并注意 \( d^2x = 0 \)，得
\[ dx^2 + 2xdy - dy^2 + (x - y)d^2y = 0. \]

故有
\[ \frac{d^2y}{dx^2} = \frac{1 + 2\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2}{y - x} = \frac{1 + 2(y + x) - (y + x)^2}{y - x} = \frac{2a^2}{(x - y)^3}. \]

3372. \( \ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x} \).

解
解法一

等式两端对 \( x \) 求导数，得
\[ \frac{x + yy'}{x^2 + y^2} = \frac{xy' - y}{x^2 + y^2}. \]

解之即得
\[ y' = \frac{x + y}{x - y}. \]

将上式再对 \( x \) 求导数，得
\[ y'' = \frac{(x - y)(1 + y') - (x + y)(1 - y')}{(x - y)^3}. \]
\[
\frac{2(xy' - y)}{(x - y)^2} = \frac{2x(x + y) - 2y(x - y)}{(x - y)^3} = \frac{2(x^2 + y^2)}{(x - y)^3}.
\]

解法二
等式两端分别微分，得
\[
\frac{xdx + ydy}{x^2 + y^2} = \frac{xdy - ydx}{x^2 + y^2}.
\]
解之即得
\[
\frac{dy}{dx} = \frac{x + y}{x - y}.
\]
对 \(xdx + ydy = xdy - ydx\) 再微分一次，得
\[
dx^2 + dy^2 + yd^2y = xdy^2,
\]
故有
\[
\frac{d^2y}{dx^2} = \frac{1}{x - y} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] = \frac{(x - y)^2 + (x + y)^2}{(x - y)^3} = \frac{2(x^2 + y^2)}{(x - y)^3}.
\]
以下各种根据情况采用直接求导法或微分法。

3373. \(y - \varepsilon \sin y = x \quad (0 \leq \varepsilon \leq 1)\).
解 等式两端对 \(x\) 求导数，得
\[
y' - \varepsilon y' \cos y = 1,
\]
故有
\[
y' = \frac{1}{1 - \varepsilon \cos y}
\]
将上式再对 $x$ 求导数，得

$$y'' = -\frac{\sin y}{(1 - \cos y)^2} + \frac{\cos y}{(1 - \cos y)^3}.$$  

3374. $x^2 = y^2$ $(x \neq y)$。

解 取对数得

$$y \ln x = x \ln y \text{ 或 } \frac{\ln x}{x} = \frac{\ln y}{y} \quad (x > 0, y > 0).$$

两端对 $x$ 求导数，得

$$\frac{1 - \ln x}{x^2} = \frac{y'(1 - \ln y)}{y^2},$$

故有

$$y' = \frac{y^2(1 - \ln x)}{x^2(1 - \ln y)}.$$  

将上式再对 $x$ 求导数，得

$$y'' = \frac{1}{x^4(1 - \ln y)^3} \left\{ x^2(1 - \ln y) \left[ 2yy'(1 - \ln x) 
- \frac{y^2}{x} \right] - y^2(1 - \ln x) \left[ 2x - 2x \ln y - \frac{x^2y'}{y} \right] \right\}$$

$$= \frac{1}{x^4(1 - \ln y)^3} \left\{ y^2 \left[ y(1 - \ln x)^2 - 2(x - y) \right.ight.$$

$$\left. \cdot (1 - \ln x)(1 - \ln y) - x(1 - \ln y)^2 \right].$$

3375. $y = 2x \arctan \frac{y}{x}$. 

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解 \( \frac{y}{x} = 2 \arctg \frac{y}{x} \)，显然 \( \frac{y}{x} \neq 1 \)。

两端微分，得

\[
d\left( \frac{y}{x} \right) = \frac{2d\left( \frac{y}{x} \right)}{1 + \left( \frac{y}{x} \right)^2}.
\]

于是，\( d\left( \frac{y}{x} \right) = 0 \)，即 \( \frac{x}{x^2} \frac{y}{x} dx = 0 \)，故有

\[
\frac{d}{dx} \frac{y}{x} = \frac{y}{x}.
\]

将上式对 \( x \) 求导数，即得

\[
\frac{d^2 y}{dx^2} = \frac{x}{x^2} \frac{d}{dx} \frac{y}{x} - \frac{y}{x^2} = 0.
\]

3376. 证明：当

\[1 + xy = k(x - y)\]

（式中 \( k \) 为常数）时，有等式

\[
\frac{d}{dx} \frac{x}{1 + x^2} = \frac{d}{dx} \frac{y}{1 + y^2}.
\]

证 将等式 \( 1 + xy = k(x - y) \) 两端微分，得

\[xdy + ydx = k(dx - dy),\]

故

\[
(x - y)(xdy + ydx) = k(x - y)(dx - dy) = (1 + xy)(dx - dy),
\]

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简化即得

$$\frac{dx}{1 + x^2} = \frac{dy}{1 + y^2}.$$  

证毕。

3377. 证明：若

$$x^2y^2 + x^2 + y^2 - 1 = 0,$$

则当 $xy > 0$ 时有等式

$$\frac{dx}{\sqrt{1 - x^4}} + \frac{dy}{\sqrt{1 - y^4}} = 0.$$

证 将所给等式两端微分，得

$$2xy^2 dx + 2x^2 y dy + 2xdx + 2ydy = 0,$$

即

$$x(y^2 + 1)dx + y(x^2 + 1)dy = 0.$$  (1)

由 $x^2y^2 + x^2 + y^2 - 1 = 0$ 可解得

$$x = \pm \sqrt{\frac{1 - y^2}{1 + y^2}}, \quad y = \pm \sqrt{\frac{1 - x^2}{1 + x^2}}.$$  (2)

因为 $xy > 0$，故知 $x, y$ 应同取正值或同取负值。不论取什么符号，当用(2)式代入(1)式后，均可得

$$\frac{dx}{\sqrt{1 - x^4}} + \frac{dy}{\sqrt{1 - y^4}} = 0.$$

3378. 证明：方程

$$(x^2 + y^2)^2 = a^2(x^2 - y^2) \quad (a \neq 0)$$

在点 $x = 0, y = 0$ 的邻域中定义两个可微分的函数：

$y = y_1(x)$ 和 $y = y_2(x)$。求 $y_1'(0)$ 及 $y_2'(0)$。

解 $(x^2 + y^2)^2 = a(x^2 - y^2)^3$ 即

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$$y^4 + (2x^2 + a^2)y^2 - (a^2x^2 - x^4) = 0.$$ 解之得

$$y^2 = \frac{-(2x^2 + a^2) + \sqrt{8a^2x^2 + a^4}}{2}$$

（根号前取正号是由于 \(y^2 \geq 0\)）。记

$$y = \pm \sqrt{\frac{\sqrt{8a^2x^2 + a^4} - 2x^2 - a^2}{2}} = \pm f(x^2).$$

不难看出 \((0, 0)\) 为极点。从点 \((0, 0)\) 出发，有多个连续的四个分枝：

$$y_1 = f(x^2), \quad 0 \leq x \leq \delta;$$
$$y_1 = f(x^2), \quad -\delta \leq x \leq 0;$$
$$y_2 = -f(x^2), \quad 0 \leq x \leq \delta;$$
$$y_4 = -f(x^2), \quad -\delta \leq x \leq 0.$$ 这几个单值分枝能否组成 \((-\delta, \delta)\) 上的可微分函数，主要是看组成的函数在 \(x = 0\) 是否可微。为此，研究各分枝在点 \(x = 0\) 处的单侧导数。

$$y'_1(0) = \lim_{x \to 0^+} \frac{y_1(x) - y_1(0)}{x - 0} = \lim_{x \to +0} \frac{f(x^2)}{x}$$

$$= \lim_{x \to +0} \frac{1}{x} \sqrt{\frac{\sqrt{8a^2x^2 + a^4} - 2x^2 - a^2}{2}}$$

$$= \lim_{x \to +0} \sqrt{\frac{\sqrt{8a^2x^2 + a^4} - 2x^2 - a^2}{2x^2}}$$

$$= \lim_{x \to +0} \sqrt{\frac{8a^2x^2 + a^4 - (2x^2 + a^4)^2}{2x^2(\sqrt{8a^2x^2 + a^4} + 2x^2 + a^2)}}$$
\[
= \lim_{x \to 0} \sqrt{\frac{4a^2 - 4x^2}{2(\sqrt{8a^2 x^2 + a^4 + 2x^2 + a^2})}} = 1.
\]

同法可得

\[
y_1^+(0) = \lim_{x \to 0} \frac{f(x^2)}{x} = -1,
\]

\[
y_1^-(0) = \lim_{x \to 0} \frac{-f(x^2)}{x} = 1.
\]

由上可以看出

\[
y_1(x) = \begin{cases} f(x^2), & 0 \leq x \leq \delta, \\ -f(x^2), & -\delta \leq x < 0 \end{cases}
\]

及

\[
y_2(x) = \begin{cases} -f(x^2), & 0 \leq x \leq \delta, \\ f(x^2), & -\delta \leq x < 0 \end{cases}
\]

是仅有的两个过点 (0, 0) 的可微分函数，且

\[
y_1'(0) = 1 \text{ 及 } y_2'(0) = -1.
\]

\[
\]

*) 此方程的图像系双纽线（图 6.28），它的极坐标方程为

\[
r^2 = a^2 \cos 2\theta.
\]

以上作法及结论

由图很容易看
3379. 设
\[(x^2 + y^2)^2 = 3x^2 y - y^3,\]
求 \(y'\) 当 \(x = 0\) 和 \(y = 0\) 时的值。

解 本题讨论方法与 3378 题类似，但由于不能直接解出 \(y = f(x)\)，故只能用隐函数表示。由
\[(x^2 + y^2)^2 = 3x^2 y - y^3,\]
得
\[x^4 + (2y^2 - 3y)x^2 + y^4 + y^3 = 0.\]
解之得
\[x^2 = \frac{(3y - 2y^2) \pm \sqrt{9y^2 - 16y^3}}{2}.\]
令
\[g(y) = \frac{3y - 2y^2 + \sqrt{9y^2 - 16y^3}}{2},\]
\[h(y) = \frac{3y - 2y^2 - \sqrt{9y^2 - 16y^3}}{2},\]
则不难验证：在 \(y = 0\) 的邻域内均有 \(g(y) \geq 0\)，而仅当 \(y \geq 0\) 时才有 \(h(y) \geq 0\)。于是，点 \((0, 0)\) 为极点，且从该点出发，有六个单值连续枝：
I. \(x_1 = \sqrt{g(y)},\ 0 \leq y \leq \varepsilon\); 它在 \(0 \leq x \leq \delta\) 上定义

II. \(x_2 = \sqrt{g(y)}, -\varepsilon \leq y \leq 0\); 它在 \(-\delta \leq x \leq 0\) 上定义

III. \(x_3 = -\sqrt{g(y)}, -\varepsilon \leq y \leq 0\); 它在 \(0 \leq x \leq \delta\) 上定义

IV. \(x_4 = -\sqrt{g(y)}, -\varepsilon \leq y \leq 0\); 它在 \(-\delta \leq x \leq 0\)
上定义隐函数 \( y = f_\varepsilon(x) \).

V. \( x_\varepsilon = \frac{h(y)}{y}, \ 0 \leq y < \varepsilon \); 它在 \( 0 \leq x < \delta \) 上定义

隐函数 \( y = f_\varepsilon(x) \).

VI. \( x_\varepsilon = -\sqrt{h(y)}, \ 0 \leq y < \varepsilon \); 它在 \(-\delta < x \leq 0\) 上定义

隐函数 \( y = f_\varepsilon(x) \).

上述隐函数的存在性，易从对右端 \( y \) 的表达式求导数而导数不为零来证. 因此，只要求上述六级在原点的单侧导数.

\[
\begin{align*}
\lim_{x \to 0^+} \frac{f_1(x) - f_1(0)}{x - 0} &= \lim_{y \to 0^+ \sqrt{g(y)}} \frac{y}{2y^2} = 0 \\
&= \lim_{y \to 0^+ \sqrt{g(y)}} \sqrt{\frac{2y^2}{3y - 2y^2 + \sqrt{9y^2 - 16y^3}}}
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0^-} \frac{f_2(x) - f_2(0)}{x - 0} &= \lim_{y \to 0^- \sqrt{g(y)}} \frac{y}{3 - 2y + \sqrt{9 - 16y}} = 0.
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0^+} \frac{f_3(x) - f_3(0)}{x - 0} &= \lim_{y \to 0^+ \sqrt{g(y)}} \frac{y}{\sqrt{g(y)}}
\end{align*}
\]

\[
\begin{align*}
&= \lim_{y \to 0^+ \sqrt{g(-y)}} \frac{-y}{2y^2} = 0 \\
&= \lim_{y \to 0^+ \sqrt{g(-y)}} \frac{\sqrt{2y^2}}{\sqrt{9y^2 + 16y^3 - 3z^2 - 2z^2}}
\end{align*}
\]

\[
\begin{align*}
&= \lim_{y \to 0^+ \sqrt{g(-y)}} \frac{2z^2(\sqrt{9y^2 + 16y^3 + 3z + 2z^2})}{(9z^2 + 16z^3) - (3z + 2z^2)^2}
\end{align*}
\]
\[
\lim_{x \to 0} \frac{f_4(x)}{x} = \lim_{y \to 0} \frac{y}{\sqrt{g(y)}} = -\sqrt{3}.
\]

\[
f'_{4-}(0) = \lim_{x \to 0} f_4(x) = \lim_{x \to 0} \frac{y}{\sqrt{g(y)}} = -\sqrt{3}.
\]

\[
f'_{5+}(0) = \lim_{x \to 0} f_5(x) = \lim_{x \to 0} \frac{y}{\sqrt{h(y)}} = \sqrt{3}.
\]

\[
f'_{5-}(0) = \lim_{y \to 0} \frac{f_5(y)}{y} = \lim_{y \to 0} \frac{2y^2}{3y^2 - 2y^2 - 9y^2 - 16y^2} = \sqrt{3}.
\]

\[
f'_{6-}(0) = \lim_{x \to 0} f_6(x) = \lim_{x \to 0} \frac{y}{\sqrt{h(y)}} = -\sqrt{3}.
\]

于是，上述六个单位连续取可组成三个（−δ, δ）上的可微函数 \( y = y_i(x) \) (\( i = 1, 2, 3 \)),

\[
y_1(x) = \begin{cases} 
    f_1(x), & x > 0 \\
    f_2(x), & x < 0
\end{cases}, \quad y'_1(0) = 0;
\]

\[
y_2(x) = \begin{cases} 
    f_5(x), & x > 0 \\
    f_6(x), & x < 0
\end{cases}, \quad y'_2(0) = -\sqrt{3};
\]

\[
y_3(x) = \begin{cases} 
    f_3(x), & x > 0 \\
    f_6(x), & x < 0
\end{cases}, \quad y'_3(0) = \sqrt{3}.
\]
*) 此方程的图象为三瓣玫瑰线（图 6.29），它的极坐标方程为

\[ r = a \sin 3\theta. \]

以上作法及结论，由图很容易看出。

3380. 设 \( x^2 + xy + y^2 = 3 \)，
求 \( y' \)，\( y'' \)及\( y''' \)。

解 等式两端对 \( x \) 求导数，得

\[ 2x + y + xy' + 2yy' = 0. \]

于是，

\[ y' = -\frac{2x + y}{x + 2y}. \]

再对上式求导数，得

\[ y'' = -\frac{1}{(x + 2y)^2} \left\{ (2 + y')(x + 2y) \right\} \]

\[ = -(1 + 2y')(2x + y) \rightarrow \frac{18}{(x + 2y)^3}; \]

\[ y''' = \frac{54}{(x + 2y)^4} (1 + 2y') = -\frac{162x}{(x + 2y)^3}. \]

3381. 设

\[ x^2 - xy + 2y^2 + x - y - 1 = 0, \]

求 \( y' \)，\( y'' \)及\( y''' \)当 \( x = 0 \)，\( y = 1 \) 时的值。

解 等式两端对 \( x \) 求导数，得
\[ 2x - y - xy' + 4y'y + 1 - y' = 0. \quad (1) \]

以 \( x = 0, \ y = 1 \) 代入 (1) 式，得

\[ y' \bigg|_{x = 0, \ y = 1} = 0. \]

将 (1) 式再对 \( x \) 求导数，得

\[ 2 - y' - y' - xy'' + 4y'y^2 + 4yy'' - y'' = 0. \quad (2) \]

以 \( x = 0, \ y = 1, \ y' = 0 \) 代入 (2) 式，得

\[ y'' \bigg|_{x = 0, \ y = 1} = -\frac{2}{3}. \]

将 (2) 式再对 \( x \) 求导数，得

\[ -3y'' - xy' + 12y'y'' + 4yy'' - y'' = 0. \quad (3) \]

以 \( x = 0, \ y = 1, \ y' = 0, \ y'' = -\frac{2}{3} \) 代入 (3) 式，得

\[ y''' \bigg|_{x = 0, \ y = 1} = -\frac{2}{3}. \]

3382. 证明：对于二次曲线

\[ ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0, \]

等式

\[ \frac{d^3}{dx^3} \left[ (y'')^{-\frac{2}{3}} \right] = 0 \]

为真。

证 原题中的二次曲线应是非退化的，即

\[ \Delta = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix} \neq 0, \]

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由 $A \neq 0$ 保证 $y'' \neq 0$.

等式两端对 $x$ 求导数，得

$$2a + 2by + 2bx y' + 2cy y' + 2d + 2ey' = 0. \quad (1)$$

于是，

$$y' = -\frac{ax + by + d}{bx + cy + e}.$$

（1）式除以 2 后，两端再对 $x$ 求导数，得

$$a + 2by' + cy'^2 + (bx + cy + e)y'' = 0.$$ 于是，

$$y'' = -\frac{a + 2by' + cy'^2}{bx + cy + e} = -\frac{1}{(bx + cy + e)^3} \cdot \left\{ a(bx + cy + e)^2 - 2b(bx + cy + e)(ax + by + d) + c(ax + by + d)^2 \right\}$$

$$= \frac{A}{(bx + cy + e)^3},$$

$$\left( y'' \right)^{-\frac{2}{3}} = A^{-\frac{2}{3}} \cdot (bx + cy + e)^2$$

$$= A^{-\frac{2}{3}} \cdot (b^2x^2 + c(cy^2 + 2bxy + 2ey) + e^2 + 2bex)$$

$$= A^{-\frac{2}{3}} \cdot (b^2x^2 - c(ax^2 + 2dxy + f) + 2bex + e^2)$$

$$= A^{-\frac{2}{3}} \cdot ((b^2 - ac)x^2 + 2(2b + c)(be - cd)x + e^2 - cf),$$

即 $\left( y'' \right)^{-\frac{2}{3}}$ 是关于 $x$ 的二次三项式，故

$$\frac{d^3}{dx^3} \left[ \left( y'' \right)^{-\frac{2}{3}} \right] = 0.$$ 由于函数 $z = z(x, y)$ 求一阶和二阶的偏导函数，设：
3383. \( x^2 + y^2 + z^2 = a^2 \).

解 等式两端微分，得

\[
2xdx + 2ydy + 2zdz = 0, \quad (1)
\]
\[
dx^2 + dy^2 + dz^2 + 2d^2z = 0, \quad (2)
\]

由 (1) 得

\[dz = -\frac{x}{z} dx - \frac{y}{z} dy,\]

故有

\[
\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}.
\]

由 (2) 得

\[
d^2z = -\frac{1}{z}(dx^2 + dy^2 + dz^2)
\]

\[
= -\frac{1}{z} dx^2 - \frac{1}{z} dy^2 - \frac{1}{z}\left(\frac{x}{z} dx + \frac{y}{z} dy\right)^2
\]

\[
= -\frac{1}{z}\left(1 + \frac{x^2}{z^2}\right)dx^2 - \frac{2xy}{z^3} dx dy - \frac{1}{z}\left(1 + \frac{y^2}{z^2}\right) dy^2,
\]

故有

\[
\frac{\partial^2 z}{\partial x^2} = -\frac{1}{z}\left(1 + \frac{x^2}{z^2}\right) = -\frac{z^2 + x^2}{z^3},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = -\frac{xy}{z^3}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{z^2 + y^2}{z^3}.
\]

3384. \( z^3 - 3xyz = a^3 \).

解 等式两端对 \( x \) 求偏导函数，得

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\[ 3z^2 \frac{\partial z}{\partial x} - 3yz - 3xy \frac{\partial z}{\partial x} = 0. \tag{1} \]

于是，

\[ \frac{\partial z}{\partial x} = \frac{yz}{z^2 - xy}. \]

同法可得

\[ \frac{\partial z}{\partial y} = \frac{xz}{z^2 - xy}. \]

（1）式除以 3 后再分别对 x 及对 y 求偏导函数，得

\[ 2z\left(\frac{\partial z}{\partial x}\right)^2 + z^2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial z}{\partial x} - xy \frac{\partial^2 z}{\partial x^2} = 0, \]

\[ \left(2z \frac{\partial z}{\partial y} - x\right) \frac{\partial z}{\partial x} + \left(z^2 - xy\right) \frac{\partial^2 z}{\partial x \partial y} \]

\[ -z - y \frac{\partial z}{\partial y} = 0. \]

将 \( \frac{\partial z}{\partial x} \) 及 \( \frac{\partial z}{\partial y} \) 代入上述两式，化简整理得

\[ \frac{\partial^2 z}{\partial x^2} = -\frac{2xy^3z}{(z^2 - xy)^3}; \]

\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{z(z^4 - 2xyz^2 - x^2y^2)}{(z^2 - xy)^3}. \]

同法可得

\[ \frac{\partial^2 z}{\partial y^2} = -\frac{2x^3yz}{(z^2 - xy)^3}. \]
3385. $x + y + z = e^z$.

解  等式两端微分，得
$$d(x + y + z) = e^z dz,$$
\[ d(x + y + z) = \frac{1}{e^z - 1} (dx + dy) = \frac{1}{x + y + z - 1} (dx + dy). \tag{1} \]

故有
$$dz = \frac{1}{e^z - 1} (dx + dy) = \frac{1}{x + y + z - 1} (dx + dy).$$

于是，
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{1}{x + y + z - 1}. \tag*{1}$$

再将 (1) 式微分一次，得
$$d^2 z = e^z d^2 z + e^z dz^2,$$

故有
$$d^2 z = -e^z (dx)^2 = -e^z (dy)^2 + 2 dx dy + dy^2).$$

于是，
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y^2} = -\frac{e^z}{(e^z - 1)^2}$$

$$= -\frac{x + y + z}{(x + y + z - 1)^2}. \tag*{2}$$

3386. $z = \sqrt{x^2 - y^2}$ $tg \frac{z}{\sqrt{x^2 - y^2}}$.

解  设 $r = \sqrt{x^2 - y^2}$，则 $\frac{z}{r} = tg \frac{z}{r}$,
\[
d(\frac{z}{r}) = \frac{d(\frac{z}{r})}{1 + \left(\frac{z}{r}\right)^2},
\]

从而有 \(d(\frac{z}{r}) = 0\)，或 \(rdz - zdr = 0\)，即

\[
dz = \frac{z}{r^2}(zdx - ydy).
\] (1)

于是，

\[
\frac{\partial z}{\partial x} = \frac{zx}{r^2} = \frac{xz}{x^2 - y^2}, \quad \frac{\partial z}{\partial y} = -\frac{yz}{r^2} = -\frac{yz}{x^2 - y^2}.
\]

由 (1) 得

\[
(x^2 - y^2)dz = xzd x - yzd y. \quad \text{(2)}
\]

(2) 式再微分一次，得

\[
(x^2 - y^2)dz = -(2xdx - 2yd y)dz + xdx dz + zd x^2 - yd y dz - zd y^2
\]

\[
= - (xdx - yd y) \left[ \frac{z(xdx - yd y)}{x^2 - y^2} \right] + zd x^2 - zd y^2
\]

\[
= \frac{z}{x^2 - y^2} \left[ -x^2 dx^2 + 2xydxdy - y^2 dy^2 \right.
\]

\[
\left. + (x^2 - y^2)d x^2 - (x^2 - y^2)dy^2 \right]
\]

\[
= \frac{z(-y^2 dx^2 + 2xydxdy - x^2 dy^2)}{x^2 - y^2}.
\]

于是，
\[
\frac{\partial^2 z}{\partial x^2} = -\frac{y^2 z}{(x^2 - y^2)^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{xyz}{(x^2 - y^2)^2},
\]
\[
\frac{\partial^2 z}{\partial y^2} = -\frac{x^2 z}{(x^2 - y^2)^2}.
\]

(3387) \(x + y + z = e^{-(x+y+z)}\).

解：等式两端对 \(x\) 求偏导函数，得
\[
1 + \frac{\partial z}{\partial x} = e^{-(x+y+z)} \cdot (-1 + \frac{\partial z}{\partial x}).
\]
于是，
\[
\frac{\partial z}{\partial x} = -1.
\]
利用对称性，得
\[
\frac{\partial z}{\partial y} = -1.
\]
显见
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y^2} = 0.
\]

(3388) 设
\[
x^2 + y^2 + z^2 - 3xyz = 0 \quad (1)
\]
及
\[
f(x, y, z) = xy^2z^3.
\]
求：(a) \(f'_x(1, 1, 1)\)，设 \(z = z(x, y)\) 是由方程 (1) 所定义的隐函数，(b) \(f'_x(1, 1, 1)\)，设 \(y = y(x, z)\) 是由方程 (1) 所定义的隐函数。说明为什么这些导数存在。
函数相异。
解 (a) 记 \( F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0 \)，
则由方程 (1) 所定义的隐函数 \( z = z(x, y) \) 的偏导函数 \( z'_i(x, y) \) 在 \((1,1)\) 点的值为

\[
z'_y(1,1) = - \frac{F'_x(1,1,1)}{F'_y(1,1,1)} = - \frac{\frac{d}{dx} F(x, 1, 1)}{\frac{d}{dy} F(1, 1, z)} \bigg|_{x=1}^{z=1} \]

\[
= - \frac{\frac{d}{dx} (x^2 + 2 - 3x)}{\frac{d}{dy} (2 + z^2 - 3z)} \bigg|_{x=1}^{z=1} = - \frac{1}{1} = -1.
\]

于是，

\[
\frac{\partial}{\partial x} \left( f(x, y, z(x, y)) \right) \bigg|_{(1,1,1)} \]

\[
= \frac{d}{dx} f(x, 1, 1) \bigg|_{x=1} + \frac{d}{dy} f(1, 1, z) \bigg|_{z=1} \cdot z'_y(1,1) \]

\[
= 1 + 3 \cdot (-1) = -2.
\]

(6) \( y'_y(1,1) = - \frac{F'_x(1,1,1)}{F'_y(1,1,1)} \)

\[
= - \frac{\frac{d}{dx} F(x, 1, 1)}{\frac{d}{dy} F(1, y, 1)} \bigg|_{x=1}^{y=1} = -1.
\]

于是，

\[
\frac{\partial}{\partial x} \left( f(x, y(x(z), z)) \right) \bigg|_{(1,1,1)}
\]

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$$= \frac{d}{dx} f(x, 1, 1) \bigg|_{x=1} + \frac{d}{dy} f(1, y, 1) \bigg|_{y=1} \cdot y'_x(1, 1)$$

$$= 1 + 2 \cdot (-1) = -1.$$ 

由 (a) 与 (6) 所求得的对 $x$ 的偏导函数在 $(1, 1, 1)$ 点的值不相等，可说明如下：

方程 $F(x, y, z) = 0$ 代表一个空间曲面，而 $f(x, y, z)$ 表示定义在这个曲面上的一个函数。函数 $G(x, y) = f(x, y, z(x, y))$ 表示把原曲面上的点投影到 $Oxy$ 平面上后，原曲面上的函数看成在 $Oxy$ 平面上定义的一个函数，$G'_x(x, y)$ 表示此函数在 $Ox$ 轴方向的变化率，它不仅包含了原来函数在 $Ox$ 轴方向的变化率，还包含了原来函数在 $Oz$ 轴方向的变化率的一部分。同样地，$H(x, z) = f(x, y(x, z), z)$ 表示把原曲面上的点投影到 $Oxz$ 平面上后，原曲面上的函数看成在 $Oxz$ 平面上定义的函数，$H'_x(x, z)$ 表示此函数在 $Ox$ 轴方向的变化率，它不仅包含了原来函数在 $Ox$ 轴方向的变化率，还包含了原来函数在 $Oy$ 轴方向的变化率的一部分。一般地，原来函数在 $Oy$ 轴和 $Oz$ 轴方向的变化率的那两部份是不相等的。

3389. 设 $x^2 + 2y^2 + 3z^2 + xy - z - 9 = 0$，求 $\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial x \partial y}$，

$$\frac{\partial^2 z}{\partial y^2}$$ 当 $x = 1$，$y = -2$，$z = 1$ 时的值。

解 等式两端微分一次，得

$$2xdx + 4ydy + 6zdz + xdy + ydx - dz = 0.$$
即

\[(1 - 6z)dz = (2x + y)dx + (4y + x)dy.\]  \hspace{1cm} (1)

再微分一次，得

\[(1 - 6z)d^2z = 6dz^2 + 2dx^2 + 2dx dy + 4dy^2.\]  \hspace{1cm} (2)

以 \(x = 1, y = -2, z = 1\) 代入 (1) 式，得 \(dz = \frac{7}{5}dy\).

再以 \(z = 1, dz = \frac{7}{5}dy\) 代入 (2) 式，得

\[d^2z = -\frac{2}{5}dx^2 - \frac{2}{5}dx dy - \frac{394}{125}dy^2.\]

于是，当 \(x = 1, y = -2, z = 1\) 时,

\[\frac{\partial^2 z}{\partial x^2} = -\frac{2}{5}, \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{5}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{394}{125}.\]

求 \(dz\) 和 \(d^2z\)，设:

\[3390. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.\]

解 等式两端微分一次，得

\[\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0.\]

于是,

\[dz = -\frac{c^2}{z}\left(\frac{xdx}{a^2} + \frac{ydy}{b^2}\right).\]

再将 \(dz\) 微分一次，得

\[d^2z = -\frac{c^2}{z^2}\left[z\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2}\right) - \left(\frac{xdx}{a^2} + \frac{ydy}{b^2}\right)dz\right].\]

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$$\begin{align*}
&= -\frac{c^4}{2^3} \left[ \left( \frac{x^2}{a^2} + \frac{z^2}{c^2} \right) \frac{d x^2}{a^2} + \frac{2 x y}{a^2 b^2} d x d y \\
&\quad + \left( \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \frac{d y^2}{b^2} \right].
\end{align*}$$

3391. \(xyz = x + y + z\).

解 等式两端微分一次，得

$$yz dx + xzd y + yzd x = dx + dy + dz, \quad (1)$$

于是，

$$dz = -\frac{(1-yz)dx + (1-xz)dy}{1-xy}. \quad (2)$$

对 (1) 式再微分一次，得

$$2z dx dy + 2xdydz + 2ydz dx + zyd^2 z = d^2 z. \quad (3)$$

以 (2) 式代入 (3) 式，化简整理得

$$d^2 z = -\frac{2}{(1-xy)^2} \left\{ y(1-yz)dx^2 + [x+y \\
- z(1+xy)]dxdy + x(1-xz)dy^2 \right\}$$

$$= -\frac{2 \left\{ y(1-yz)dx^2 - 2xdydz + x(1-xz)dy^2 \right\}}{(1-xy)^2}.$$ 

3392. \(\frac{x}{z} = \ln \frac{z}{y}\).

解 等式两端微分一次，得

$$\frac{zdx - xdz}{z^2} = dz - \frac{dy}{y}.$$
于是，

$$dz = \frac{z(ydx + zd\gamma)}{y(x+z)}.$$  

对 $(x+z)dz = zd\gamma + \frac{z^2}{y}d\gamma$ 再微分一次，得

$$(x+z)d^2z = -(dz + d\gamma)dz + zd\gamma$$

$$+ \frac{2z}{y}dzd\gamma - \frac{z^2}{y^2}d\gamma^2$$

$$= -dz^2 + \frac{2z}{y}d\gamma dz - \frac{z^2}{y^2}d\gamma^2 = -(dz - \frac{z}{y}d\gamma)^2$$

$$= -\frac{z^2[(ydx + zd\gamma) - (x+z)d\gamma]^2}{y^2(x+z)^2}$$

$$= -\frac{z^2(ydx - xd\gamma)}{y^2(x+z)^2}.$$  

于是，

$$d^2z = -\frac{z^2(ydx - xd\gamma)^2}{y^2(x+z)^3}.$$  

3393. $z = x + \arctg \frac{y}{z-x}$.  

解  等式两端微分一次，得

$$dz = dx + \frac{1}{1 + \frac{y^2}{(z-x)^2}}(z-x)dy - y(zdx - dx).$$

化简整理，得
\[dz = dx + \frac{z-x}{(z-x)^2 + y(y+1)} \, dy,\]

再对上式微分一次，得

\[d^2z = \frac{1}{[(z-x)^2 + y(y+1)]^2} \left\{ [(z-x)^2 + y(y+1)] \, dy \cdot (dz - dx) - (z-x) \, dy \cdot 2(z-x)(dz - dx) + 2y \, dy + dy \right\}.\]

将 \(dz\) 代入化简整理，即有

\[d^2z = \frac{2(z-x)(y+1)[(z-x)^2 + y^2]}{[(z-x)^2 + y(y+1)]^3} \, dy^2.\]

3394. 设 \(u^3 - 3(x+y)u^2 + z^3 = 0\)，求 \(du\)。

解 等式两端微分，得

\[3u^2 \, du - 3u^2(dx + dy) - 6u(x+y)du + 3z^2 \, dz = 0.\]

于是，

\[du = \frac{u^2(dx + dy) - z^2 \, dz}{u[u - 2(x+y)]}.\]

3395. 设 \(F(x+y+z, x^2 + y^2 + z^2) = 0\)，求 \(\frac{\partial^2 z}{\partial x \partial y}\)。

解 等式两端对 \(x\) 求偏导函数，得

\[F'_1 \cdot \left( 1 + \frac{\partial z}{\partial x} \right) + F'_2 \cdot \left( 2x + 2z \frac{\partial z}{\partial x} \right) = 0.\]

于是，

\[\frac{\partial z}{\partial x} = - \frac{F'_1 + 2xF'_2}{F'_1 + 2xF'_2}. \quad (1)\]
同法可得
\[
\frac{\partial z}{\partial y} = - \frac{F'_1 + 2yF'_2}{F'_1 + 2zF'_2}.
\]

(1) 式两端对 \( y \) 求偏导函数，得
\[
\frac{\partial^2 z}{\partial x \partial y} = - \frac{1}{(F'_1 + 2zF'_2)^2} \left\{ (F'_1 + 2zF'_2)(F'_2)_{yy} - (F'_1 + 2xF'_2)(F'_1)_{yy} + 2(z - x)F'_2 \right\}
\]
\[
= - \frac{1}{(F'_1 + 2zF'_2)^2} \left[ 2(z - x)F'_1 \cdot (F'_2)_{yy} + 2(z - x)F'_2 \right.
\]
\[
\left. \cdot (F'_1)_{yy} - 2(F'_1 F'_2 + x(F'_2)^2)z'_y \right\}
\]
\[
= - \frac{2(z - x)}{(F'_1 + 2zF'_2)^2} \left\{ F'_1 \cdot (F'_2)_{yy} - F'_2 \cdot (F'_1)_{yy} \right\}
\]
\[
- \frac{2F'_2 \cdot (F'_1 + 2xF'_2) \cdot (F'_1 + 2yF'_2)}{(F'_1 + 2zF'_2)^2}.
\]

现分别求 \( (F'_1)_{yy} \) 及 \( (F'_2)_{yy} \):
\[
(F'_1)_{yy} = F''_{11} \cdot (1 + z'_y) + F''_{12} \cdot (2y + 2zz'_y) ,
\]
\[
(F'_2)_{yy} = F''_{21} \cdot (1 + z'_y) + F''_{22} \cdot (2y + 2zz'_y) .
\]

注意到
\[
1 + z'_y = \frac{2(z - y)F'_1}{F'_1 + 2zF'_2} , \quad 2y + 2zz'_y = \frac{2(y - x)F'_1}{F'_1 + 2zF'_2} ,
\]
即得
\[
F'_1 \cdot (F'_2)_{yy} - F'_2 \cdot (F'_1)_{yy} = F'_1 F''_{21} \cdot \frac{2(z - y)F'_1}{F'_1 + 2zF'_2}
\]

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$$+rac{F_{12}'}{F_1'+2zF_2'}\cdot\frac{2(y-z)F_2'}{F_1'+2zF_2'}$$

$$-F_2'F_{11}' \cdot \frac{2(z-y)F_2'}{F_1'+2zF_2'} - F_2'F_{12}' \cdot \frac{2(y-z)F_1'}{F_1'+2zF_2'}$$

$$= \frac{2(y-z)}{F_1'+2zF_2'} \left\{ (F_1')^2F_{22}'' - 2F_1'F_2'F_{12}'' + (F_2')^2F_{11}'' \right\} .$$

于是，

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{4(x-z)(y-z)}{(F_1'+2zF_2')^3} \left\{ (F_1')^2F_{22}'' - 2F_1'F_2'F_{12}'' + (F_2')^2F_{11}'' \right\}$$

$$- \frac{2F_2' \cdot (F_1'+2xF_2') \cdot (F_1'+2yF_2')}{(F_1'+2zF_2')^8} .$$

3396. 设 $F(x-y, y-z, z-x) = 0$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$。

解 等式两端对 $x$ 求偏导函数，得

$$F_1' + F_2' \cdot \left( -\frac{\partial z}{\partial x} \right) + F_3' \cdot \left( \frac{\partial z}{\partial x} - 1 \right) = 0 .$$

于是，

$$\frac{\partial z}{\partial x} = \frac{F_1' - F_3'}{F_2' - F_3'} .$$

同法可得

$$\frac{\partial z}{\partial y} = \frac{F_2' - F_1'}{F_2' - F_3'} .$$

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3397. 设 \( F(x, x+y, x+y+z) = 0 \)，求 \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \) 和 \( \frac{\partial^2 z}{\partial x^2} \)。

解  等式两端分别对 \( x \) 及对 \( y \) 求偏导函数，得

\[
F'_1 + F'_2 + F'_3 \cdot (1 + \frac{\partial z}{\partial x}) = 0,
\]

\[
F'_2 + F'_3 \cdot (1 + \frac{\partial z}{\partial y}) = 0.
\]

于是，

\[
\frac{\partial z}{\partial x} = -\left(1 + \frac{F'_1 + F'_2}{F'_3}\right), \quad \frac{\partial z}{\partial y} = -\left(1 + \frac{F'_2}{F'_3}\right).
\]

再将 \( \frac{\partial z}{\partial x} \) 对 \( x \) 求偏导函数，得

\[
\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(F'_3)^2} \left\{ F'_3 \cdot \left[ F''_{11} + F''_{12} + F''_{13} \cdot (1 + \frac{\partial z}{\partial x}) \right] + F''_{21} + F''_{22} + F''_{23} \cdot \left(1 + \frac{\partial z}{\partial x}\right) \right\}.
\]

将 \( \frac{\partial z}{\partial x} \) 代入化简整理得

\[
\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(F'_3)^3} \left\{ (F'_3)^2 \cdot (F''_{11} + 2F''_{12} + F''_{22}) - 2(F'_1 + F'_2)F'_3 \cdot (F''_{13} + F''_{23}) + (F'_1 + F'_2)^2 F''_{33} \right\}.
\]

3398. 设 \( F(xz, yz) = 0 \)，求 \( \frac{\partial^2 z}{\partial x^2} \)。
解  等式两端对 $x$ 求偏导函数，得

$$F'_{1} \cdot (z + x \frac{\partial z}{\partial x}) + F'_{2} \cdot y \frac{\partial z}{\partial x} = 0.$$  

于是，

$$\frac{\partial z}{\partial x} = -\frac{zF'_{1}}{xF'_{1} + yF'_{2}}.$$  

将 $\frac{\partial z}{\partial x}$ 再对 $x$ 求偏导函数，得

$$\frac{\partial^{2} z}{\partial x^{2}} = -\frac{1}{(xF'_{1} + yF'_{2})^{2}} \left\{ (xF'_{1} + yF'_{2}) \cdot \left[ F'_{1} \frac{\partial z}{\partial x} \right. \right.$$  

$$+ z \left( F''_{11} \cdot (z + x \frac{\partial z}{\partial x}) + F''_{12}y \frac{\partial z}{\partial x} \right)$$  

$$- \left[ F'_{1} + x \left( F'_{11} \cdot (z + x \frac{\partial z}{\partial x}) + F'_{12}y \frac{\partial z}{\partial x} \right) \right.$$$$
+ y \left( F'_{21} \cdot (z + x \frac{\partial z}{\partial x}) + F'_{22}y \frac{\partial z}{\partial x} \right) \left] \right\} 2F'_{1} \}.$$  

将 $\frac{\partial z}{\partial x}$ 代入化简整理得

$$\frac{\partial^{2} z}{\partial x^{2}} = -\frac{1}{(xF'_{1} + yF'_{2})^{2}} \{ y^{2}z^{2} \left( (F'_{1})^{2}F''_{12} \right.$$$$
- 2F'_{1}F'_{2}F'_{12} + (F'_{2})^{2}F'_{11} \right] - 2z(F'_{1})^{2}$$  

$$\cdot \left( xF'_{1} + yF'_{2} \right) \}.$$  

3399. 设 $F(x + z, y + z) = 0$；

(6) $F \left( \frac{x}{z}, \frac{y}{z} \right) = 0$，求 $d^{2}z$.  

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解 (a) 等式两端微分，得
\[ F'_1 \cdot (dx + dz) + F'_2 \cdot (dy + dz) = 0. \quad (1) \]

于是，
\[ dz = - \frac{F'_1 dx + F'_2 dy}{F'_1 + F'_2}, \]
\[ dx + dz = \frac{F'_2 \cdot (dx - dy)}{F'_1 + F'_2}, \]
\[ dy + dz = - \frac{F'_1 \cdot (dx - dy)}{F'_1 + F'_2}. \]

对 (1) 式再求一次微分，得
\[ F''_{11} \cdot (dx + dz)^2 + 2F''_{12} \cdot (dx + dz)(dy + dz) + F''_{22} \cdot (dy + dz)^2 + (F'_1 + F'_2)dz^2 = 0. \]

于是，
\[ dz^2 = - \frac{1}{F'_1 + F'_2} \left[ F''_{11} \cdot (dx + dz)^2 + 2F''_{12} \cdot (dx + dz)(dy + dz) + F''_{22} \cdot (dy + dz)^2 \right] \]
\[ = - \frac{1}{(F'_1 + F'_2)^2} \left[ F''_{11} \cdot (F'_2)^2 - 2F'_{12} \cdot F'_1 F'_2 \right. \]
\[ \left. + F''_{22} \cdot (F'_1)^2 \right] (dx - dy)^2. \]

(6) 等式两端微分，得
\[ F'_1 \cdot \frac{2dx - xdz}{z^2} + F'_2 \cdot \frac{2dy - ydz}{z^2} = 0. \quad (2) \]

于是，
\[ dz = \frac{z(F'_1 dx + F'_2 dy)}{xF'_1 + yF'_2}. \]
\[
zd x - x d z = \frac{2F'_2 \cdot (yd x - xd y)}{xF'_1 + yF'_2},
\]
\[
zd y - y d z = -\frac{2F'_1 \cdot (yd x - xd y)}{xF'_1 + yF'_2}.
\]

(2) 式乘以 2 后再微分一次，得

\[
F''_{11} \cdot \frac{(zd x - xd z)^2}{z^2} + 2F''_{12} \cdot \frac{(zd x - xd z)(zd y - yd z)}{z^2} + F''_{22} \cdot \frac{(zd y - yd z)^2}{z^2} - (xF'_1 + yF'_2) d^2 z = 0.
\]

于是，

\[
d^2 z = \frac{1}{z^2 (xF'_1 + yF'_2)} \left[ F''_{11} \cdot (zd x - xd z)^2 + 2F''_{12} (zd x - xd z)(zd y - yd z) + F''_{22} \cdot (zd y - yd z)^2 \right]
\]

\[
= \frac{(yd x - xd y)^2}{(xF'_1 + yF'_2)^2} \left[ F''_{11} \cdot (F'_2)^2 - 2F'_1 F'_2 F''_{12} + F''_{22} \cdot (F'_1)^2 \right].
\]

3400. 设 \( x = x(y, z), \ y = y(x, z), \ z = z(x, y) \) 为由方程 \( F(x, y, z) = 0 \) 所定义的函数。证明，

\[
\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.
\]

证  根据隐函数求导法，有

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\[
\frac{\partial x}{\partial y} = -\frac{\partial F}{\partial z}, \quad \frac{\partial y}{\partial z} = -\frac{\partial F}{\partial x}, \quad \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial y}.
\]

三式相乘即得

\[
\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.
\]

3401. 设 \(x + y + z = 0, \ x^2 + y^2 + z^2 = 1\)，求 \(\frac{dx}{dz}\) 和 \(\frac{dy}{dz}\)。

解 对 \(z\) 求导数，得

\[
\begin{cases}
\frac{dx}{dz} + \frac{dy}{dz} + 1 = 0, \\
2x\frac{dx}{dz} + 2y\frac{dy}{dz} + 2z = 0.
\end{cases}
\]

联立方程求解，得

\[
\frac{dx}{dz} = \frac{y - z}{x - y}, \quad \frac{dy}{dz} = \frac{z - x}{x - y}.
\]

3402. 设 \(x^2 + y^2 = \frac{1}{2}z^2, \ x + y + z = 2\)，求 \(\frac{dx}{dz}, \ \frac{dy}{dz}, \ \frac{d^2x}{dz^2}\)

和 \(\frac{d^3y}{dz^3}\) 当 \(x = 1, \ y = -1, \ z = 2\) 时的值。

解 对 \(z\) 求导数，得

\[
\begin{cases}
2x\frac{dx}{dz} + 2y\frac{dy}{dz} = z, \quad (1) \\
dx \frac{dx}{dz} + dy \frac{dy}{dz} + 1 = 0. \quad (2)
\end{cases}
\]

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\[
\begin{align*}
\begin{cases}
2 \left( \frac{dx}{dz} \right)^2 + 2x \frac{d^2 x}{dz^2} + 2 \left( \frac{dy}{dz} \right)^2 + 2 \frac{d^2 y}{dz^2} = 1, \\
\frac{d^2 x}{dz^2} + \frac{d^2 y}{dz^2} = 0.
\end{cases}
\end{align*}
\]

将 \(x = 1, y = -1, z = 2\) 代入 (1), (2), 解得
\[
\frac{dx}{dz} = 0, \quad \frac{dy}{dz} = -1.
\]

将上述结果及 \(x, y, z\) 值联同由 (4) 式所决定的式子
\[
\frac{d^2 x}{dz^2} = -\frac{d^2 y}{dz^2}
\]
一起代入 (3) 式，即得
\[
\frac{d^2 x}{dz^2} = -\frac{1}{4}, \quad \frac{d^2 y}{dz^2} = \frac{1}{4}.
\]

3403. 设 \(xu - yv = 0, yu + xv = 1\)，求 \(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}\) 和 \(\frac{\partial v}{\partial y}\)。

解 微分得
\[
\begin{align*}
\begin{cases}
xdx - ydv = vdy - udx, \\
ydu + xdv = -vdx - udy.
\end{cases}
\end{align*}
\]

于是，
\[
\begin{align*}
du &= \frac{1}{x^2 + y^2} [-(xu + yv)dx + (xy - yu)dy], \\
\frac{\partial u}{\partial x} &= -\frac{xu + yv}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{xy - yu}{x^2 + y^2}.
\end{align*}
\]
同法可得
\[
\frac{\partial v}{\partial x} = \frac{yu - xv}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2} \quad (x^2 + y^2 > 0).
\]

3404. 设 \( u + v = x + y \), \( \frac{\sin u}{\sin v} = \frac{x}{y} \), 求 \( du, dv, d^2 u \) 和 \( d^2 v \).

解 将原式改写为
\[
\begin{cases}
  u + v = x + y, \\
  y \sin u = x \sin v.
\end{cases}
\]
微分得
\[
\begin{align*}
  du + dv &= dx + dy, \\
  \sin u dy + y \cos u du &= \sin v dx + x \cos v dv. \tag{1}
\end{align*}
\]
联立求解，得
\[
\begin{align*}
  du &= \frac{1}{x \cos v + y \cos u} [(\sin v + x \cos v) dx \\
  &\quad - (\sin u - x \cos u) dy],
\end{align*}
\]
\[
\begin{align*}
  dv &= \frac{1}{x \cos v + y \cos u} [-(\sin v - y \cos u) dx \\
  &\quad + (\sin u + y \cos u) dy].
\end{align*}
\]
对 (1), (2) 式再微分一次，得
\[
\begin{align*}
  d^2 u + d^2 v &= 0, \\
  y \cos u \cdot d^2 u + 2 \cos u \cdot dy du - y \sin u \cdot du^2 \\
  &= x \cos v \cdot d^2 v + 2 \cos v \cdot dx dv - x \sin v \cdot dv^2.
\end{align*}
\]
联立求解，得
\[ d^2u = -d^2v = \frac{1}{x\cos v + y\cos u} \left( (2\cos vdx \\
- x\sin vdv)dy - (2\sin udy - y\sin udu)du \right) . \]

\[ e^\frac{x}{y} \cos v = \frac{x}{\sqrt{2}}, \quad e^\frac{x}{y} \sin v = \frac{y}{\sqrt{2}}. \]

求 \( du, dv, d^2u \) 和 \( d^2v \) 当 \( x = 1, y = 1, u = 0, \quad v = \frac{\pi}{4} \) 时的表达式。

解 将所给二式相减及平方相加，分别得

\[ \begin{aligned}
\frac{\tan \frac{u}{y}}{\sec^2 \frac{v}{y}} &= \frac{y}{x}, \\
\frac{2u}{e^x} &= \frac{x^2 + y^2}{2}.
\end{aligned} \]  \hspace{1cm} (1)  \hspace{1cm} (2)

微分 (1) 式：

\[ \sec^2 \frac{v}{y} \cdot \frac{ydv - vdy}{y^2} = \frac{xdy - ydx}{x^2}. \]  \hspace{1cm} (3)

以 \( x = 1, \quad y = 1, \quad v = \frac{\pi}{4} \) 代入 (3) 代，得

\[ du = \frac{1}{4} dy - \frac{1}{2} (dx - dy). \]

微分 (3) 式：

\[ 2\sec^2 \frac{v}{y} \cdot \frac{\tan \frac{u}{y}}{y} \cdot \left( \frac{ydv - vdy}{y^2} \right)^2 + \sec^2 \frac{v}{y} \]
\[ \frac{y^2 d^2 u}{y^3} = -2(y dv - v dy) dy \\
= -\frac{2(x dy - y dx)}{x^3}. \quad (4) \]

以 \( x = 1, y = 1, v = \frac{\sigma^2}{4} \) 及 \( dv \) 值代入 (4) 式，得

\[ d^2 u = \frac{1}{2} (dx - dy)^2. \]

微分 (2) 式，

\[ 2e^{\frac{2u}{x}} \cdot \frac{x du - udx}{x^2} = xdx + ydy. \quad (5) \]

以 \( x = 1, y = 1, u = 0 \) 代入 (5) 式，得

\[ du = \frac{dx + dy}{2}. \]

微分 (5) 式，

\[ 4e^{\frac{2u}{x}} \left( \frac{x du - udx}{x^2} \right)^2 + 2e^{\frac{2u}{x}} \frac{x^2 d^2 u - 2(x du - udx) dx}{x^3} \\
= dx^2 + dy^2. \quad (6) \]

以 \( x = 1, y = 1, u = 0 \) 及 \( du \) 代入 (6) 式，得

\[ d^2 u = dx^2. \]

3406. 设：

\[ x = t + t^{-1}, \quad y = t^2 + t^{-2}, \quad z = t^3 + t^{-3}. \]

求 \( \frac{dy}{dx}, \frac{dz}{dx}, \frac{d^2 y}{dx^2} \) 和 \( \frac{d^2 z}{dx^2} \).
解 \[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2t - \frac{2}{t^2}}{1 - \frac{1}{t^2}} = 2 \left( t + \frac{1}{t} \right);
\]
\[
\frac{dz}{dx} = \frac{dz}{dt} \cdot \frac{dt}{dx} = \frac{3t^2 - \frac{3}{t^4}}{1 - \frac{1}{t^2}} = 3 \left( t^2 + \frac{1}{t^2} + 1 \right);
\]
\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{2 \left( 1 - \frac{1}{t^2} \right)}{1 - \frac{1}{t^2}} = 2;
\]
\[
\frac{d^2z}{dx^2} = \frac{d}{dt} \left( \frac{dz}{dx} \right) = \frac{3 \left( 2t - \frac{2}{t^3} \right)}{1 - \frac{1}{t^2}} = 6 \left( t + \frac{1}{t} \right).
\]

注 本题也可消去 t 以求 \( \frac{dy}{dx} \), \( \frac{dz}{dx} \), \( \frac{d^2y}{dx^2} \) 和 \( \frac{d^2z}{dx^2} \). 事实上，

\[
y = \left( t + \frac{1}{t} \right)^2 - 2 = x^2 - 2,
\]
\[
z = \left( t + \frac{1}{t} \right) \left( t^2 - 1 + \frac{1}{t^2} \right) = x(x^2 - 3) = x^3 - 3x.
\]

于是，

\[
\frac{dy}{dx} = 2x, \quad \frac{dz}{dx} = 3x^2 - 3,
\]
\[
\frac{d^2y}{dx^2} = 2, \quad \frac{d^2z}{dx^2} = 6x.
\]
再将 $x = t + \frac{1}{t}$ 代入上述结果，即得

\[ \frac{dy}{dx} = 2 \left( t + \frac{1}{t} \right), \quad \frac{dz}{dx} = 3 \left( t^2 + \frac{1}{t^2} + 1 \right), \]

\[ \frac{d^2 y}{dx^2} = 2, \quad \frac{d^2 z}{dx^2} = 6 \left( t + \frac{1}{t} \right). \]

3407. 在 $Oxy$ 平面上怎样的域内方程组

\[ x = u + v, \quad y = u^2 + v^2, \quad z = u^3 + v^3 \]

（式中参数 $u$ 和 $v$ 取一切可能的实数值）定义 $z$ 为变量 $x$ 和 $y$ 的函数？求导函数 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$。

解 由 $u + v = x, \quad u^2 + v^2 = y$ 解得

\[ u = \frac{x \pm \sqrt{2y - x^2}}{2}, \quad v = \frac{x \mp \sqrt{2y - x^2}}{2}, \]

其中 $2y - x^2 \geq 0$ 或 $y \geq \frac{x^2}{2}$，此即所求之域。

再由 $x = u + v$ 及 $y = u^2 + v^2$ 分别对 $x$ 求偏导函数，得

\[ 1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \quad 0 = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x}. \]

联立求解得

\[ \frac{\partial u}{\partial x} = \frac{v}{v - u}, \quad \frac{\partial v}{\partial x} = -\frac{u}{v - u} \quad (u \neq v). \]

又由 $z = u^3 + v^3$ 对 $x$ 求偏导函数，即可得
\[
\frac{\partial z}{\partial x} = 3u^2 \frac{\partial u}{\partial x} + 3v^2 \frac{\partial v}{\partial x} = 3u^2 \cdot \frac{v}{v-u} - 3v^2 \cdot \frac{u}{v-u} = -3uv.
\]

同法求得
\[
\frac{\partial z}{\partial y} = \frac{3}{2}(u+v).
\]

注 本题也可消去 \(u,v\) 求 \(\frac{\partial z}{\partial x}\) 及 \(\frac{\partial z}{\partial y}\)。事实上，
\[
x^2 - y = 2uv,
\]
\[
z = (u+v)(u^2-uv+v^2) = x\left(\frac{3}{2} y - \frac{x^2}{2}\right)
\]
\[
= \frac{x}{2}(3y-x^2).
\]

于是，
\[
\frac{\partial z}{\partial x} = \frac{3}{2} y - \frac{3}{2} x^2 = -3uv,
\]
\[
\frac{\partial z}{\partial y} = \frac{3}{2} x = \frac{3}{2} (u+v).
\]

但一般说来，用参数表示的函数和消去参数后的函数，它们的定义域是不同的。

3408. 设 \(x = \cos \phi \cos \psi, y = \cos \phi \sin \psi, z = \sin \phi\)，求 \(\frac{\partial^2 z}{\partial x^2}\)。

解 由 \(x = \cos \phi \cos \psi, y = \cos \phi \sin \psi\) 对 \(x\) 求偏导，得
\[
\begin{align*}
1 &= -\sin \varphi \cos \psi \frac{\partial \varphi}{\partial x} - \cos \varphi \sin \psi \frac{\partial \psi}{\partial x}, \\
0 &= -\sin \varphi \sin \psi \frac{\partial \varphi}{\partial x} + \cos \varphi \cos \psi \frac{\partial \psi}{\partial x}.
\end{align*}
\]

联立求解，得
\[
\frac{\partial \varphi}{\partial x} = -\frac{\cos \varphi}{\sin \varphi}, \quad \frac{\partial \psi}{\partial x} = -\frac{\sin \psi}{\cos \varphi}.
\]

于是，
\[
\frac{\partial z}{\partial x} = \cos \varphi \frac{\partial \varphi}{\partial x} = -\cot \varphi \cos \psi,
\]
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial \varphi} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial \psi} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial \psi}{\partial x}
\]
\[
= -\frac{\cos \psi}{\sin^2 \varphi} \left( -\frac{\cos \psi}{\sin \varphi} \right) + \cot \varphi \sin \psi \left( -\frac{\sin \psi}{\cos \varphi} \right)
\]
\[
= -\frac{\cos^2 \psi + \sin^2 \psi \sin^2 \varphi}{\sin^3 \varphi} = -\frac{\sin^2 \varphi + \cos^2 \varphi \cos^2 \psi}{\sin^3 \varphi}.
\]

注 本题也可消去 \( \varphi \), \( \psi \) 求 \( \frac{\partial^2 z}{\partial x^2} \). 事实上，
\[
x^2 + y^2 + z^2 = \cos^2 \varphi \cos^2 \psi + \cos^2 \varphi \sin^2 \psi + \sin^2 \varphi
\]
\[
= \cos^2 \varphi + \sin^2 \varphi = 1.
\]

于是，
\[
2x + 2z \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = -\frac{x}{z},
\]
\[
\frac{\partial^2 z}{\partial x^2} = -\frac{z - x \frac{\partial z}{\partial x}}{z^2} = -\frac{z^2 + x^2}{z^3}
\]

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\[
\frac{\sin^2 \varphi + \cos^2 \varphi \cos^2 \varphi}{\sin^2 \varphi}.
\]

3409. 设 \( x = u \cos v, y = u \sin v, z = v \), 求 \( \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \) 及 \( \frac{\partial^2 z}{\partial y^2} \).

解 本题求微分，可将所有的二阶偏导函数一起求出。

\[
dx = \cos v du - u \sin v dv, \\
dy = \sin v du + u \cos v dv.
\]

联立求解，得

\[
du = \cos v dx + \sin v dy, \\
dv = \frac{1}{u} (-\sin v dx + \cos v dy),
\]

\[
u dv = -\sin v dx + \cos v dy.
\]

再对上式微分一次，得

\[
u d^2 v + dudv = -\cos v du dx - \sin v du dy
\]

\[
= -dudv,
\]

于是，

\[
d^2 z = d^2 v = -\frac{2}{u} dudv = -\frac{2}{u^2} (\cos v dx + \sin v dy) \\
\cdot (-\sin v dx + \cos v dy)
\]

\[
= \frac{2}{u^2} (\sin v \cos v dx^2 - \cos v du dx y - \sin v \cos v du y^2),
\]

从而有

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\[
\frac{\partial^2 z}{\partial x^2} = \frac{2\sin v \cos v}{u^2} = \frac{\sin 2v}{u^2},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = -\frac{\cos 2v}{u^2}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{\sin 2v}{u^2}.
\]

注：本题也可消去u, v, 由z = v = \arctg \frac{y}{x} 获解。

3410. 设 \( z = z(x, y) \) 为由方程组:

\[
x = e^{u + v}, \quad y = e^{u - v}, \quad z = uv
\]

（u 及 v 为参数）所定义的函数，求当 \( u = 0 \) 及 \( v = 0 \) 时的 \( dz \) 及 \( d^2 z \).

解

\[
dx \bigg|_{v=0} = e^{u+v}(du+dv) \bigg|_{v=0} = du + dv,
\]

\[
dy \bigg|_{v=0} = e^{u-v}(du-dv) \bigg|_{v=0} = du - dv.
\]

于是，当 \( u = 0 \) 及 \( v = 0 \) 时，

\[
du = \frac{1}{2} (dx + dy), \quad dv = \frac{1}{2} (dx - dy);
\]

\[
dz = udv + vdu = 0;
\]

\[
d^2z = ud^2v + 2udu + vdu = 2udv
\]

\[
= 2 \left( \frac{dx + dy}{2} \right) \left( \frac{dx - dy}{2} \right) = \frac{1}{2} (dx^2 - dy^2).
\]

3411. 设 \( z = x^2 + y^2 \)，其中 \( y = y(x) \) 为由方程 \( x^2 - xy + y^2 \) = 1 所定义的函数，求 \( \frac{dz}{dx} \) 及 \( \frac{d^2 z}{dx^2} \).

解

先由 \( x^2 - xy + y^2 = 1 \) 求 \( \frac{dy}{dx} \) 及 \( \frac{d^2 y}{dx^2} \)。
\[ \begin{align*}
2x - y - xy' + 2yy' &= 0, \\
2 - 2y' - xy'' + 2y' + 2yy'' &= 0. 
\end{align*} \]

于是，

\[ y' = \frac{2x - y}{x - 2y}, \quad y'' = \frac{6(x^2 - xy + y^2)}{(x - 2y)^3} = \frac{6}{(x - 2y)^2}. \]

下面求 \( \frac{dz}{dx} \) 及 \( \frac{d^2 z}{dx^2} \).

\[ \begin{align*}
\frac{dz}{dx} &= 2x + 2yy' = 2x + 2y, \\
\frac{2x - y}{x - 2y} &= \frac{2(x^2 - y^2)}{x - 2y}, \\
\frac{d^2 z}{dx^2} &= 2 + 2y' + 2y''y = 2y' + xy'' \\
&= \frac{2(2x - y)}{x - 2y} + \frac{6x}{(x - 2y)^3}. 
\end{align*} \]

3412. 设 \( u = \frac{x + z}{y + z} \)，其中 \( z \) 为由方程 \( ze^z = xe^x + ye^y \) 所

定义的函数，求 \( \frac{du}{dx} \) 及 \( \frac{du}{dy} \).

解 将 \( ze^z = xe^x + ye^y \) 两端微分，得

\[ e^z(1 + z)dz = e^x(1 + x)dx + e^y(1 + y)dy. \]

又因

\[ du = \frac{1}{(y + z)^2}[(y + z)dx + (y + z)dy - (z + x)dy - (z + x)dx] \\
= \frac{1}{(y + z)^2}[(y + z)dx - (x + z)dy - (y - x)dz] \]

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\[
\frac{1}{(y+z)^2}[(y+z)dx - (x+z)dy
\]
\[
+ \frac{(y-x)e^{y(1+x)}}{e^z(1+z)} dx + \frac{(y-x)e^{y(1+y)}}{e^z(1+z)} dy],
\]

故得

\[
\frac{\partial u}{\partial x} = \frac{1}{y+z} + \frac{(x+1)(y-x)}{(z+1)(y+z)^2} e^{x-y},
\]

\[
\frac{\partial u}{\partial y} = -\frac{x+z}{(y+z)^2} + \frac{(y+1)(y-x)}{(z+1)(y+z)^2} e^{y-x}.
\]

3413. 设方程:

\[x = \phi(u,v), \quad y = \psi(u,v), \quad z = \chi(u,v)\]

定义 \(z\) 为 \(x\) 和 \(y\) 的函数。求 \(\frac{\partial z}{\partial x}\) 和 \(\frac{\partial z}{\partial y}\)。

解 对 \(x\) 求偏导函数，得

\[1 = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x}, \quad (1)\]

\[0 = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v} \frac{\partial v}{\partial x}, \quad (2)\]

\[\frac{\partial z}{\partial x} = \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x}. \quad (3)\]

由 (1) 及 (2) 解得

\[\frac{\partial u}{\partial x} = \frac{1}{l} \frac{\partial \psi}{\partial v}, \quad \frac{\partial v}{\partial x} = -\frac{1}{l} \frac{\partial \psi}{\partial u}, \quad (4)\]

其中
\[ I = \begin{vmatrix} \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \\ \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \\ \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \end{vmatrix} = \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \frac{\partial \varphi}{\partial v}. \]

再将（4）的结果代入（3），即得

\[ \frac{\partial z}{\partial x} = -\frac{1}{I} \left( \frac{\partial \psi}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial \psi}{\partial v} \frac{\partial x}{\partial u} \right), \]

同法求得

\[ \frac{\partial z}{\partial y} = -\frac{1}{I} \left( \frac{\partial \varphi}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial \varphi}{\partial v} \frac{\partial x}{\partial u} \right). \]

3414. 设：

\[ x = \varphi(u, v), \quad y = \psi(u, v). \]

求反函数：\( u = u(x, y) \) 和 \( v = v(x, y) \) 的一阶和二阶偏导函数。

解 微分二次，得

\[ dx = \varphi'_1 du + \varphi'_2 dv, \]

\[ dy = \psi'_1 du + \psi'_2 dv, \]

\[ 0 = \varphi''_{11} du^2 + 2 \varphi''_{12} dudv + \varphi''_{22} dv^2 + \varphi'_1 d^2 u + \varphi'_2 d^2 v, \]

\[ 0 = \psi''_{11} du^2 + 2 \psi''_{12} dudv + \psi''_{22} dv^2 + \psi'_1 d^2 u + \psi'_2 d^2 v. \]

其中下标号 1，2 分别代表对 \( u, v \) 的偏导函数，余类推。

令 \( I = \varphi \psi'_2 - \varphi'_2 \psi'_1 \)，则由（1），（2）可解得
\[ du = \frac{1}{I} (\psi_2^t dx - \psi_2^t dy), \]  

\[ dv = \frac{1}{I} (\psi_1^t dy - \psi_1^t dx). \]  

于是，
\[ \frac{\partial u}{\partial x} = \frac{1}{I} \psi_2^t = \frac{1}{I} \frac{\partial \psi}{\partial v}, \quad \frac{\partial u}{\partial y} = -\frac{1}{I} \frac{\partial \varphi}{\partial v}, \]
\[ \frac{\partial v}{\partial x} = -\frac{1}{I} \frac{\partial \psi}{\partial u}, \quad \frac{\partial v}{\partial y} = \frac{1}{I} \frac{\partial \varphi}{\partial u}. \]

由 (3), (4) 解出 \( d^2 u, d^2 v \), 并把 (5), (6) 的结果代入，即得
\[ d^2 u = \frac{1}{I} \left[ \psi_2^t (\psi_1^t d u^2 + 2 \psi_1^{t^2} du dv + \psi_2^{t^2} dv^2) ight. \]
\[ \left. - \psi_2^t (\psi_1^{t^2} d u^2 + 2 \psi_1^{t^2} du dv + \psi_2^{t^2} dv^2) \right] \]
\[ = \frac{1}{I^3} \left[ (\psi_2^t \psi_1^{t^2} - \psi_2 \psi_1^{t^2}) (\psi_2^t dx - \psi_2^t dy)^2 \right. \]
\[ + 2 (\psi_2^t \psi_1^{t^2} - \psi_2 \psi_1^{t^2}) (\psi_2^t dx - \psi_2^t dy)(\psi_1^t dy - \psi_1^t dx) \]
\[ + (\psi_2^t \psi_1^{t^2} - \psi_2 \psi_1^{t^2}) (\psi_1^t dy - \psi_1^t dx)^2 \]  
\[ = \frac{\partial^2 u}{\partial x^2} d x^2 + 2 \frac{\partial^2 u}{\partial x \partial y} d x d y + \frac{\partial^2 u}{\partial y^2} d y^2. \]

比较上式两端的系数，即得
\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{I^3} \left[ \left( \frac{\partial \varphi}{\partial v} \frac{\partial^2 \psi}{\partial u^2} - \frac{\partial \psi}{\partial v} \frac{\partial^2 \varphi}{\partial u^2} \right) \right. \]
\[ \left. \cdot \left( \frac{\partial \psi}{\partial v} \right)^2 - 2 \left( \frac{\partial \varphi}{\partial v} \frac{\partial^2 \psi}{\partial u \partial v} - \frac{\partial \psi}{\partial v} \frac{\partial^2 \varphi}{\partial u \partial v} \right) \right] \]
\begin{align*}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{1}{I^2} \left[ \left( \frac{\partial \psi}{\partial v} \frac{\partial^2 \psi}{\partial v^2} - \frac{\partial \psi}{\partial u} \frac{\partial^2 \psi}{\partial u^2} \right) \frac{\partial \varphi}{\partial u} \frac{\partial \varphi}{\partial u} \right] \\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{I^2} \left[ \left( \frac{\partial \varphi}{\partial u} \frac{\partial^2 \varphi}{\partial u^2} - \frac{\partial \varphi}{\partial v} \frac{\partial^2 \varphi}{\partial v^2} \right) \frac{\partial \varphi}{\partial v} \frac{\partial \varphi}{\partial v} \right] \\
\text{同法可求得} \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}.
\end{align*}

3415. 设 (a) \( x = u \cos \frac{v}{u} \), \( y = u \sin \frac{v}{u} \);

(b) \( x = e^u + u \sin v \), \( y = e^u - u \cos v \),

求 \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \).

解  利用 3414 题的结果求之。
(a) \( \varphi(u,v) = u \cos \frac{v}{u}, \quad \psi(u,v) = u \sin \frac{v}{u} \). 于是，

\[
\frac{\partial \varphi}{\partial u} = \cos \frac{v}{u} + \frac{v}{u} \sin \frac{v}{u}, \quad \frac{\partial \varphi}{\partial v} = -\sin \frac{v}{u},
\]

\[
\frac{\partial \psi}{\partial u} = \sin \frac{v}{u} - \frac{v}{u} \cos \frac{v}{u}, \quad \frac{\partial \psi}{\partial v} = \cos \frac{v}{u},
\]

\[
I = \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial v} \frac{\partial \psi}{\partial u} = \left( \cos \frac{v}{u} \right) + \frac{v}{u} \sin \frac{v}{u} \cos \frac{v}{u} - \left( -\sin \frac{v}{u} \right)
\]

\[
\cdot \left( \sin \frac{v}{u} - \frac{v}{u} \cos \frac{v}{u} \right) = 1.
\]

从而得

\[
\frac{\partial u}{\partial x} = \frac{1}{I} \frac{\partial \psi}{\partial v} = \cos \frac{v}{u}, \quad \frac{\partial u}{\partial y} = -\frac{1}{I} \frac{\partial \varphi}{\partial v} = \sin \frac{v}{u},
\]

\[
\frac{\partial v}{\partial x} = -\frac{1}{I} \frac{\partial \varphi}{\partial u} = \frac{v}{u} \cos \frac{v}{u} - \sin \frac{v}{u},
\]

\[
\frac{\partial v}{\partial y} = \frac{1}{I} \frac{\partial \varphi}{\partial u} = \frac{v}{u} \sin \frac{v}{u} + \cos \frac{v}{u}.
\]

(6) \( \varphi(u,v) = e^x + u \sin v, \quad \psi(u,v) = e^x - u \cos v. \)

于是，

\[
\frac{\partial \varphi}{\partial u} = e^x + \sin v, \quad \frac{\partial \varphi}{\partial v} = u \cos v,
\]

\[
\frac{\partial \psi}{\partial u} = e^x - \cos v, \quad \frac{\partial \psi}{\partial v} = u \sin v,
\]

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I = (e^x + \sin u)u \sin u - (e^x - \cos u)u \cos u
= u(e^x(\sin u - \cos u) + 1).

从而得

\[
\frac{\partial u}{\partial x} = \frac{\sin u}{e^x(\sin u - \cos u) + 1},
\]

\[
\frac{\partial u}{\partial y} = -\frac{\cos u}{e^x(\sin u - \cos u) + 1},
\]

\[
\frac{\partial v}{\partial x} = -\frac{e^x - \cos u}{u(e^x(\sin u - \cos u) + 1)},
\]

\[
\frac{\partial v}{\partial y} = \frac{e^u + \sin u}{u(e^x(\sin u - \cos u) + 1)}.
\]

3476. 函数 \( u = u(x) \) 由方程组

\[ u = f(x, y, z), \quad g(x, y, z) = 0, \]

\[ h(x, y, z) = 0 \]

定义，求 \( \frac{du}{dx} \) 和 \( \frac{d^2 u}{dx^2} \).

解 微分得

\[
du = f_x \, dx + f_y \, dy + f_z \, dz = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} +
\right.
\]

\[
+ dz \frac{\partial}{\partial z} \right) f,
\]

\[ 0 = g_x \, dx + g_y \, dy + g_z \, dz = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} +
\right.
\]

\[
+ dz \frac{\partial}{\partial z} \right) g,
\]

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\[ 0 = h'_x dx + h'_y dy + h'_z dz = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) h. \]  \hspace{1cm} (3)

令 \( \frac{\partial (g, h)}{\partial (y, z)} = I_1, \frac{\partial (g, h)}{\partial (z, x)} = I_2, \frac{\partial (g, h)}{\partial (x, y)} = I_3 \), 则

由 (2), (3) 可解得

\[ dy = \frac{I_2}{I_1} dx, \quad dz = \frac{I_3}{I_1} dx. \]

将 \( dy, dz \) 代入 (1), 得

\[
du = f'_x dx + f'_y \frac{I_2}{I_1} dx + f'_z \frac{I_3}{I_1} dx = \frac{1}{I_1} \left( I_1 f'_x + I_2 f'_y + I_3 f'_z \right) dx = \frac{I}{I_1} dx,
\]

其中 \( I = \frac{D(f, g, h)}{D(x, y, z)} \). 于是,

\[ \frac{du}{dx} = \frac{I}{I_1}. \]

对 (1), (2), (3) 式再求一次微分，得

\[
d^2 u = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 f + f'_x dx^2 y
\]

\[ + f'_y dx^2 z, \]  \hspace{1cm} (4)

\[
0 = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 g + g'_x dx^2 y
\]

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\begin{align*}
+ g'_z d^2 z, \quad & \text{(5)} \\
0 = & \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 h + h'_z d^2 y \\
& + h'_z d^2 z. \quad & \text{(6)}
\end{align*}

于是，

\begin{align*}
d^2 y = & \frac{1}{I_1} \left[ g'_x \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 h \\
& - h'_z \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 g \right],
\end{align*}

\begin{align*}
d^2 z = & \frac{1}{I_1} \left[ h'_z \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 g - e'_z \\
& \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 h \right].
\end{align*}

令 \( \frac{\partial (h, f)}{\partial (y, z)} = I_4, \quad \frac{\partial (f, g)}{\partial (y, z)} = I_5 \)，并将 \( d^2 y \) 及 \( d^2 z \) 代入(4)，即得

\begin{align*}
d^2 u = & \frac{1}{I_1} \left[ I_1 \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 f \\
& + I_4 \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 g \\
& + I_5 \cdot \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^2 h \right],
\end{align*}

再以 \( dy = \frac{I_2}{I_1} dx \) 及 \( dz = \frac{I_3}{I_1} dx \) 代入上式，即得
\[
\frac{d^2 u}{dx^2} = \frac{1}{I_1^2} \left[ I_1 \left( I_1 \frac{\partial}{\partial x} + I_2 \frac{\partial}{\partial y} + I_3 \frac{\partial}{\partial z} \right)^2 f + I_4 \left( I_1 \frac{\partial}{\partial x} + I_2 \frac{\partial}{\partial y} + I_3 \frac{\partial}{\partial z} \right)^2 g + I_5 \left( I_1 \frac{\partial}{\partial x} + I_2 \frac{\partial}{\partial y} + I_3 \frac{\partial}{\partial z} \right)^2 h \right].
\]

3417. 函数 \( u = u(x, y) \) 由方程组
\[
u(x, y, z, t), \quad g(y, z, t) = 0, \quad h(z, t) = 0
\]
定义，求 \( \frac{\partial u}{\partial x} \) 和 \( \frac{\partial u}{\partial y} \).

解 微分得
\[
du = f' x dx + f' y dy + f' z dz + f' t dt, \quad (1)
\]
\[
0 = g' y dy + g' z dz + g' t dt, \quad (2)
\]
\[
0 = h' z dz + h' t dt. \quad (3)
\]
令 \( I_1 = \frac{\partial (g', h')}{\partial (z, t)} \)，则由(2)，(3)可解得
\[
dz = \frac{1}{I_1} \cdot (-g' h'') dy, \quad dt = \frac{1}{I_1} \cdot (g' h') dy.
\]
将 \( dz \) 及 \( dt \) 代入(1)式，得
\[
du = f' x dx + f' y dy - \frac{g'}{I_1} \cdot (f' h' - f' h'') dy.
\]
于是，
\[
\frac{\partial u}{\partial x} = f' x, \quad \frac{\partial u}{\partial y} = f' y + g' y, \quad \frac{I_2}{I_1}.
\]

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其中 $I_2 = \frac{\partial (h, f)}{\partial (z, t)}$.

3418. 设:

$$x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w).$$

求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 和 $\frac{\partial u}{\partial z}$.

解  微分得

$$dx = f'_u du + f'_v dv + f'_w dw,$$
$$dy = g'_u du + g'_v dv + g'_w dw,$$
$$dz = h'_u du + h'_v dv + h'_w dw.$$

令 $I = \frac{D(f, g, h)}{D(u, v, w)}$，则有

$$du = \frac{1}{I} \begin{vmatrix} dx & f'_u & f'_w \\ dy & g'_u & g'_w \\ dz & h'_u & h'_w \end{vmatrix} = \frac{I_1}{I} dx + \frac{I_2}{I} dy + \frac{I_3}{I} dz,$$

其中 $I_1 = \frac{\partial (g, h)}{\partial (v, w)}, \quad I_2 = \frac{\partial (h, f)}{\partial (v, w)}, \quad I_3 = \frac{\partial (f, g)}{\partial (v, w)}$.

于是，

$$\frac{\partial u}{\partial x} = \frac{I_1}{I}, \quad \frac{\partial u}{\partial y} = \frac{I_2}{I}, \quad \frac{\partial u}{\partial z} = \frac{I_3}{I}.$$

3419. 设函数 $z = z(x, y)$ 满足方程组

$$f(x, y, z, t) = 0, \quad g(x, y, z, t) = 0,$$

式中 $t$ 为参数量。求 $dz$.

解  微分得
\[ f'_x dx + f'_y dy + f'_z dz + f'_t dt = 0, \]
\[ g'_x dx + g'_y dy + g'_z dz + g'_t dt = 0. \]

把 \( dz, dt \) 看作未知数，其它为系数。解之得

\[
dz = \frac{1}{I_3} \left[ (f'_x g'_x + g'_y f'_x) dx + (f'_x g'_y - g'_x f'_y) dy \right]
= \frac{1}{I_3} \left[ (f'_x g'_x - g'_x f'_y) dx + (f'_x g'_y - g'_x f'_y) dy \right]
= - \frac{I_1 dx + I_2 dy}{I_3},
\]

其中 \( I_1 = \frac{\partial (f, g)}{\partial (x, t)} \), \( I_2 = \frac{\partial (f, g)}{\partial (y, t)} \), \( I_3 = \frac{\partial (f, g)}{\partial (z, t)} \).

3420. 设 \( u = f(z) \)，其中 \( z \) 由方程式 \( z = x + y \varphi(z) \) 所定义的为变数 \( x \) 和 \( y \) 的隐函数。证明拉格朗日公式

\[
\frac{\partial^n u}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}} \left\{ [\varphi(z)]^n \frac{\partial u}{\partial x} \right\}.
\]

证

\[ dz = dx + \varphi(z) dy + y \varphi'(z) dz. \]

于是,

\[
\frac{\partial z}{\partial x} = \frac{1}{1 - y \varphi'(z)},
\]

\[
\frac{\partial z}{\partial y} = \frac{\varphi(z)}{1 - y \varphi'(z)} = \varphi(z) \frac{\partial z}{\partial x}.
\]

从而得

\[
\frac{\partial u}{\partial y} = f'(z) \frac{\partial z}{\partial y} = f'(z) \varphi(z) \frac{\partial z}{\partial x} = \varphi(z) \frac{\partial u}{\partial x},
\]

即当 \( n = 1 \) 时，拉格朗日公式为真.
对于任意可微函数 \( g(z) \)，有

\[
\frac{\partial}{\partial y} \left[ g(z) \frac{\partial u}{\partial x} \right] = g'(z) \frac{\partial z}{\partial y} \frac{\partial u}{\partial x} + g(z) \frac{\partial^2 u}{\partial x \partial y}
\]

\[
= \varphi(z) g'(z) \frac{\partial z}{\partial x} \frac{\partial u}{\partial x} + g(z) \frac{\partial}{\partial x} \left[ \varphi(z) \frac{\partial u}{\partial x} \right]
\]

\[
= \varphi(z) g'(z) \frac{\partial z}{\partial x} \frac{\partial u}{\partial x} + \varphi'(z) g(z) \frac{\partial z}{\partial x} \frac{\partial u}{\partial x} + \varphi(z) g(z) \frac{\partial z}{\partial x} \frac{\partial^2 u}{\partial x^2}
\]

\[
= \frac{\partial}{\partial x} \left[ \varphi(z) g(z) \frac{\partial u}{\partial x} \right].
\]

令 \( g(z) = \varphi(z) \)，得

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \varphi(z) \frac{\partial u}{\partial x} \right]
\]

\[
= \frac{\partial}{\partial x} \left[ \varphi^2(z) \frac{\partial u}{\partial x} \right],
\]

即当 \( n = 2 \) 时，拉格朗日公式也为真。设当 \( n = k \) 时，公式为真，即有

\[
\frac{\partial^{k+1} u}{\partial y^{k+1}} = \frac{\partial^k}{\partial x^{k+1}} \left[ \varphi^k(z) \frac{\partial u}{\partial x} \right].
\]

于是，

\[
\frac{\partial^{k+1} u}{\partial y^{k+1}} = \frac{\partial}{\partial y} \left\{ \frac{\partial^{k-1}}{\partial x^{k-1}} \left[ \varphi^{k-1}(z) \frac{\partial u}{\partial x} \right] \right\}
\]

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\[
\frac{\partial^{t-1}}{\partial x^{t-1}} \left\{ \frac{\partial}{\partial y} \left[ \varphi^t(z) \frac{\partial u}{\partial x} \right] \right\} = \\
= -\frac{\partial^{t-1}}{\partial x^{t-1}} \left\{ \frac{\partial}{\partial x} \left[ \varphi^{t-1}(z) \frac{\partial u}{\partial x} \right] \right\} = \\
= \frac{\partial^t}{\partial x^t} \left[ \varphi^{t+1}(z) \frac{\partial u}{\partial x} \right].
\]

即当 \( n = k + 1 \) 时，拉格朗日公式也为真。于是，对于一切自然数 \( n \)，均有

\[
\frac{\partial^n u}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}} \left[ \varphi^n(z) \frac{\partial u}{\partial x} \right].
\]

3421. 证明：由方程

\[
\Phi(x-az, y-bz) = 0 \quad (1)
\]

[其中 \( \Phi(u, v) \) 是变量 \( u, v \) 的任意可微分函数，\( a \) 和 \( b \)为常数] 所定义的函数 \( z = z(x, y) \) 为方程

\[
a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 1
\]

的解。说明曲面(1)的几何性质。

解 由于

\[
\Phi' \cdot (1 - a \frac{\partial z}{\partial x}) - b \Phi' \cdot \frac{\partial z}{\partial x} = 0,
\]

\[
-\Phi' \cdot a \frac{\partial z}{\partial y} + \Phi' \cdot (1 - b \frac{\partial z}{\partial y}) = 0,
\]

故有

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\[
\frac{\partial z}{\partial x} = \frac{\Phi'_1}{a\Phi'_1 + b\Phi'_2}, \quad \frac{\partial z}{\partial y} = \frac{\Phi'_2}{a\Phi'_1 + b\Phi'_2}.
\]

将上面二个等式依次乘以 \(a, b\), 然后相加, 即得

\[
a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 1,
\]

这就说明 \(z = z(x, y)\) 为方程 \(a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 1\) 的解.

等式 \(a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 1\) 表示曲面 (1) 上任一点 \(P_1(x_1, y_1, z_1)\) 的法向量 

\[
\mathbf{n}_1 = \left\{ \left. \frac{\partial z}{\partial x} \right|_{P_1}, \frac{\partial z}{\partial y} \right|_{P_1} \right\} - 1
\]

皆与向量 \(r_1 = \{a, b, 1\}\) 垂直. 过点 \(P_1\) 作平行于 

\(r_1\) 的直线 \(l_1\):

\[
\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{1}.
\]

易知 \(l_1\) 上的点皆在曲面 (1) 上. 于是, 曲面 (1) 是母线平行于 \(r_1\) 的柱面.

3422. 证明：由方程

\[
\Phi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right) = 0
\]

（其中 \(\Phi(u, v)\) 是变数 \(u\) 和 \(v\) 的任意可微分函数）所定义的函数 \(z = z(x, y)\) 满足方程式

\[
(x-x_0) \frac{\partial z}{\partial x} + (y-y_0) \frac{\partial z}{\partial y} = z-z_0.
\]
说明曲面（2）的几何性质。

解 由于

$$
\Phi_1 \cdot \frac{z-z_0-(x-x_0) \frac{\partial z}{\partial x}}{(z-z_0)^2} - \Phi_2 \cdot \frac{(y-y_0) \frac{\partial z}{\partial x}}{(z-z_0)^2} = 0,
$$

$$
\Phi_1 \cdot \frac{(x-x_0) \frac{\partial z}{\partial y}}{(z-z_0)^2} + \Phi_2 \cdot \frac{z-z_0-(y-y_0) \frac{\partial z}{\partial y}}{(z-z_0)^2} = 0,
$$

故有

$$
\frac{\partial z}{\partial x} = \frac{(z-z_0) \Phi_1}{(x-x_0) \Phi_1 + (y-y_0) \Phi_2},
$$

$$
\frac{\partial z}{\partial y} = \frac{(z-z_0) \Phi_2}{(x-x_0) \Phi_1 + (y-y_0) \Phi_2}.
$$

将上面二个等式依次乘以 $x-x_0$ 及 $y-y_0$，然后相加，即得

$$
(x-x_0) \frac{\partial z}{\partial x} + (y-y_0) \frac{\partial z}{\partial y} = z-z_0,
$$

本题获证。

等式 $(x-x_0) \frac{\partial z}{\partial x} + (y-y_0) \frac{\partial z}{\partial y} - (z-z_0) = 0$ 表示曲面 (2) 在其上任一点 $P_2 (x_2, y_2, z_2)$ 的法向量

$$
\vec{n}_2 = \left\{ \frac{\partial z}{\partial x} \bigg|_{P_2}, \frac{\partial z}{\partial y} \bigg|_{P_2}, -1 \right\}
$$

与向量 $\vec{r}_2 = \{x_2-x_0, y_2-y_0, z_2-z_0\}$ 垂直。作过点 $P_0 (x_0, y_0, z_0), P_2 (x_2, y_2, z_2)$ 的直线 $l_2$. 

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\[ \frac{x-x_0}{x_2-x_0} = \frac{y-y_0}{y_2-y_0} = \frac{z-z_0}{z_2-z_0}. \]

易知点在任一点皆在曲面(2)上。于是，曲面(2)是顶点在 \( P_0 \) 的锥面。

3423. 证明：由方程

\[ ax + by + cz = \Phi(x^2 + y^2 + z^2) \quad (3) \]

（其中 \( \Phi(u) \) 是变数 \( u \) 的任意可微分函数， \( a, b \) 和 \( c \) 为常数）所定义的函数 \( z = z(x, y) \) 满足方程

\[ (cy-bz) \frac{\partial z}{\partial x} + (az-cx) \frac{\partial z}{\partial y} = bx - ay. \]

说明曲面（3）的几何性质。

解 由于

\[ a + c \frac{\partial z}{\partial x} = \Phi' \cdot \left( 2x + 2z \frac{\partial z}{\partial x} \right), \]

\[ b + c \frac{\partial z}{\partial y} = \Phi' \cdot \left( 2y + 2z \frac{\partial z}{\partial y} \right), \]

故有

\[ \frac{\partial z}{\partial x} = \frac{2x\Phi' - a}{c - 2z\Phi'}, \quad \frac{\partial z}{\partial y} = \frac{2y\Phi' - b}{c - 2z\Phi'}. \]

将上面二个等式依次乘以 \( (cy-bz) \) 及 \( (az-cx) \)，然后相加，即得

\[ (cy-bz) \frac{\partial z}{\partial x} + (az-cx) \frac{\partial z}{\partial y} = \frac{(2x\Phi' - a)(cy-bz) + (2y\Phi' - b)(az-cx)}{c - 2z\Phi'}. \]
\[ -\frac{(c-2z\Phi')(bx-ay)}{c-2z\Phi'} = bx-ay, \]

本题证
设 \( P_3(x_3, y_3, z_3) \) 是曲面 (3) 上任意一点，并记 \( \vec{r}_3 = \{a, b, c\} \)。由于曲面 (3) 在 \( P_3 \) 点的法向量为
\[ \vec{n}_3 = \left\{ \frac{\partial z}{\partial x} \bigg|_{P_3}, \frac{\partial z}{\partial y} \bigg|_{P_3}, -1 \right\}, \]
故由方程
\[ (c, y-bz) \frac{\partial z}{\partial x} + (az-cx) \frac{\partial z}{\partial y} - (bx-ay) = 0 \]
知
\[ \vec{n}_3 \perp (\vec{P}_3 \times \vec{r}_3), \]
其中 \( \vec{P}_3 = \{x_3, y_3, z_3\} \).

设由原点到 \( P_3 \) 的距离为 \( d \)，即
\[ x_3^2 + y_3^2 + z_3^2 = d^2. \]
考虑平面
\[ \Pi: \quad ax + by + cz = d \]
和过点 \( P_3 \) 的球面
\[ S: \quad x^2 + y^2 + z^2 = d^2, \]
并设平面 \( \Pi \) 与球面 \( S \) 的交线为 \( C \)，则
1° 由点 \( P_3 \) 在曲面 (3) 上可知
\[ ax_3 + by_3 + cz_3 = \Phi(x_3^2 + y_3^2 + z_3^2), \]
即
\[ d = \Phi(d^2). \]

这表明曲线 \( C \) 上的点的坐标皆满足方程 (3)，即曲线 \( C \) 位于曲面 (3) 上。
2°由Ⅱ为平面，S为球面知交线C为一圆周曲线。

3°圆C的圆心Q即为由原点到平面Ⅱ的垂足，故Q点位于过原点且与平面Ⅱ垂直的直线l上。

综上所述，可见曲面（3）是以直线

\[ l: \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \]

为旋转轴的旋转曲面。

3424. 函数 \( z = z(x, y) \) 由方程

\[ x^2 + y^2 + z^2 = yf\left( \frac{x}{y} \right) \]

所给出，证明：

\[ (x^2 - y^2 - z^2) \frac{\partial z}{\partial x} + 2xy \frac{\partial z}{\partial y} = 2xz. \]

证 由于

\[ 2x + 2z \frac{\partial z}{\partial x} = f'(\frac{z}{y}) \frac{\partial z}{\partial x}, \]

故有

\[ \frac{\partial z}{\partial x} = \frac{2x}{f'(\frac{z}{y}) - 2z}. \]

同法可求得

\[ \frac{\partial z}{\partial y} = \frac{x^2 - y^2 + z^2 - z f'(\frac{z}{y})}{2y z - y f'(\frac{z}{y})}. \]

于是，

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\[(x^2 - y^2 - z^2) \frac{\partial z}{\partial x} + 2xy \frac{\partial z}{\partial y} = \frac{2xy(y^2 + z^2 - x^2) + 2xy(x^2 - y^2 + z^2 - zf')}{y(2z - f')} = \frac{2xyz(2z - f')}{y(2z - f')} = 2xz.\]

本题获证。

3425. 函数 \(z = z(x, y)\) 由方程

\[F'(x + z y^{-1}, y + z x^{-1}) = 0\]

所给出，证明；

\[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z - xy.\]

证 由于

\[F'_1 \cdot \left(1 + \frac{1}{y} \frac{\partial z}{\partial x}\right) + F'_2 \cdot \left(\frac{x \frac{\partial z}{\partial x} - z}{x^2}\right) = 0,\]

\[F'_1 \cdot \left(\frac{y \frac{\partial z}{\partial y} - z}{y^2}\right) + F'_2 \cdot \left(1 + \frac{1}{x} \frac{\partial z}{\partial y}\right) = 0,\]

故有

\[\frac{\partial z}{\partial x} = \frac{y z F'_2 - x^2 y F'_1}{x(x F'_1 + y F'_2)} \quad \frac{\partial z}{\partial y} = \frac{x z F'_1 - x y^2 F'_2}{y(x F'_1 + y F'_2)}.\]

于是，

\[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{y z F'_2 - x^2 y F'_1 + x z F'_1 - x y^2 F'_2}{x F'_1 + y F'_2}.\]
(z - xy)(xF'_1 + yF'_2) = z - xy,

本题证。

3426．证明：由方程组
\[
\begin{align*}
xcos\alpha + ysin\alpha + lnz &= f(\alpha), \\
- xsin\alpha + ycos\alpha &= f'(\alpha)
\end{align*}
\]
(其中 \(\alpha = \alpha(x, y)\) 为参数及 \(f(\alpha)\) 为任意可微分的函数) 所定义的函数 \(z = z(x, y)\) 满足方程式
\[
\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2.
\]

证 由 \(xcos\alpha + ysin\alpha + lnz = f(\alpha)\) 两端对 \(x\) 求偏导函数，得
\[
\begin{align*}
cos\alpha - xsin\alpha \frac{\partial \alpha}{\partial x} + ycos\alpha \frac{\partial \alpha}{\partial x} + \frac{1}{z} \frac{\partial z}{\partial x}
\end{align*}
\]
\[
= f'(\alpha) \frac{\partial \alpha}{\partial x}.
\]
由于 \(- xsin\alpha + ycos\alpha = f'(\alpha)\)，代入上式，即得
\[
cos\alpha + \frac{1}{z} \frac{\partial z}{\partial x} = 0 \text{ 或 } \frac{\partial z}{\partial x} = -zcos\alpha. \quad (1)
\]

同法可求得
\[
\frac{\partial z}{\partial y} = -zsin\alpha. \quad (2)
\]

将 \((1)\)，\((2)\) 两式依次平方，然后相加，即得
\[
\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2,
\]

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本题求证。

3427. 证明：由方程组

\[
\begin{align*}
    z &= \alpha x + \frac{y}{\alpha} + f(\alpha), \\
    0 &= x - \frac{y}{\alpha^2} + f'(\alpha)
\end{align*}
\]

所给出的函数 \( z = z(x, y) \) 满足方程

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 1.
\]

证 由于

\[
\begin{align*}
    dz &= \alpha dx + \frac{1}{\alpha} dy + \left[ x - \frac{y}{\alpha^2} + f'(\alpha) \right] d\alpha \\
    &= \alpha dx + \frac{1}{\alpha} dy,
\end{align*}
\]

故有

\[
\frac{\partial z}{\partial x} = \alpha, \quad \frac{\partial z}{\partial y} = \frac{1}{\alpha}.
\]

于是，

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \alpha \cdot \frac{1}{\alpha} = 1,
\]

本题求证。

3428. 证明：由方程组

\[
\begin{align*}
    [z - f(\alpha)]^2 &= x^2(y^2 - \alpha^2), \\
    [z - f(\alpha)] f'(\alpha) &= \alpha x^2
\end{align*}
\]
所定义的函数 \( z = z(x, y) \) 满足方程

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = xy.
\]

证 \( 2(z-f(\alpha)) [dz-f'(\alpha) d\alpha] = (y^2-\alpha^2) 2xdx + x^2(2ydy - 2\alpha d\alpha) \). 于是，

\[
(z-f(\alpha)) dz = x(y^2-\alpha^2) dx + x^2 ydy
\]

\[
+ \{x^2 - (z-f(\alpha)) f'(\alpha)\} d\alpha
\]

\[
= x(y^2-\alpha^2) dx + x^2 ydy,
\]

从而得

\[
\frac{\partial z}{\partial x} = \frac{x(y^2-\alpha^2)}{z-f(\alpha)}, \quad \frac{\partial z}{\partial y} = \frac{x^2 y}{z-f(\alpha)}.
\]

从而得

\[
\frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \frac{x^2 y(y^2-\alpha^2)}{(z-f(\alpha))^2}
\]

\[
= xy \cdot \frac{x^2 (y^2-\alpha^2)}{(z-f(\alpha))^2} = xy,
\]

本题证成。

3429. 证明：由方程组

\[
\begin{align*}
z &= \alpha x + y\varphi(\alpha) + \psi(\alpha), \\
0 &= x + y\varphi'(\alpha) + \psi'(\alpha)
\end{align*}
\]

所给出的函数 \( z = z(x, y) \) 满足方程

\[
\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0.
\]

证

\[
\frac{\partial z}{\partial x} = a + \frac{x}{\partial x} + y\varphi'(\alpha) \frac{\partial}{\partial x} + \psi'(\alpha) \frac{\partial}{\partial x}
\]

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\[ = a + \left[ x + y \varphi'(a) + \psi'(a) \right] \frac{\partial a}{\partial x} = a, \]

\[ \frac{\partial^2 z}{\partial x^2} = \frac{\partial a}{\partial x}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial a}{\partial y}. \]

又 \[ \frac{\partial z}{\partial y} = x \frac{\partial a}{\partial y} + \varphi(a) + y \varphi'(a) \frac{\partial a}{\partial y} + \psi'(a) \frac{\partial a}{\partial y} = \varphi(a), \]

\[ \frac{\partial^2 z}{\partial y^2} = \varphi'(a) \frac{\partial a}{\partial y}, \quad \frac{\partial^2 z}{\partial y \partial x} = \varphi'(a) \frac{\partial a}{\partial x}. \]

而 \[ \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = \frac{\partial a}{\partial x} \frac{\partial a}{\partial y} \varphi'(a) - \left( \frac{\partial a}{\partial y} \right)^2 \]

\[ = \frac{\partial a}{\partial y} \left[ \varphi'(a) \frac{\partial a}{\partial x} - \frac{\partial a}{\partial y} \right], \]

或由于 \[ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}, \]

故 \[ \frac{\partial a}{\partial y} = \varphi'(a) \frac{\partial a}{\partial x}. \]

于是，

\[ \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0 \]

本题证毕.

*) 此式也可由原方程组第二式两端分别对 \( x \) 和 \( y \) 求偏导函数而获得.

3430. 证明：由方程

\[ y = x \varphi(z) + \psi(z) \]

所定义的隐函数 \( z = z(x, y) \) 满足方程
\[(\frac{\partial^2 z}{\partial y^2})^2 \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x \partial y} + (\frac{\partial z}{\partial x})^2 \frac{\partial^2 z}{\partial y^2} = 0.\]

证 记 \(\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s,\)

\[\frac{\partial^2 z}{\partial y^2} = t.\]

将所给方程两端分别对 \(x\) 和对 \(y\) 逐次求偏导数，得
\[
\begin{align*}
\varphi(z) + [x\varphi'(z) + \psi'(z)]p &= 0, \\
[x\varphi'(z) + \psi'(z)]q &= 1; \\
2\varphi'(z)p + [x\varphi''(z) + \psi''(z)]r^2 + [x\varphi'(z) + \psi'(z)]r &= 0, \\
\varphi'(z)q + [x\varphi''(z) + \psi''(z)]pq + [x\varphi'(z) + \psi'(z)]s &= 0, \\
(x\varphi''(z) + \psi''(z))q^2 + [x\varphi'(z) + \psi'(z)]t &= 0.
\end{align*}
\]
(1), (2), (3) 三式依次乘以 \(q^2\), \((-2pq)\) 及 \(p^2\)，然后相加，并注意到 \(x\varphi'(z) + \psi'(z) \neq 0\)（因为 \([x\varphi'(z) + \psi'(z)]q = 1\)），即得
\[r q^2 - 2pqs + t p^2 = 0,\]

此即所要证明的。

§4. 变量代换

1° 在含有导函数的式子中的变量代换。设于式

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\[ A = \Phi(x, y, y', y''_x, \ldots) \]

中需要把 \( x, y \) 换为新的变量：\( t \) (自变量) 及 \( u \) (函数)，这些变量由方程

\[ x = f(t, u), \quad y = g(t, u) \]  \( (1) \)

与原来的变量 \( x \) 和 \( y \) 联系起来。

把方程式 \( (1) \) 微分，便有:

\[ y'_x = \frac{\frac{\partial g}{\partial t} + \frac{\partial g}{\partial u} u'}{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} u'} \cdot \]

同样地可表示出高阶的导函数 \( y''_x \), \( \ldots \) 因此我们得:

\[ A = \Phi_1(t, u, u', u''_t, \ldots). \]

2° 在含有偏导函数的式子中自变量的代换，于下式中

\[ B = F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \]

\[ \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \ldots) \]

令

\[ x = f(u, v), \quad y = g(u, v), \]  \( (2) \)

其中 \( u \) 和 \( v \) 为新的自变量，则按次的偏导函数 \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \ldots \)

由下列方程所确定:

\[ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial g}{\partial u}, \]

\[ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial f}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial g}{\partial v}. \]
等等。

3° 在含有偏导函数的式子中自变量和函数的代数。在一般的情况下，设有方程

\[ x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w), \quad (3) \]

其中 \( u, v \) 为新的自变量及 \( w = w(u, v) \) 为新的函数，则对于偏导函数 \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \ldots \) 得到这样的方程:

\[
\frac{\partial z}{\partial x} \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial u} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial g}{\partial u} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial u} \right)
\]

\[ = \frac{\partial h}{\partial u} + \frac{\partial h}{\partial w} \frac{\partial w}{\partial u}, \]

\[
\frac{\partial z}{\partial x} \left( \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial v} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial g}{\partial v} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial v} \right)
\]

\[ = \frac{\partial h}{\partial v} + \frac{\partial h}{\partial w} \frac{\partial w}{\partial v}. \]

等等。

在某些情况下，使用全微分法进行变量代换是方便的。

3431. 把 \( y \) 看作新的自变量，变换方程

\[ y', y^3 - 3y''^2 = x. \]

解 函数 \( y = y(x) \) 的各阶导函数 \( y', y'', y''' \), \ldots 与其反函数 \( x = x(y) \) 的各阶导函数 \( x', x'', x''' \), \ldots 之间有下述关系。

\[ y' = \frac{1}{x'}, \quad \text{公式 1} \]
\[ y'' = (y')' = \left(\frac{1}{x'}\right)' \cdot y' = -\frac{x''}{x'^{\frac{3}{2}}} \cdot \frac{1}{x} \]

\[ y''' = (y'')' = -\left[\frac{x''}{(x')^3}\right]' \cdot y' \]

\[ = \frac{3(x'')^2 - x'x''}{(x')^5}. \]

公式 3

以公式 1、2、3 代入所给方程，化简整理即得

\[ x'' + x(x')^5 = 0. \]

3432. 用同样的方法变换方程

\[ (y')^2 y^{(4)} - 10y'y''y''' + 15(y'')^3 = 0. \]

解 一

由公式 3 可得

\[ y^{(4)} = (y'')' = \left[\frac{3(x'')^2 - x'x''}{(x')^5}\right]' \cdot y' \]

\[ = \frac{6x'x''x''' - (x')^2 x^{(4)} - x'x''x''' - 5(3(x'')^2 - x'x''x'')}{(x')^6}. \]

\[ \cdot \frac{1}{x'} = \frac{10x'x''x''' - (x')^2 x^{(4)} - 15x''^3}{(x')^7}. \]

公式 4

以公式 1、2、3、4 代入所给方程，化简整理即得

\[ x^{(4)} = 0. \]

解法二

由公式 4 可看出

\[ x^{(4)} = \frac{10y'y''y''' - (y')^2 y^{(4)} - 15(y'')^3}{(y')^7}. \]
因此，所给方程可改写为
\[ -x^{(4)} (y')^7 = 0. \]
由于 \( y' \neq 0 \)，故得
\[ x^{(4)} = 0. \]

3433. 取 \( x \) 作函数，\( t = xy \) 作自变量，变换方程
\[ y'' + 2x y' + y = 0. \]

解 将 \( t = t(x) \) 看作 \( x \) 的函数。对 \( t = xy \) 两端分别求 \( x \) 的一阶、二阶导数，得
\[
\frac{dt}{dx} = y + xy',
\]
\[
\frac{d^2t}{dx^2} = 2y' + xy''.
\]

由于 \( \frac{dx}{dt} = \frac{1}{dt} \)，故由 (1) 式得
\[
y' = \frac{1 - y \frac{dx}{dt}}{x \frac{dx}{dt}}.
\]

由公式 2 及 (2) 式可得
\[
-\frac{d^2x}{dt^2} \frac{x}{(\frac{dx}{dt})^3} = 2y' + xy'',
\]
\[
y'' = -\frac{d^2x}{dt^2} \frac{2y'}{x}.
\]
将（4）式代入所给方程，得
\[- \frac{d^2 x}{dt^2} + xy \left( \frac{dx}{dt} \right)^3 = 0 \text{ 或 } \frac{d^2 x}{dt^2} - t \left( \frac{dx}{dt} \right)^3 = 0.\]

引入新变量，变换下列常微分方程:

3434. \( x^2 y'' + xy' + y = 0 \), 若 \( x = e^t \).

解 当函数 \( y \) 不变，只作自变量的代换 \( x = x(t) \) 时，

注意到对 \( \frac{dt}{dx}, \frac{d^2 t}{dx^2} \) 运用公式 1 及 2，即得

\[
y' = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dx}, \quad \text{公式 5}
\]

\[
y'' = \frac{d}{dx} \left( \frac{dy}{dt} \frac{dt}{dx} \right) = \frac{d^2 y}{dt^2} \left( \frac{dt}{dx} \right)^2 + \frac{dy}{dt} \frac{d^2 t}{dx^2} \]

\[
= - \frac{d^2 y}{dt^2} \frac{d}{dx} - \frac{dy}{dt} \frac{d^2 x}{dx^2} \left( \frac{dx}{dt} \right)^3. \quad \text{公式 6}
\]

在本题中，\( x = e^t \)，故有

\[
\frac{dx}{dt} = e^t = x, \quad \frac{d^2 x}{dt^2} = e^t = x,
\]

从而有

\[
y' = \frac{dy}{dt},
\]

\[
y'' = \frac{x \frac{d^2 y}{dt^2} - x \frac{dy}{dt}}{x^3} = \frac{1}{x^2} \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right).
\]

将 \( y' \) 及 \( y'' \) 代入所给方程，即得

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\[
\frac{d^2 y}{dt^2} + y = 0.
\]

3435. \( y^{(n)} = \frac{6y}{x^3} \), 若 \( t = \ln |x| \).

解

应用复合函数求导公式，有

\[
y' = \frac{d}{dt} \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt},
\]

\[
y'' = \frac{d}{dx} \left( \frac{1}{x} \frac{dy}{dt} \right) = \frac{1}{x^2} \left( x \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)
\]

\[
= \frac{d^2 y}{dt^2} - \frac{dy}{dt}
\]

\[
y'' = \frac{1}{x^3} \left[ x^2 \left( \frac{d^3 y}{dt^3} - \frac{d^2 y}{dt^2} \right) \frac{dt}{dx} - 2x \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \right]
\]

\[
= \frac{1}{x^3} \left( \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} \right).
\]

将 \( y'' \) 代入所给方程，即得

\[
\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 6y = 0.
\]

3438. \((1-x^2) y'' - xy' + n^2 y = 0\)，若 \( x = \cos t \).

解

注意到 \( \frac{dx}{dt} = -\sin t \), \( \frac{d^2 x}{dt^2} = -\cos t \)，用公式 5 及 6，就有

\[
y' = -\frac{dy}{dt}, \quad y'' = -\sin t \frac{d^2 y}{dt^2} + \cos t \frac{dy}{dt} \over -\sin^3 t.
\]
将 $y', y''$ 及 $x$ 代入所给方程，即得

$$\frac{d^2 y}{dt^2} + n^2 y = 0.$$  

3437. $y'' + y' \text{th} x + \frac{m^2}{\text{ch}^2 x} y = 0$, 若 $x = \ln \tan \frac{t}{2}$.

解 仍用公式 5 及 6，注意到

$$\frac{dx}{dt} = \frac{1}{\sin t}, \quad \frac{d^2 x}{dt^2} = -\frac{\cos t}{\sin^2 t},$$

$$\text{ch} x = \frac{1}{\sin t}, \quad \text{th} x = -\cos t,$$

就有

$$y' = \sin t \frac{dy}{dt}, \quad y'' = \sin^2 t \frac{d^2 y}{dt^2} + \sin t \cos t \frac{dy}{dt}.$$  

将 $y', y'', \text{ch} x$ 及 $\text{th} x$ 代入所给方程，即得

$$\frac{d^2 y}{dt^2} + m^2 y = 0.$$  

3438. $y'' + p(x) y' + q(x) y = 0$, 令 $y = u e^{-\int x_0^t p(t) dt}$.

解 $y' = \frac{du}{dx} e^{-\int x_0^t p(t) dt} - \frac{1}{2} u \cdot p(x) e^{-\int x_0^t p(t) dt}$

$$y'' = \frac{d^2 u}{dx^2} e^{-\int x_0^t p(t) dt} - p(x) \frac{du}{dx} e^{-\int x_0^t p(t) dt}$$

$$+ \frac{1}{4} u \cdot p^2(x) e^{-\int x_0^t p(t) dt}$$.  

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$$-\frac{1}{2} u \cdot p'(x) e^{-\frac{1}{2} \int_{x_0}^{x} p'(t) dt}.$$ 

将 $y', y''$ 代入所给方程，化简整理即得

$$\frac{d^2 u}{d x^2} + \left[ q(x) - \frac{1}{4} p'(x) - \frac{1}{2} p'(x) \right] u = 0.$$ 

3439. $x^4 y'' + x y y' - 2 y^2 = 0.$ 令

$$x = e^t, \quad y = u e^{2t},$$

其中

$$u = u(t).$$

解

$$y' = \frac{dy}{dx} \frac{dx}{dt} = \frac{e^{2t}}{e^t} (2u + u') = e^t (2u + u'),$$

$$y'' = \frac{d^2 y}{dx^2} \frac{dx}{dt} = e^t (u'' + 3u' + 2u).$$

其中 $u'$ 及 $u''$ 表示 $u$ 对 $t$ 的一阶及二阶导函数，以下各题类似，不再说明。

将 $y', y''$ 及 $x, y$ 代入所给方程，化简整理即得

$$u'' + (u + 3) u' + 2u = 0.$$ 

3440. $(1 + x^2)^2 y'' = y,$ 若

$$x = \tan t, \quad y = \frac{u}{\cos t},$$

其中

$$u = u(t).$$

解

$$y' = \frac{u' \cos t + u \sin t}{\cos^2 t} = u' \cos t + u \sin t,$$

$$1 = \frac{1}{\cos^2 t}.$$ 

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\[ y'' = \frac{u'' \cos t + uc \cos t}{\cos^2 t} = (u'' + u) \cos^2 t. \]

将 \( y', y'' \) 及 \( x, y \) 代入所给方程，化简整理即得

\[ u'' = 0. \]

3441. \((1 - x^2)^2 y'' = -y\), 若

\[ x = \theta t, \quad y = \frac{u}{\cosh t}, \]

其中 \( u = u(t) \).

解

\[ y' = \frac{u' \cosh t - u \sinh t}{\cosh^2 t} = u' \cosh t - u \sinh t, \]

\[ y'' = \frac{u'' \cosh t - u \cosh t}{\cosh^2 t} = (u'' - u) \cosh t. \]

将 \( y'' \) 及 \( x, y \) 代入所给方程，化简整理即得

\[ u'' = 0. \]

3442. \( y'' + (x + y)(1 + y')^2 = 0 \), 若 \( x = u + t, \quad y = u - t \), 其中 \( u = u(t) \).

解

\[ y' = \frac{u' - 1}{u' + 1}, \]

\[ y'' = \frac{u''(u' + 1) - u''(u' - 1)}{(u' + 1)^2} = \frac{2u''}{(u' + 1)^3}. \]
将 $y', y''$ 及 $x, y$ 代入所给方程，化简整理即得

$$u'' + 8u(u')^3 = 0.$$  

3443. $y'' - x^3 y'' + x y' - y = 0$，若 $x = \frac{1}{t}$ 及 $y = \frac{u}{t}$，其中 $u = u(t)$。

解 $y' = -\frac{t^2}{t^2} = u - t u'$，

$$y'' = -\frac{t u''}{t^2} = t^2 u'',$$

$$y''' = \frac{3t^2 u'' + t^3 u'''}{t^2} = -t^4 (3u'' + tu').$$

将 $y', y'', y'''$ 及 $x, y$ 代入所给方程，化简整理即得

$$t^4 u + (3t^4 + 1) u'' + u' = 0.$$  

3444. 假定

$$u = \frac{y}{x - b}, t = \ln \left| \frac{x - a}{x - b} \right|,$$

并取 $u$ 作为变量 $t$ 的函数，以变换斯托克斯方程

$$y'' = \frac{A y}{(x - a)^2 (x - b)^2}.$$  

解 由于 $t = \ln |x - a| - \ln |x - b|$，故有

$$\frac{d^2 t}{dx^2} = \frac{1}{x - a} - \frac{1}{x - b} = \frac{a - b}{(x - a)(x - b)}.$$
\[
\frac{dx}{dt} = \frac{(x-a)(x-b)}{a-b}.
\]  

（1）

又因 \(u = \frac{y}{x-b}\)，故 \(y = u(x-b)\)，

\[
y' = (x-b) \frac{du}{dx} + u = \frac{du}{dt} (x-b) + u
\]

\[
= \frac{(a-b)u'}{x-a} + u,
\]  

（2）

\[
y'' = \frac{dy'}{dx} = \left[ \frac{(a-b)u''}{x-a} + u' - \frac{(a-b)u'}{(x-a)^2} \frac{dx}{dt} \right]
\]

\[
\cdot \frac{b-a}{(x-a)(x-b)} = \frac{(a-b)^2(u'' - u')}{(x-a)^2(x-b)}.
\]  

（3）

将（3）式代入所给方程，即得

\[
u'' - u' = \frac{Au}{(a-b)^2} \quad (a \neq b).
\]

3445．证明：若方程

\[
\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = 0,
\]

由代换 \(x = \varphi(\xi)\) 变换为方程

\[
\frac{d^2 y}{d\xi^2} + P(\xi) \frac{dy}{d\xi} + Q(\xi) y = 0,
\]

则

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\[
\left(2P(\xi)Q(\xi) + Q'(\xi)\right)[Q(\xi)]^{-\frac{3}{2}}
= [2p(x)q(x) + q'(x)][q(x)]^{-\frac{3}{2}}.
\]

证 \( \frac{d x}{d \xi} = \varphi'(\xi), \frac{d^2 x}{d \xi^2} = \varphi''(\xi). \) 由公式 5 及 6，得

\[
\frac{d y}{d x} = \frac{d y}{d \xi} \frac{d \xi}{d x} = \frac{d y}{d \xi} \frac{1}{\varphi'((\xi))} = \frac{d^2 y}{d \xi^2} \frac{1}{[\varphi'((\xi))]^2} \frac{d x}{d \xi} = \frac{d y}{d \xi} - \frac{\varphi''((\xi))}{[\varphi'((\xi))]^2} \frac{d y}{d \xi}.
\]

代入原方程，两端同乘 \( \varphi'((\xi))^2 \)，即得

\[
\frac{d^2 y}{d \xi^2} + \left\{ p \left[ \varphi((\xi)) \varphi'((\xi)) - \frac{\varphi''((\xi))}{\varphi'((\xi))} \right] \frac{d y}{d \xi}
+ q(\varphi((\xi)) \varphi'(\xi))^2 \frac{d y}{d \xi} = 0.
\]

于是，

\[
P(\xi) = p \varphi' - \frac{\varphi''}{\varphi'}, \quad Q(\xi) = q \cdot (\varphi')^2;
Q'(\xi) = q' \cdot (\varphi')^2 + 2q \varphi' \varphi''.
\]

从而得知

\[
\left(2P(\xi)Q(\xi) + Q'(\xi)\right)[Q(\xi)]^{-\frac{3}{2}}
= \left\{ 2 \left( p \varphi' - \frac{\varphi''}{\varphi'} \right) q \cdot (\varphi')^2 + q' \cdot (\varphi')^3
+ 2q \varphi' \varphi'' \right\} [q \cdot (\varphi')^2]^{-\frac{3}{2}}
\]

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\begin{align*}
&= \left\{ 2pq \cdot (\varphi')^3 + q' \cdot (\varphi')^3 \right\} q^{-3} \cdot (\varphi')^{-3}, \\
&= \left\{ 2p(x)q(x) + q'(x) \right\} \frac{x}{q(x)}^{\frac{3}{2}},
\end{align*}

本题获证。

3446. 在方程
\[ \Phi(y, y', y'') = 0 \]
（其中\(\Phi\)为变量\(y, y', y''\)的齐次函数）中令\(y = e^{\int x \, dx} \).

解
\[ y' = u \cdot e^{\int x \, dx}, \quad y'' = (u' + u^2) e^{\int x \, dx}. \]

代入方程\(\Phi(y, y', y'') = 0\)，由于\(\Phi\)关于\(y, y', y''\)是齐次的，因此，各项含有的因式\(e^{\int x \, dx}\)均可约去，最后得
\[ \Phi(1, u, u' + u^2) = 0. \]

3447. 在方程
\[ F(x^2 y'', xy', y) = 0 \]
（其中\(F\)为其变量的齐次函数）中令\(u = x \cdot \frac{y'}{y} \).

解
\[ y' = \frac{yu}{x}, \quad y'' = \frac{x(u'y + y'u) - yu}{x^2} \]
\[ = \frac{yu + (u^2 - u)}{x^2} \].

于是，
\[ xy' = uy, \quad x^2 y'' = y(u^2 + (u^2 - u)) \).

由于\(F\)为其变量的齐次函数，因此，各项含有的因子\(y\)均可约去，最后得
\[ F(xu' + u^2 - u, u, 1) = 0. \]
证明：经射影变换

\[ x = \frac{a_1 \xi + b_1 \eta + c_1}{a_5 + b \eta + c}, \quad y = \frac{a_2 \xi + b_2 \eta + c_2}{a_5 + b \eta + c}, \]

方程式

\[ y''(1 + y'^2) - 3y'y''^2 = 0 \]

不变其形状。

证  本题似有误，事实上，作压缩变换

\[ \xi = \xi, \quad \eta = a \eta \quad (a \neq 0) \]

(它是射影变换的特例)，则原方程变为

\[ a \eta''(1 + a \eta'^2) - 3a^3 \eta' \eta''^2 = 0, \]

显然形式已改变。

证明：

\[ S(x(t)) = \frac{x''(t)}{x'(t)} - \frac{3}{2} \left( \frac{x''(t)}{x'(t)} \right)^2 \]

对于线性分式变换

\[ y = \frac{a x(t) + b}{c x(t) + d} \quad (ad - bc \neq 0), \]

其值不变。

证  已知的变换

\[ y = \frac{a x + b}{c x + d} = \frac{a(x + \frac{d}{c}) + (b - \frac{ad}{c})}{c x + d} = \frac{a}{c} + \frac{bc - ad}{c(c x + d)} \]

可由下述变换所构成。
\[ y = \alpha + \beta y_2, \quad y_2 = \frac{1}{y_1}, \quad y_1 = cx + d. \]

只要证明在上述各种变换下 \( S \) 的值不变即可。

1° 令 \( y_1 = cx + d \)，则 \( y_1'(t) = cx'(t) \)，\( y_1''(t) = cx''(t) \)，\( y_1'''(t) = cx'''(t) \)。于是，

\[
S(y_1(t)) = \frac{y_1'''(t)}{y_1'(t)} - \frac{3}{2} \left[ \frac{y_1''(t)}{y_1'(t)} \right]^2 - S(x(t));
\]

2° 令 \( y_2 = \frac{1}{y_1} \)，则 \( y_2'(t) = -\frac{y_1'}{y_1^2} \)，

\[
y_2''(t) = \frac{y_1'' y_1^2 - 2y_1 y_1'}{y_1^3},
\]

\[
y_2'''(t) = \frac{y_1''' y_1^2 - 6y_1'' y_1' y_1 + 6y_1'^3}{y_1^4}. \]

于是，

\[
S(y_2(t)) = \frac{y_2'''(t)}{y_2'(t)} - \frac{3}{2} \left[ \frac{y_2''(t)}{y_2'(t)} \right]^2 - S(x(t));
\]

\[
= \frac{y_1''' y_1^2 - 6y_1'' y_1' y_1 + 6y_1'^3}{y_1^4} - \frac{3}{2} \left[ \frac{y_1 y_1'' - 2y_1'^2}{y_1^2} \right]^2
\]

\[
= \frac{y_1'' y_1 - 6y_1' y_1 + 6y_1'^2}{y_1} - \frac{3}{2} \left( \frac{y_1'}{y_1} - \frac{2y_1'}{y_1} \right)^2
\]

\[
= \frac{y_1''' - 3}{2} (\frac{y_1'}{y_1})^2 = S(y_1(t)) = S(x(t));
\]

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3° 由1°及2° 即知

\[ S(y(t)) = S(\alpha + \beta y_2) = \frac{(\alpha + \beta y_2)''}{(\alpha + \beta y_2)'}, \]

\[ = \frac{3}{2} \left( \frac{y_2''}{y_2'} \right)^2 = S(y_2(t)) = S(x(t)). \] 证毕。

将下列方程式改变为极坐标 \( r \) 与 \( \varphi \) 所表示的方程，即

令 \( x = r \cos \varphi, \ y = r \sin \varphi; \)

3450. \( \frac{d^2 y}{dx^2} = \frac{x + y}{x - y}. \)

解  当 \( x = r \cos \varphi, \ y = r \sin \varphi \) 时，

\[ \frac{dx}{d\varphi} = \cos \varphi \frac{dr}{d\varphi} - r \sin \varphi, \quad \frac{dy}{d\varphi} = \sin \varphi \frac{dr}{d\varphi} + r \cos \varphi, \]

\[ \frac{d^2 x}{d\varphi^2} = \cos \varphi \frac{d^2 r}{d\varphi^2} - 2 \sin \varphi \frac{dr}{d\varphi} - r \cos \varphi, \]

\[ \frac{d^2 y}{d\varphi^2} = \sin \varphi \frac{d^2 r}{d\varphi^2} + 2 \cos \varphi \frac{dr}{d\varphi} - r \sin \varphi. \]

由公式 5 及 6，即得

\[ \frac{dy}{dx} = \frac{\sin \varphi \frac{dr}{d\varphi} + r \cos \varphi}{\cos \varphi \frac{dr}{d\varphi} - r \sin \varphi}, \]  公式 7

\[ \frac{d^2 y}{dx^2} = \frac{\frac{d^2 y}{d\varphi^2} \frac{dx}{d\varphi} - \frac{dy}{d\varphi} \frac{d^2 x}{d\varphi^2}}{(\frac{dx}{d\varphi})^3}. \]

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\[
= r^2 + 2\left(\frac{dr}{d\varphi}\right)^2 - r\frac{d^2r}{d\varphi^2} \quad \text{公式 8}
\]

将公式 7 及 \(x, y\) 代入所给方程，化简整理即得

\[
\frac{dr}{d\varphi} = r \text{ 或 } r' = r.
\]

以下各题，\(\frac{dr}{d\varphi}\) 及 \(\frac{d^2r}{d\varphi^2}\) 均简记为 \(r'\) 及 \(r''\)。

3451. \((xy' - y)^2 = 2xy(1 + y'^2)\).

解

\[
xy' - y = r\cos\varphi \cdot \frac{r'\sin\varphi + r\cos\varphi}{r'\cos\varphi - r\sin\varphi} = r\sin\varphi
\]

\[
= r\left(\frac{r'\sin\varphi \cos\varphi + r\cos^2\varphi - r'\sin\varphi \cos\varphi + r\sin^2\varphi}{r'\cos\varphi - r\sin\varphi}\right)
\]

\[
= \frac{r^2}{r'\cos\varphi - r\sin\varphi}.
\]

\[1 + y'^2 = 1 + \left(\frac{r'\sin\varphi + r\cos\varphi}{r'\cos\varphi - r\sin\varphi}\right)^2
\]

\[= \frac{r'^2 + r^2}{(r'\cos\varphi - r\sin\varphi)^2}.
\]

将 \(xy' - y, 1 + y'^2\) 及 \(x, y\) 代入所给方程，化简整理即得

\[
r'^2 = \frac{1 - \sin 2\varphi}{\sin 2\varphi} \cdot r^2.
\]
3452. \( (x^2 + y^2)^2 y'' = (x + yy')^3 \).

解
\[ x + yy' = r \cos \phi + r \sin \phi \cdot \frac{r' \sin \phi + r \cos \phi}{r' \cos \phi - r \sin \phi} \]
\[ = \frac{r' \cos^2 \phi - r^2 \sin \phi \cos \phi + r' \sin^2 \phi + r^2 \sin \phi \cos \phi}{r' \cos \phi - r \sin \phi} \]
\[ = \frac{rr'}{r' \cos \phi - r \sin \phi}. \]

将公式8，\( x + yy' \)及\( x, y \)代入所给方程，化简整理即得
\[ r(r^2 + 2r'^2 - rr'') = r'^3. \]

3453. 把式子
\[ \frac{x + yy'}{xy' - y} \]
变换为极坐标的式子。
解 将3451题中\( xy' - y \)的结果及3452题中\( x + yy' \)的结果代入所给式子，即得
\[ \frac{x + yy'}{xy' - y} = \frac{r'}{r}. \]

3454. 把平面曲线的曲率
\[ K = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}} \]
用极坐标\( r \)及\( \phi \)表示之。
解 将3451题中\( 1 + y'^2 \)的结果及公式8代入，化简整理即得
\[ K = \frac{|r^2 + 2r_1^2 - rr''|}{(r^2 + r_1^2)^{\frac{3}{2}}} \]

3455. 将方程组

\[
\begin{align*}
\frac{dx}{dt} &= y + kx(x^2 + y^2), \\
\frac{dy}{dt} &= -x + ky(x^2 + y^2)
\end{align*}
\]

\( \frac{d^2x}{dt^2} = -x + ky(x^2 + y^2) \)

改为极坐标方程。

解 由原方程组得

\[
\begin{align*}
\cos \varphi \frac{dr}{dt} - r \sin \varphi \frac{d\varphi}{dt} &= r \sin \varphi + kr^3 \cos \varphi, \\
\sin \varphi \frac{dr}{dt} + r \cos \varphi \frac{d\varphi}{dt} &= -r \cos \varphi + kr^3 \sin \varphi.
\end{align*}
\]

联立解之，即得

\[
\begin{align*}
\frac{dr}{dt} &= \frac{1}{r} \left( r \cos \varphi \cdot (r \sin \varphi + kr^3 \cos \varphi) \right. \\
&\quad \left. - (r \sin \varphi) (r \cos \varphi + kr^3 \sin \varphi) \right) = kr^5,
\end{align*}
\]

\[
\frac{d\varphi}{dt} = \frac{1}{r} \left[ \cos \varphi \cdot (r \cos \varphi + kr^3 \sin \varphi) \right. \\
&\quad \left. - \sin \varphi \cdot (r \sin \varphi + kr^3 \cos \varphi) \right] = -1,
\]

即原方程组转化为
\[
\begin{aligned}
\frac{dr}{dt} &= kr^3, \\
\frac{d\varphi}{dt} &= -1.
\end{aligned}
\]

3456. 引用新函数 \( r = \sqrt{x^2 + y^2} \), \( \varphi = \arctg \frac{y}{x} \)，变换式子

\[
W = x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2}.
\]

解 由 \( r = \sqrt{x^2 + y^2} \) 两端微分，得

\[
dr = \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{x}{r} \, dx + \frac{y}{r} \, dy
\]

或

\[
r \, dr = x \, dx + y \, dy. \quad (1)
\]

由 \( \varphi = \arctg \frac{y}{x} \) 两端微分，得

\[
d\varphi = -\frac{xdy - ydx}{x^2 + y^2} = \frac{x}{r^2} \, dy - \frac{y}{r^2} \, dx
\]

或

\[
r^2 \, d\varphi = x \, dy - y \, dx. \quad (2)
\]

于是，由 (1) 及 (2) 可得

\[
x \, rdr - yr^2 \, d\varphi = (x^2 \, dx + xy \, dy) - (xy \, dy - y^2 \, dx)
\]

\[
= (x^2 + y^2) \, dx = r^2 \, dx,
\]

\[
dx = \frac{x}{r} \, dr - y \, d\varphi. \quad (3)
\]

同理可得
\[ dy = \frac{y}{r} dr + x d\varphi. \tag{4} \]

从而由 (3) 及 (4)，得

\[
\begin{align*}
xd^2 y - yd^2 x &= x \left( \frac{y}{r} \right. d^2 r - \frac{y}{r^2} dr^2 \\
&+ \frac{1}{r} drd y + dxd \varphi + x d^2 \varphi \\
- y \left( \frac{x}{r} d^2 r - \frac{x}{r^2} dr^2 + \frac{1}{r} dxd r - dyd \varphi - y d^2 \varphi \right) \\
&= \frac{dr}{r} (xdy - ydx) + (xdx + ydy) d\varphi \\
&+ (x^2 + y^2) d^2 \varphi \\
&= \frac{dr}{r} (r^2 d\varphi) + (rdr) d\varphi + r^2 d^2 \varphi \\
&= 2r drd \varphi + r^2 d^2 \varphi,
\end{align*}
\]

于是，

\[
W = x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = 2r \frac{dr}{dt} \frac{d\varphi}{dt} + r^2 \frac{d^2 \varphi}{dt^2}
\]

\[
= \frac{d}{dt} \left( r^2 \frac{d\varphi}{dt} \right).
\]

3457. 在勒武德变换中曲线 \( y = y(x) \) 的每一点 \((x, y)\) 对应于点 \((X, Y)\)，其中

\[ X = y', \ Y = x y' - y. \]

求 \( Y' \), \( Y'' \) 及 \( Y''' \).
解  \[
Y' = \frac{dY}{dX} = \frac{dY}{dx} \cdot \frac{dx}{dX} = \frac{xy''}{y''} = x;
\]

\[
Y'' = \frac{dY'}{dx} = \frac{1}{y''};
\]

\[
Y''' = -\frac{1}{y''^2} = -\frac{y''}{y''^2}.
\]

引入新变量 \(\xi\) 及 \(\eta\)，解下列方程：

3458. \(\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\)，令 \(\xi = x + y, \eta = x - y\).

解  当仅作自变量代换，引入新自变量

\(\xi = \xi(x, y), \eta = \eta(x, y)\)

这个最简单的情形时，只要把 \(\xi, \eta\) 看作中间变量，用复合函数求偏导函数的公式，即可求出：

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial x},
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial y}.
\]

代入原方程，即得变换后的方程。本题中，

\[
\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 1, \quad \frac{\partial \eta}{\partial y} = -1.
\]

于是，
\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}. \]

代入原方程，得

\[ \frac{\partial z}{\partial \xi} + \frac{2x}{\partial \eta} = \frac{\partial z}{\partial \xi} - \frac{2y}{\partial \eta} \text{ 或 } \frac{\partial z}{\partial \eta} = 0, \]

即

\[ z = \varphi(\xi) = \varphi(x + y), \]

其中 \( \varphi \) 为任意的函数。

3459. \( y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0, \) 令 \( \xi = x, \eta = x^2 + y^2. \)

解

\[ \frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} = 2x, \quad \frac{\partial \eta}{\partial y} = 2y. \]

于是，

\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} + 2x \frac{\partial z}{\partial \eta}, \quad \frac{\partial z}{\partial y} = 2y \frac{\partial z}{\partial \eta}. \]

代入原方程，得

\[ y \left( \frac{\partial z}{\partial \xi} + 2x \frac{\partial z}{\partial \eta} \right) - 2xy \frac{\partial z}{\partial \eta} = 0 \text{ 或 } y \frac{\partial z}{\partial \xi} = 0. \]

由于 \( y \neq 0 \)，故 \( \frac{\partial z}{\partial \xi} = 0, \) 即

\[ z = \varphi(\eta) = \varphi(x^2 + y^2), \]

其中 \( \varphi \) 为任意的函数。

3460. \( a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = 1 \quad (a \neq 0), \) 令 \( \xi = x, \eta = y - bz. \)
解 当变量间的变换关系比较复杂时，用全微分法较好，首先，根据新旧变元之间的关系，求出它们微分之间的关系

$$d\xi = dx, \quad d\eta = dy - bdz.$$  
(1)

其次，将所求得的微分式代入表示新变元关系的全微分式，并按旧变元关系重新整理。

$$dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta = \frac{\partial z}{\partial \xi} dx + \frac{\partial z}{\partial \eta} (dy - bdz),$$

$$(1 + b \frac{\partial z}{\partial \eta}) dz = \frac{\partial z}{\partial \xi} dx + \frac{\partial z}{\partial \eta} dy,$$

$$dz = \frac{\partial z}{\partial \xi} dx + \frac{\partial z}{\partial \eta} dy.$$

把整理后的式子与表示旧变元的全微分式

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

比较，即得

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial z}{\partial \xi}}{1 + b \frac{\partial z}{\partial \eta}}, \quad \frac{\partial z}{\partial y} = \frac{\frac{\partial z}{\partial \eta}}{1 + b \frac{\partial z}{\partial \eta}}.$$  

代入原方程，得

$$a \frac{\partial z}{\partial \xi} + b \frac{\partial z}{\partial \eta} = 1 + b \frac{\partial z}{\partial \eta} \text{或} \frac{\partial z}{\partial \xi} = \frac{1}{a}.$$  

于是，

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\[ z = \frac{x}{a} + \varphi(y - bx). \]

3461. \( x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z \). 令 \( \xi = x \) 及 \( \eta = \frac{y}{x} \).

解 \[ \frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{x}. \]

\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \frac{x}{x^2} \frac{\partial z}{\partial \eta}, \quad \frac{\partial z}{\partial y} = \frac{1}{x} \frac{\partial z}{\partial \eta}. \]

代入原方程，得

\[ x \left( \frac{\partial z}{\partial \xi} - \frac{y}{x^2} \frac{\partial z}{\partial \eta} \right) + \frac{y}{x} \frac{\partial z}{\partial \eta} = z, \]

\[ \frac{\partial z}{\partial \xi} = z \text{ 或 } \frac{\partial z}{\partial \xi} = z. \]

解之，得

\[ z = \xi \varphi(\eta) = x \varphi\left(\frac{y}{x}\right). \]

取 \( u \) 与 \( v \) 作新的变量，变换下列方程式

3462. \( x \frac{\partial z}{\partial x} + \sqrt{1 + y^2} \frac{\partial z}{\partial y} = xy \), 若 \( u = \ln x \),

\[ v = \ln(y + \sqrt{1 + y^2}). \]

解 \[ \frac{\partial u}{\partial x} = \frac{1}{x}, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1 + y^2}}. \]
\[
\frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial u}, \quad \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 + y^2}} \frac{\partial z}{\partial v}.
\]

注意到 \(x = e^u\) 及 \(y = shv\)，代入原方程，即得

\[
\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = e^u shv.
\]

3463. \((x + y) \frac{\partial z}{\partial x} - (x - y) \frac{\partial z}{\partial y} = 0\)，若 \(u = \ln \sqrt{x^2 + y^2}\)，

\[
v = \arctg \frac{y}{x}.
\]

解

\[
\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},
\]

\[
\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}.
\]

\[
\frac{\partial z}{\partial x} = \frac{x}{x^2 + y^2} \frac{\partial z}{\partial u} - \frac{y}{x^2 + y^2} \frac{\partial z}{\partial v},
\]

\[
\frac{\partial z}{\partial y} = \frac{y}{x^2 + y^2} \frac{\partial z}{\partial u} + \frac{x}{x^2 + y^2} \frac{\partial z}{\partial v}.
\]

代入原方程，得

\[
\frac{x + y}{x^2 + y^2} \left( x \frac{\partial z}{\partial u} - y \frac{\partial z}{\partial v} \right) = \frac{x - y}{x^2 + y^2},
\]

\[
( y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} ) = 0,
\]

\[
\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0 \quad \text{或} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial v}.
\]
3464. \( \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z + \sqrt{x^2 + y^2 + z^2} \), 若 \( u = \frac{y}{x} \),

\( v = z + \sqrt{x^2 + y^2 + z^2} \).

解 本题用微分法较好。

\[
du = \frac{xdy - ydx}{x^2},
\]

\[
dv = dz + \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}}
\]

\[
= dz + \frac{xdx + ydy + zdz}{r}
\]

\( (r = \sqrt{x^2 + y^2 + z^2}) \).

\[
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{\partial z}{\partial u} \left( \frac{dy}{x} - \frac{ydx}{x^2} \right)
\]

\[
+ \frac{\partial z}{\partial v} \left( dx + \frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz \right).
\]

于是，

\[
(1 - \frac{\partial z}{\partial v} - \frac{z}{r} \frac{\partial z}{\partial v}) dz = (-\frac{y}{x^2} \frac{\partial z}{\partial u} + \frac{x}{r} \frac{\partial z}{\partial v}) dx
\]

\[
+ (\frac{1}{x} \frac{\partial z}{\partial u} + \frac{y}{r} \frac{\partial z}{\partial v}) dy,
\]

\[
\frac{\partial z}{\partial x} = (-\frac{y}{x^2} \frac{\partial z}{\partial u} + \frac{x}{r} \frac{\partial z}{\partial v}) (1 - \frac{\partial z}{\partial v} - \frac{z}{r} \frac{\partial z}{\partial v})^{-1},
\]
\[
\frac{\partial z}{\partial y} = \left( \frac{1}{x} \frac{\partial z}{\partial u} + \frac{y}{r} \frac{\partial z}{\partial v} \right) \left( 1 - \frac{\partial z}{\partial v} - \frac{z}{r} \frac{\partial z}{\partial v} \right)^{-1}.
\]

代入原方程，得

\[
x \left( -\frac{y}{x^2} \frac{\partial z}{\partial u} + \frac{x}{r} \frac{\partial z}{\partial v} \right) + y \left( \frac{1}{x} \frac{\partial z}{\partial u} + \frac{y}{r} \frac{\partial z}{\partial v} \right)
\]

\[= (z+r) \left( 1 - \frac{\partial z}{\partial v} - \frac{z}{r} \frac{\partial z}{\partial v} \right),\]

\[2(z+r) \frac{\partial z}{\partial v} = z+r.\]

如果z+r=0，则可推得x^2+y^2=0，但由于x≠0，所以x^2+y^2不可能为零。于是，z+r≠0，从而得

\[
\frac{\partial z}{\partial v} = \frac{1}{2}.
\]

3465. \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{x}{z}, \quad \text{若} \quad u=2x-z^2, \quad v=\frac{y}{z}.

解 \quad du=2dx-2zd\!z, \quad dv=\frac{dy}{z}-\frac{y}{z^2}d\!z.

\[
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{\partial z}{\partial u} (2dx-zd\!z)
\]

\[+ \frac{\partial z}{\partial v} \left( \frac{1}{z} dy - \frac{y}{z^2} d\!z \right).
\]

于是，

\[
(1+2z\frac{\partial z}{\partial u} + \frac{y}{z^2} \frac{\partial z}{\partial v}) dz = 2 \frac{\partial z}{\partial u} dx + \frac{1}{z} \frac{\partial z}{\partial v} dy,
\]

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\[
\frac{\partial z}{\partial x} = 2 \frac{\partial z}{\partial u} \left( 1 + 2z \frac{\partial z}{\partial u} + z^2 \frac{\partial z}{\partial v} \right)^{-1},
\]

\[
\frac{\partial z}{\partial y} = \frac{1}{z} \frac{\partial z}{\partial v} \left( 1 + 2z \frac{\partial z}{\partial u} + z^2 \frac{\partial z}{\partial v} \right)^{-1}.
\]

代入原方程，得

\[
2x \frac{\partial z}{\partial u} + y \cdot \frac{1}{z} \frac{\partial z}{\partial v} = \frac{x}{z} \left( 1 + 2z \frac{\partial z}{\partial u} + z^2 \frac{\partial z}{\partial v} \right),
\]

\[
\left( \frac{y}{z} - \frac{xy}{z^2} \right) \frac{\partial z}{\partial v} = \frac{x}{z}.
\]

再以 \( y = zv, x = \frac{1}{2} (u + z^2) \) 代入上式，最后得

\[
\frac{\partial z}{\partial v} = \frac{z}{v} \frac{z^2 + u}{z^2 - u}.
\]

\[3466^+.\ (x + z) \frac{\partial z}{\partial x} + (y + z) \frac{\partial z}{\partial y} = x + y + z, \text{ 若 } u = x + z, \]

\[v = y + z.\]

解得

\[
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{\partial z}{\partial u} (dx + dz) + \frac{\partial z}{\partial v} (dy + dz).
\]

于是，

\[
\left( 1 - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) dz = \frac{\partial z}{\partial u} dx + \frac{\partial z}{\partial v} dy,
\]

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \left( 1 - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)^{-1},
\]

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\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \left( 1 - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)^{-1}.
\]

将 \( \frac{\partial z}{\partial x} \) 及 \( \frac{\partial z}{\partial y} \) 代入原方程，并注意到 \( x + y + z = u + v - z \)，即得

\[
u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} = (u + v - z) \left( 1 - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right),
\]

\[
(2u + v - z) \frac{\partial z}{\partial u} + (2v + u - z) \frac{\partial z}{\partial v} = u + v - z.
\]

3467. 取

\[\xi = y + ze^{-x}, \quad \eta = x + z e^{-y}\]

作为新的自变量，变换式子

\[
(z + e^x) \frac{\partial z}{\partial x} + (z + e^y) \frac{\partial z}{\partial y} - (z^2 - e^{x+y}).
\]

解

\[
dz = \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta
\]

\[
= \frac{\partial z}{\partial \xi} \left( dy + e^{-x}dz - ze^{-x}dx \right) + \frac{\partial z}{\partial \eta} \cdot \left( dx + e^{-y}dz - ze^{-y}dy \right).
\]

于是，

\[
\left( 1 - e^{-x} \frac{\partial z}{\partial \xi} - e^{-y} \frac{\partial z}{\partial \eta} \right) dz = \left( \frac{\partial z}{\partial \eta} - ze^{-x} \frac{\partial z}{\partial \xi} \right) dx
\]

\[+ \left( \frac{\partial z}{\partial \xi} - ze^{-y} \frac{\partial z}{\partial \eta} \right) dy,
\]
\[
\frac{\partial z}{\partial x} = \left( \frac{\partial z}{\partial \xi} - ze^{-x} \frac{\partial z}{\partial \eta} \right) \left( 1 - e^{-x} \frac{\partial z}{\partial \xi} - e^{-x} \frac{\partial z}{\partial \eta} \right)^{-1},
\]

\[
\frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial \xi} - ze^{-y} \frac{\partial z}{\partial \eta} \right) \left( 1 - e^{-y} \frac{\partial z}{\partial \xi} - e^{-y} \frac{\partial z}{\partial \eta} \right)^{-1}.
\]

代入原式，化简整理即得

\[
\text{原式} = \frac{e^{x+y} - z^2}{1 - e^{-x} \frac{\partial z}{\partial \xi} - e^{-y} \frac{\partial z}{\partial \eta}}.
\]

3468. 假定

\[x = uv, \ y = \frac{1}{2} (u^2 - v^2)\]

变换式子

\[
\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2.
\]

解

\[dx = vdu + udv, \ dy = udu - vdv.\]

解之，得

\[du = \frac{vdx + udy}{u^2 + v^2}, \ dv = \frac{udx - vdy}{u^2 + v^2}.
\]

于是，

\[
\frac{dz}{du} du + \frac{dz}{dv} dv = \frac{1}{u^2 + v^2} \left[ \frac{\partial z}{\partial u} (vdx + udy),
\right.
\]

\[+ \left. \frac{\partial z}{\partial v} (udx - vdy) \right] = \frac{1}{u^2 + v^2} \left[ (v \frac{dz}{du} + u \frac{dz}{dv}) dx + (u \frac{dz}{du} - v \frac{dz}{dv}) dy \right],
\]

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\[
\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = \frac{1}{(u^2 + v^2)^2} \left[(v \frac{\partial z}{\partial u} + u \frac{\partial z}{\partial v})^2 + (u \frac{\partial z}{\partial u} - v \frac{\partial z}{\partial v})^2 \right]
= \frac{1}{u^2 + v^2} \left[(\frac{\partial z}{\partial u})^2 + (\frac{\partial z}{\partial v})^2 \right].
\]

3469. 于方程

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0
\]

令 \( \xi = x, \eta = y - x, \zeta = z - x \).

解 \[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x}
= \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \zeta},
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}, \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial \zeta}.
\]

三式相加即得

\[
\frac{\partial u}{\partial \xi} = 0.
\]

3470. 取 \( \nu \) 作为函数，而 \( y \) 和 \( z \) 作为自变量，变换方程

\[
(x - z) \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.
\]

解 \[
dx = \frac{\partial x}{\partial y} dy + \frac{\partial x}{\partial z} dz, \quad dz = \frac{1}{\frac{\partial x}{\partial z} - \frac{\partial y}{\partial z}} dx - \frac{\partial y}{\partial z} dy.
\]
于是，
\[
\frac{\partial z}{\partial x} = \frac{1}{\frac{\partial x}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\partial x}{\partial z}.
\]

代入原方程，得
\[
(x-z) \cdot \frac{1}{\frac{\partial x}{\partial z}} - y \cdot \frac{\partial x}{\partial z} = 0,
\]
即
\[
\frac{\partial x}{\partial y} = \frac{x-z}{y} \quad (y \neq 0).
\]

3471. 取 \( x \) 作为函数，而 \( u = y - z, \ v = y + z \) 作为自变量，变换方程
\[
(y-z) \frac{\partial z}{\partial x} + (y+z) \frac{\partial z}{\partial y} = 0.
\]
解  \( du = dy - dz, \ dv = dy + dz \).

\[
dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = \frac{\partial x}{\partial u} (dy - dz) + \frac{\partial x}{\partial v} (dy + dz).
\]

于是，
\[
(\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}) dz = -dx + (\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}) dy,
\]
\[
\frac{\partial z}{\partial x} = -\frac{1}{\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}}, \quad \frac{\partial z}{\partial y} = \frac{\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}}{\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}},
\]

代入原方程，去分母，即得

\[
\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} = \frac{u}{v} \quad (v \neq 0).
\]

3472^+ 取 \( x \) 作为函数及 \( u = xz, \ v = yz \) 作为自变量，变换形式

\[
A = (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2.
\]

解 \( du = zdz + xdx, \ dv = ydz + zdy \).

\[
dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = \frac{\partial x}{\partial u} (xdz + zdz) + \frac{\partial x}{\partial v} (ydz + zdy).
\]

于是，

\[
(x \frac{\partial x}{\partial u} + y \frac{\partial x}{\partial v}) dz = (1 - z \frac{\partial x}{\partial u}) dx - z \frac{\partial x}{\partial v} dy,
\]

\[
\frac{\partial z}{\partial x} = \frac{1 - z \frac{\partial x}{\partial u}}{x \frac{\partial x}{\partial u} + y \frac{\partial x}{\partial v}}, \quad \frac{\partial z}{\partial y} = - \frac{z \frac{\partial x}{\partial v}}{x \frac{\partial x}{\partial u} + y \frac{\partial x}{\partial v}}.
\]

代入原式，即得
\[ A = \frac{(1-z^2 \frac{\partial x}{\partial u})^2 + z^2 (\frac{\partial x}{\partial u})^2}{(x \frac{\partial x}{\partial u} + y \frac{\partial x}{\partial v})^2} \]

\[ = 1 - 2z \frac{\partial x}{\partial u} + z^2 \left[ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2 \right] \]

\[ = \frac{1 - 2 \cdot \frac{u}{x} \frac{\partial x}{\partial u} + \left( \frac{u}{x} \right)^2 \left[ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2 \right]}{x^2 \left( \frac{\partial x}{\partial u} + \frac{v}{u} \frac{\partial x}{\partial v} \right)^2} \]

\[ = \frac{u^2 \left\{ x^2 - 2xu \frac{\partial x}{\partial u} + u^2 \left[ \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2 \right] \right\}}{x^4 \left( \frac{\partial x}{\partial u} + \frac{v}{u} \frac{\partial x}{\partial v} \right)^2} \]

3473. 于方程

\[ (y + z + u) \frac{\partial u}{\partial x} + (x + z + u) \frac{\partial u}{\partial y} \]

\[ + (x + y + u) \frac{\partial u}{\partial z} = x + y + z \]

中，令 \( e^x = x - u \), \( e^y = y - u \), \( e^z = z - u \).

解 \( \frac{du}{d\xi} = \frac{\partial u}{\partial \xi} d\xi + \frac{\partial u}{\partial \eta} d\eta + \frac{\partial u}{\partial \zeta} d\zeta \)

\[ = \frac{du}{d\xi} e^{-t} (dx - du) + \frac{du}{d\eta} e^{-t} (dy - du) \]

\[ + \frac{du}{d\zeta} e^{-t} (dz - du). \]

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于是，

\[(1 + e^{-\xi} \frac{\partial u}{\partial \xi} + e^{-\eta} \frac{\partial u}{\partial \eta} + e^{-z} \frac{\partial u}{\partial z})d\eta \]

\[= e^{-\xi} \frac{\partial u}{\partial \xi} d\xi + e^{-\eta} \frac{\partial u}{\partial \eta} d\eta + e^{-z} \frac{\partial u}{\partial z} dz.\]

将由上式所确定的 \(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\) 及 \(\frac{\partial u}{\partial z}\) 代入原方程，即得

\[(y + z + u)e^{-\xi} \frac{\partial u}{\partial \xi} + (x + z + u)e^{-\eta} \frac{\partial u}{\partial \eta} + (x + y + u)e^{-z} \frac{\partial u}{\partial z} = (x + y + z)\left(1 + e^{-\xi} \frac{\partial u}{\partial \xi} + e^{-\eta} \frac{\partial u}{\partial \eta} + e^{-z} \frac{\partial u}{\partial z}\right).\]

消去同类项，得

\[(x - u)e^{-\xi} \frac{\partial u}{\partial \xi} + (y - u)e^{-\eta} \frac{\partial u}{\partial \eta} + (z - u)e^{-z} \frac{\partial u}{\partial z} + (x + y + z) = 0,\]

即

\[\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial z} + 3u + e^\xi + e^\eta + e^z = 0.\]

于下列方程中，代入新的变量 \(u, v, w\)，其中 \(w = w(u, v)\):

3474. \(y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = (y - x)z, \quad u = x^2 + y^2, \quad v = \frac{1}{x} + \frac{1}{y},\)

\(w = \ln z - (x + y).\)

解：\(du = 2xdx + 2yd\gamma, \quad dv = -\frac{1}{x^2}dx - \frac{1}{y^2}dy,\)
\[dw = \frac{1}{z} \, dz - dx - dy,\]

另一方面，
\[dw = \frac{\partial w}{\partial u} \, du + \frac{\partial w}{\partial v} \, dv,\]

故有
\[\frac{1}{z} \, dz - dx - dy = \frac{\partial w}{\partial u} \left( 2x \, dx + 2y \, dy \right)\]

\[+ \frac{\partial w}{\partial v} \left( -\frac{1}{x^2} \, dx - \frac{1}{y^2} \, dy \right).\]

整理得
\[dz = \left( 2xz \frac{\partial w}{\partial u} - \frac{z}{x^2} \frac{\partial w}{\partial v} + z \right) \, dx\]

\[+ \left( 2yz \frac{\partial w}{\partial u} - \frac{z}{y^2} \frac{\partial w}{\partial v} + z \right) \, dy.\]

将由上式所确定的 \(\frac{\partial z}{\partial x}\) 及 \(\frac{\partial z}{\partial y}\) 代入原方程，得

\[yz \left( 2x \frac{\partial w}{\partial u} - \frac{1}{x^2} \frac{\partial w}{\partial v} + 1 \right)\]

\[-xz \left( 2y \frac{\partial w}{\partial u} - \frac{1}{y^2} \frac{\partial w}{\partial v} + 1 \right)\]

\[= (y - x)z,\]

即

\[\frac{\partial w}{\partial v} = 0.\]

3475. \(x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = 2z,\) 令 \(u = x, \quad v = \frac{1}{y} - \frac{1}{z}.\)
\[ w = \frac{1}{z} - \frac{1}{x} \cdot \]

解得

\[ du = dx, \quad dv = \frac{1}{x^2} dx - \frac{1}{y^2} dy, \quad dw = \frac{1}{x^2} dx - \frac{1}{z^2} dz. \]

于是，

\[ \frac{1}{x^2} dx - \frac{1}{z^2} dz = \frac{\partial w}{\partial u} dx + \frac{\partial w}{\partial v} \left( \frac{1}{x^2} dx - \frac{1}{y^2} dy \right), \]

\[ dz = z^2 \left( \frac{1}{x^2} \frac{\partial w}{\partial u} - \frac{1}{x^2} \frac{\partial w}{\partial v} \right) dx + \frac{z^2}{y^2} \frac{\partial w}{\partial v} dy, \]

\[ \frac{\partial z}{\partial x} = z^2 \left( \frac{1}{x^2} \frac{\partial w}{\partial u} - \frac{1}{x^2} \frac{\partial w}{\partial v} \right), \quad \frac{\partial z}{\partial y} = \frac{z^2}{y^2} \frac{\partial w}{\partial v}. \]

代入原方程，得

\[ z^2 \left( 1 - x^2 \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \right) + z^2 \frac{\partial w}{\partial v} = z^2 \]

或

\[ x^2 z^2 \frac{\partial w}{\partial u} = 0. \]

由于 \( z \neq 0, \quad x \neq 0 \)，故得

\[ \frac{\partial w}{\partial u} = 0. \]

3476. \( (xy + z) \frac{\partial z}{\partial x} + (1 - y^2) \frac{\partial z}{\partial y} = x + yz \)，设 \( u = yz - x \)，

\( v = xz - y, \quad w = xy - z. \)

解得

\[ dw = y dx + x dy - dz = \frac{\partial w}{\partial u} (2 dy + y dz - dx) \]

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\[ + \frac{\partial w}{\partial v} (zdx + xdz - dy). \]

整理得

\[ (1 + x \frac{\partial w}{\partial u} + y \frac{\partial w}{\partial u}) \, dz = (y + z \frac{\partial w}{\partial u} - z \frac{\partial w}{\partial u}) \, dx \]

于是，

\[ \frac{\partial z}{\partial x} = (y + \frac{\partial w}{\partial u} - z \frac{\partial w}{\partial u})(1 + x \frac{\partial w}{\partial v} + y \frac{\partial w}{\partial u})^{-1}, \]

\[ \frac{\partial z}{\partial y} = (x + \frac{\partial w}{\partial u} - z \frac{\partial w}{\partial u})(1 + x \frac{\partial w}{\partial v} + y \frac{\partial w}{\partial u})^{-1}. \]

代入原方程，得

\[ (xy + z)(y + \frac{\partial w}{\partial u} - z \frac{\partial w}{\partial v}) \]

\[ + (1 - y^2)(x + \frac{\partial w}{\partial u} - z \frac{\partial w}{\partial u}) \]

\[ = (x + yz)(1 + x \frac{\partial w}{\partial v} + y \frac{\partial w}{\partial u}), \]

即

\[ (1 - x^2 - y^2 - z^2 - 2xyz) \frac{\partial w}{\partial v} = 0. \]

不难验证，由方程 \(1 - x^2 - y^2 - z^2 - 2xyz = 0\) 确定的隐函数不是原方程的解（证略）。于是，
\[ \frac{\partial w}{\partial v} = 0. \]

3477. \( (x \frac{\partial z}{\partial x})^2 + (y \frac{\partial z}{\partial y})^2 = z^2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}, \) 令 \( x = ue^v, y = ve^u, \)

\[ z = we^v. \]

解 \[ dx = e^u du + ue^v dw, \quad dy = e^v dv + ve^u dw, \]

\[ dz = e^w (1 + w) dw. \]

于是，有

\[ e^u dw = \frac{1}{1 + w} dz, \]

\[ e^v du = dx - ue^v dw = dx - \frac{u}{1 + w} dz, \]

\[ e^w dv = dy - ve^u dw = dy - \frac{v}{1 + w} dz. \]

在全微分式 \( dw = \frac{\partial w}{\partial u} du + \frac{\partial w}{\partial v} dv \) 的两端都乘以 \( e^w \)，并
将上述结果代入，得

\[ \frac{dz}{1 + w} = \frac{\partial w}{\partial u} \left( dx - \frac{u}{1 + w} dz \right) \]

\[ + \frac{\partial w}{\partial v} \left( dy - \frac{v}{1 + w} dz \right) \]

或

\[ (1 + u \frac{\partial w}{\partial u} + v \frac{\partial w}{\partial v}) dz = (1 + w) \frac{\partial w}{\partial u} dx \]
\[ (1 + w) \frac{\partial w}{\partial u} dy. \]

将由上式所确定的 \( \frac{\partial z}{\partial x} \) 及 \( \frac{\partial z}{\partial y} \) 代入原方程，得

\[
\left[ ue^{w}(1+w) \frac{\partial w}{\partial u} \right]^{2} + \left[ ve^{w}(1+w) \frac{\partial w}{\partial v} \right]^{2}
\]

\[ = \left( ve^{w}(1+w) \right)^{2} \frac{\partial w}{\partial u} \frac{\partial w}{\partial v}. \]

消去 \( v^{w}(1+w) \) 2，即得

\[ u^{2} \left( \frac{\partial w}{\partial u} \right)^{2} + v^{2} \left( \frac{\partial w}{\partial v} \right)^{2} = w^{2} \frac{\partial w}{\partial u} \frac{\partial w}{\partial v}. \]

3478. 假定 \( u = \ln \sqrt{x^{2} + y^{2}}, v = \arctg z \), \( w = x + y + z \),
其中 \( w = w(u, v) \)，变换式子

\( (x - y) : \left( \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right). \)

解：
\[ dw = dx + dy + dz = \frac{\partial w}{\partial u} du + \frac{\partial w}{\partial v} dv \]

\[ = \frac{\partial w}{\partial u} \left( \frac{xdx + ydy}{x^{2} + y^{2}} \right) + \frac{\partial w}{\partial v} \left( \frac{dz}{1 + z^{2}} \right). \]

于是，

\[ (1 - \frac{1}{1 + z^{2}} \frac{\partial w}{\partial u}) dz = \left( \frac{x}{x^{2} + y^{2}} \frac{\partial w}{\partial u} - 1 \right) dx \]

\[ + \left( \frac{y}{x^{2} + y^{2}} \frac{\partial w}{\partial u} - 1 \right) dy. \]
将由上式所确定的 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$ 代入所给式子，即得

$$
\frac{x-y}{\frac{\partial z}{\partial x}} = \frac{(x-y)(1 - \frac{1}{1 + z^2})}{\frac{\partial w}{\partial u}}
$$

$$
\frac{(1 - \cos^2 y)\frac{\partial w}{\partial u}}{\frac{\partial w}{\partial u}}
$$

3479. 假定 $u = xe^z, v = ye^z, w = ze^z$，其中 $w = w(u, v)$。变换式子

$$
A = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}
$$

解

$$
dw = e^z(1 + x) \, dz = \frac{\partial w}{\partial u} \, du + \frac{\partial w}{\partial v} \, dv
$$

$$
= \frac{\partial w}{\partial u} (e^z \, dx + xe^z \, dz) + \frac{\partial w}{\partial v} (e^z \, dy + ye^z \, dz).
$$

于是，

$$
(1 + z - x \frac{\partial w}{\partial u} - y \frac{\partial w}{\partial v}) \, dz = \frac{\partial w}{\partial u} \, dx + \frac{\partial w}{\partial v} \, dy,
$$

$$
\frac{\partial z}{\partial x} = \frac{\frac{\partial w}{\partial u}}{1 + z - x \frac{\partial w}{\partial u} - y \frac{\partial w}{\partial v}},
$$

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\[
\frac{\partial z}{\partial y} = \frac{\frac{\partial w}{\partial v}}{1 + z - x \frac{\partial w}{\partial u} - y \frac{\partial w}{\partial v}},
\]

\[
A = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial w}{\partial v}.
\]

3480. 在方程

\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u + \frac{xy}{z}
\]

中令：\( \xi = \frac{x}{z}, \eta = \frac{y}{z}, \zeta = z, w = \frac{u}{z} \).

其中 \( w = w(\xi, \eta, \zeta) \).

解

\[
dw = \frac{z du - u dz}{z^2} = \frac{\partial w}{\partial \xi} d\xi + \frac{\partial w}{\partial \eta} d\eta + \frac{\partial w}{\partial \zeta} d\zeta
\]

\[
= \frac{\partial w}{\partial \xi} \left( \frac{z dx - x dz}{z^2} \right) + \frac{\partial w}{\partial \eta} \left( \frac{z dy - y dz}{z^2} \right)
\]

\[
+ \frac{\partial w}{\partial \zeta} dz.
\]

两端同乘 \( z^2 \)，整理得

\[
z du = z \frac{\partial w}{\partial \xi} dx + z \frac{\partial w}{\partial \eta} dy + \left( u - x \frac{\partial w}{\partial \xi} - y \frac{\partial w}{\partial \eta} \right) dz
\]

+ \left( z^2 \frac{\partial w}{\partial \zeta} \right) dz.

将由上式所确定的 \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \) 及 \( \frac{\partial u}{\partial z} \) 代入原方程，得

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\[ x \frac{\partial w}{\partial \xi} + y \frac{\partial w}{\partial \eta} + (u - x \frac{\partial w}{\partial \xi} - y \frac{\partial w}{\partial \eta} + z^2 \frac{\partial w}{\partial \xi}) \]
\[ = u + \frac{x y}{z}, \]

即

\[ \frac{\partial w}{\partial \zeta} = \frac{x y}{z^2} = \frac{\xi \eta}{\zeta}. \]

假定 \( x = r \cos \varphi, \ y = r \sin \varphi \)，改变下列各式为极坐标 \( r \) 和 \( \varphi \) 所表示的式子。

3481. \( w = x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x}. \)

解

\[ dx = \cos \varphi dr - r \sin \varphi d\varphi, \]
\[ dy = \sin \varphi dr + r \cos \varphi d\varphi. \]

联立解之，得

\[ dr = \frac{x}{r} dx + \frac{y}{r} dy, \quad d\varphi = \frac{x}{r^2} dy - \frac{y}{r^2} dx. \]

于是，

\[ du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \varphi} d\varphi \]
\[ = \left( \frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi} \right) dx + \left( \frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi} \right) dy, \]

\[ \begin{cases} \frac{\partial u}{\partial x} = \frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi}, \\
\frac{\partial u}{\partial y} = \frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi}. \end{cases} \quad \text{公式9} \]
将公式 9 代入原式，即得

\[ w = x \left( \frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi} \right) - y \left( \frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi} \right) \]

\[ = \frac{\partial u}{\partial \varphi}. \]

3482. \( w = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}. \)

解 将公式 9 代入，即得

\[ w = x \left( \frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi} \right) + y \left( \frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi} \right) \]

\[ = r \frac{\partial u}{\partial r}. \]

3483. \( w = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2. \)

解 \( w = \left( \frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi} \right)^2 + \left( \frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi} \right)^2 \)

\[ = \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial u}{\partial \varphi} \right)^2. \]

3484. \( w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \)

解 先导出极坐标变换的所有二阶偏导函数的变换式。将 \( r, \varphi \) 看作中间变量，\( x, y \) 看作自变量。由于

\[ d^2 r = d(d r) = d \left( \frac{x}{r} d x + \frac{y}{r} d y \right) \]

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\[
\frac{1}{r} (d^2 x + dy^2) = \frac{xdx + ydy}{r^2} dr
\]

\[
= \frac{1}{r} (dx^2 + dy^2) - \frac{1}{r^3} (xdx + ydy)^2
\]

\[
= \frac{1}{r^3} (yd - xdx)^2,
\]

\[
d^2 \varphi = d^2 (d \varphi) = \frac{x}{r^2} dy - \frac{y}{r^2} dx
\]

\[
= - \frac{2(xdy - ydx)}{r^3} dr
\]

\[
= -\frac{2}{r^4} (xdy - ydx)(xdx + ydy),
\]

故有

\[
d^2 u = \frac{\partial^2 u}{\partial r^2} dr^2 + 2 \frac{\partial^2 u}{\partial r \partial \varphi} dr d\varphi + \frac{\partial^2 u}{\partial \varphi^2} d\varphi^2
\]

\[
+ \frac{\partial u}{\partial r} d^2 r + \frac{\partial u}{\partial \varphi} d^2 \varphi
\]

\[
= \frac{\partial^2 u}{\partial r^2} \left( \frac{xdx + ydy}{r} \right)^2 + 2 \frac{\partial^2 u}{\partial r \partial \varphi} \left( \frac{xdy - ydx}{r^2} \right)
\]

\[
\cdot \left( \frac{xdx + ydy}{r} \right) \left( \frac{xdy - ydx}{r^2} \right)
\]

\[
+ \frac{\partial^2 u}{\partial \varphi^2} \left( \frac{xdy - ydx}{r^2} \right)^2 + \frac{\partial u}{\partial r} \frac{(xdy - ydx)^2}{r^3}
\]

\[
+ \frac{\partial u}{\partial \varphi} \left( -\frac{2}{r^4} \right) (xdy - ydx)(xdx + ydy).
\]

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将上式右端按 $dx^2$, $dxdy$, $dy^2$ 合并同类项，并与全微分式

$$d^2u = \frac{\partial^2 u}{\partial x^2} dx^2 + 2 \frac{\partial^2 u}{\partial x \partial y} dxdy + \frac{\partial^2 u}{\partial y^2} dy^2$$

比较，即得

$$\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= \frac{x^2}{r^2} \frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3} \frac{\partial^2 u}{\partial r \partial \varphi} + \frac{y^2}{r^4} \frac{\partial^2 u}{\partial \varphi^2} \\
\frac{\partial^2 u}{\partial y^2} &= -\frac{y^2}{r^2} \frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3} \frac{\partial^2 u}{\partial r \partial \varphi} + \frac{x^2}{r^4} \frac{\partial^2 u}{\partial \varphi^2}
\end{align*}$$

$$+ \frac{y^2}{r^3} \frac{\partial u}{\partial r} + \frac{2xy}{r^4} \frac{\partial u}{\partial \varphi}$$

公式10

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{xy}{r^2} \frac{\partial^2 u}{\partial r^2} + \frac{x^2 - y^2}{r^3} \frac{\partial^2 u}{\partial r \partial \varphi} - \frac{xy}{r^4} \frac{\partial^2 u}{\partial \varphi^2}$$

$$- \frac{xy}{r^3} \frac{\partial u}{\partial r} - \frac{x^2 - y^2}{r^2} \frac{\partial u}{\partial \varphi}$$

将公式10代入原式，即得

$$w = -\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{r} \frac{\partial u}{\partial r}$$

3485. $w = x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$.

解  将公式10代入原式，化简整理得

$$w = r^2 \frac{\partial^2 u}{\partial r^2}$$
3486. \[ w = y \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} \]
\[-\left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}\right).\]

解：将公式10中的u换成z，然后代入原式，化简整理得
\[ w = \frac{\partial^2 z}{\partial \varphi^2}. \]

3487. 在式子
\[ I = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \]

中，令 \( x = r \cos \varphi, \ y = r \sin \varphi. \)

解：对函数u及v分别用公式9，即得
\[ I = \left(\frac{x}{r} \frac{\partial u}{\partial r} - \frac{y}{r^2} \frac{\partial u}{\partial \varphi}\right) \left(\frac{y}{r} \frac{\partial v}{\partial r} + \frac{x}{r^2} \frac{\partial v}{\partial \varphi}\right) \]
\[-\left(\frac{y}{r} \frac{\partial u}{\partial r} + \frac{x}{r^2} \frac{\partial u}{\partial \varphi}\right) \left(\frac{x}{r} \frac{\partial v}{\partial r} - \frac{y}{r^2} \frac{\partial v}{\partial \varphi}\right) \]
\[ = \frac{1}{r} \left(\frac{\partial u}{\partial r} \frac{\partial v}{\partial \varphi} - \frac{\partial u}{\partial \varphi} \frac{\partial v}{\partial r}\right). \]

3488. 引用新的自变量
\[ \xi = x - at, \ \eta = x + at. \]

解方程
\[ \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \]
解
\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = -a \frac{\partial u}{\partial \xi} + a \frac{\partial u}{\partial \eta}, \]

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-a \frac{\partial u}{\partial \xi} + a \frac{\partial u}{\partial \eta}), \]

\[ = a^2 \frac{\partial^2 u}{\partial \xi^2} - 2a^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 u}{\partial \eta^2}, \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}. \]

于是，由 \[ \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \]

得

\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \]

求之，得 \[ \frac{\partial u}{\partial \xi} = f(\xi), \]

从而

\[ u = \varphi(\xi) + \psi(\eta) = \varphi(x - at) + \psi(x + at), \]

其中 \( \varphi \)及 \( \psi \)为任意的函数

取 \( u \)及 \( v \)作新的自变量，变换下列方程

\[ 3489. \quad 2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0, \]

设 \( u = x + 2y + 2 \)及 \( v = x - y - 1 \).

解

\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \]
\[
\frac{\partial z}{\partial y} = 2 \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}.
\]
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2},
\]
\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = 2 \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2},
\]
\[
\frac{\partial^2 z}{\partial y^2} = 4 \frac{\partial^2 z}{\partial u^2} + 4 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}.
\]
代入原方程，化简整理即得
\[
3 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} = 0.
\]

3490. \((1 + x^2) \frac{\partial^2 z}{\partial x^2} + (1 + y^2) \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.\)

设 \(u = \ln(x + \sqrt{1 + x^2})\) 及 \(v = \ln(y + \sqrt{1 + y^2}).\)

则
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{du}{dx} = \frac{1}{\sqrt{1 + x^2}} \frac{\partial z}{\partial u},
\]
\[
\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1 + y^2}} \frac{\partial z}{\partial v},
\]
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial u} \right) = -\frac{x}{(1 + x^2)^{\frac{3}{2}}} \frac{\partial z}{\partial u} + \frac{1}{1 + x^2} \frac{\partial^2 z}{\partial u^2},
\]
\[
\frac{\partial^2 z}{\partial y^2} = -\frac{y}{(1 + y^2)^{\frac{3}{2}}} \frac{\partial z}{\partial v} + \frac{1}{1 + y^2} \frac{\partial^2 z}{\partial v^2}.
\]
代入原方程，化简整理得

\[ \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0. \]

\[ 3491^* : a x^2 \frac{\partial^2 z}{\partial x^2} + 2b x y \frac{\partial^2 z}{\partial x \partial y} + c y^2 \frac{\partial^2 z}{\partial y^2} = 0 \] (a, b, c 为常数)，设 \( u = \ln x, \; v = \ln y \).

解

\[ \frac{\partial z}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial u}, \quad \frac{\partial z}{\partial y} = \frac{1}{y} \frac{\partial z}{\partial v}, \]

\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy} \frac{\partial^2 z}{\partial u \partial v}, \]

\[ \frac{\partial^2 z}{\partial x^2} = -\frac{1}{x^2} \frac{\partial z}{\partial u} + \frac{1}{x^2} \frac{\partial^2 z}{\partial u^2}, \]

\[ \frac{\partial^2 z}{\partial y^2} = -\frac{1}{y^2} \frac{\partial z}{\partial v} + \frac{1}{y^2} \frac{\partial^2 z}{\partial v^2}. \]

代入原方程，化简整理得

\[ a \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u} \right) + 2b \frac{\partial^2 z}{\partial u \partial v} + c \left( \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v} \right) = 0. \]

\[ 3492 . \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \] 设 \( u = \frac{x}{x^2 + y^2}, \; v = -\frac{y}{x^2 + y^2}. \)

解

\[ \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \]

\[ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}. \]
\[
\begin{align*}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial^2 z}{\partial u^2} \left( \frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \\
+ \frac{\partial^2 z}{\partial v^2} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial x^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 z}{\partial u^2} \left( \frac{\partial u}{\partial y} \right)^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \\
+ \frac{\partial^2 z}{\partial v^2} \left( \frac{\partial v}{\partial y} \right)^2 + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial y^2}.
\end{align*}
\]

本题中，
\[
\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial u}{\partial x} = \frac{2xy}{(x^2 + y^2)^2},
\]
\[
\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial u}{\partial x},
\]
\[
\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial u}{\partial x},
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right)
\]
\[
= \frac{\partial}{\partial y} \left( -\frac{\partial u}{\partial y} \right) = -\frac{\partial^2 u}{\partial y^2},
\]

同法可得
\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}.
\]

注意到
\[
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2,
\]

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\[
\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y},
\]
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,
\]
则由公式11，即得
\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \cdot \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) = 0.
\]
由于\( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \neq 0 \)，故得变换后的方程
\[
\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0.
\]
3493. \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + m^2 z = 0 \)，设\( x = e^u \cos v, y = e^u \sin v \)。
解 由于 \( x = e^u \cos v, y = e^u \sin v \)，故有
\[
x^2 + y^2 = e^{2u}, \quad u = \ln \sqrt{x^2 + y^2},
\]
\[
\tan v = \frac{y}{x}, \quad v = \arctan \frac{y}{x} \quad (v的多值性不影响求导所得的结果)。于是，
\[
\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{x}{\partial y},
\]
\[
\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.
\]
由3492题得
\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + m^2 z = \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right]
\]
\[
\cdot \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) + m^2 z
\]
\[
= \left[ \frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} \right]
\]
\[
\cdot \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) + m^2 z
\]
\[
= e^{-2u} \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) + m^2 z = 0,
\]
即
\[
\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + m^2 e^{2u} z = 0.
\]
3494.  \(\frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} = \frac{1}{2} \frac{\partial z}{\partial y} \) \((y \gg 0)\), 设 \(u = x - 2\sqrt{y}\) 及 \(v = \frac{1}{2} \sqrt{y}\)。

解
\[
\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{y}}, \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{y}},
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{2y^2}, \quad \frac{\partial^2 v}{\partial y^2} = -\frac{1}{2y^2}.
\]

由公式11得

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\[ \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}, \]

\[ \frac{\partial^2 z}{\partial y^2} = \frac{1}{2y^2} \frac{\partial z}{\partial u} - \frac{1}{2y^2} \frac{\partial z}{\partial v} + \frac{1}{y} \frac{\partial^2 z}{\partial u^2} \]

\[ -\frac{2}{y} \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y} \frac{\partial^2 z}{\partial v^2}, \]

\[ \frac{\partial z}{\partial y} = -\frac{1}{\sqrt{y}} \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v}. \]

代入原方程，化简整理得

\[ \frac{\partial^2 z}{\partial u \partial v} = 0. \]

3495. \[ x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = 0, \text{ 设 } u = xy, v = \frac{x}{y}. \]

解

\[ \frac{\partial u}{\partial x} = y, \quad \frac{\partial v}{\partial x} = \frac{1}{y}, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial y} = \frac{x}{y^2}, \]

\[ \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = \frac{2x}{y^3}. \]

由公式11得

\[ \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2} \frac{\partial^2 z}{\partial v^2}, \]

\[ \frac{\partial^2 z}{\partial y^2} = x^2 \frac{\partial^2 z}{\partial u^2} - 2x^2 \frac{\partial^2 z}{\partial u \partial v}. \]

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\[
+ \frac{x^2}{y^4} \frac{\partial^2 z}{\partial v^2} + \frac{2x}{y^3} \frac{\partial z}{\partial v}.
\]

代入原方程，化简整理得

\[
\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{2u} \frac{\partial z}{\partial v}.
\]

3496. \( x^2 \frac{\partial^2 z}{\partial x^2} - (x^2 + y^2) \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0 \),

设\( u = x + y \), \( v = \frac{1}{x} + \frac{1}{y} \).

解：
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} - \frac{1}{x^2} \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{1}{y^2} \frac{\partial z}{\partial v}.
\]

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} - \frac{2}{x^2} \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{x^4} \frac{\partial^2 z}{\partial v^2} + \frac{2}{x^8} \frac{\partial z}{\partial v},
\]

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} - \frac{2}{y^2} \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^4} \frac{\partial^2 z}{\partial v^2} + \frac{2}{y^8} \frac{\partial z}{\partial v},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - \left( \frac{1}{x^2} + \frac{1}{y^2} \right) \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{x^2 y^2} \frac{\partial^2 z}{\partial v^2}.
\]

代入原方程，得

\[
\frac{(x^2 - y^2)^2}{x^2 y^2} \frac{\partial^2 z}{\partial u \partial v} + 2 \left( \frac{1}{x} + \frac{1}{y} \right) \frac{\partial z}{\partial v} = 0.
\]

注意到\( v = \frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy} = \frac{u}{xy} \)，即\( xy \cdot \frac{u}{v} \)，于是

就有

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\[
\frac{(x^2 - y^2)^2}{x^2 y^2} = \left(\frac{x+y}{x^2 y^2}\right)^2 (x-y)^2 \\
= \left(\frac{1}{x} + \frac{1}{y}\right)^2 ((x+y)^2 - 4xy) \\
= uv^2 \left(u^2 - 4\frac{v}{u}\right) = uv(uv - 4).
\]

从而得变换后的方程

\[
\frac{\partial^2 z}{\partial u \partial v} = \frac{2}{u(4-uv)} \frac{\partial z}{\partial v}.
\]

3497. \(xy\frac{\partial^2 z}{\partial x^2} - (x^2 + y^2) \frac{\partial^2 z}{\partial x \partial y} + xy \frac{\partial^2 z}{\partial y^2} + y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y}\)

\(= 0\)，设\(u = \frac{1}{2}(x^2 + y^2)\)及\(v = xy\)。

解 \(\frac{\partial z}{\partial x} = x \frac{\partial z}{\partial u} + y \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v},\)

\[
\frac{\partial^2 z}{\partial x^2} = x \frac{\partial^2 z}{\partial u^2} + 2xy \frac{\partial^2 z}{\partial u \partial v} + y^2 \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u},
\]

\[
\frac{\partial^2 z}{\partial y^2} = y \frac{\partial^2 z}{\partial u^2} + 2xy \frac{\partial^2 z}{\partial u \partial v} + x^2 \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = xy \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) + (x^2 + y^2)
\]

\[
\cdot \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v}.
\]

代入原方程，得
\[
((x^2 + y^2)^2 - 4x^2y^2)\frac{\partial^2 z}{\partial u \partial v} - 4xy\frac{\partial z}{\partial u},
\]

即

\[
(u^2 - v^2)\frac{\partial^2 z}{\partial u \partial v} = v\frac{\partial z}{\partial u}.
\]

3498. \( x^2 \frac{\partial^2 z}{\partial x^2} - 2x\sin y \frac{\partial^2 z}{\partial x \partial y} + \sin^2 y \frac{\partial^2 z}{\partial y^2} = 0 \),

设\( u = x\tan \frac{y}{2} \), \( v = x \).

解

\[
\frac{\partial z}{\partial x} = \tan \frac{y}{2} \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{x}{2} \sec^2 \frac{y}{2} \frac{\partial z}{\partial u},
\]

\[
\frac{\partial^2 z}{\partial x^2} = \tan^2 \frac{y}{2} \frac{\partial^2 z}{\partial u^2} + 2\tan \frac{y}{2} \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2},
\]

\[
\frac{\partial^2 z}{\partial y^2} = \frac{x}{2} \sec^2 \frac{y}{2} \tan \frac{y}{2} \frac{\partial z}{\partial u} + \frac{x^2}{4} \sec^4 \frac{y}{2} \frac{\partial^2 z}{\partial u^2},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2} \sec^2 \frac{y}{2} \frac{\partial z}{\partial u} + \frac{x}{2} \sec^2 \frac{y}{2} \tan \frac{y}{2} \frac{\partial^2 z}{\partial u^2}
\]

\[
+ \frac{x}{2} \sec^2 \frac{y}{2} \frac{\partial^2 z}{\partial u \partial v}.
\]

代入原方程，得

\[
x^2 \frac{\partial^2 z}{\partial v^2} = \left( x\sin y \sec^2 \frac{y}{2} - \frac{x}{2} \sin^2 y \sec^2 \frac{y}{2} \tan \frac{y}{2} \right)
\]

\[
\frac{\partial z}{\partial u} = \left( 2x \tan \frac{y}{2} - 2x \tan \frac{y}{2} \sin \frac{y}{2} \right) \frac{\partial z}{\partial u}.
\]
$$-2x \cdot \tan \frac{y}{2} \cos^2 \frac{y}{2} \frac{\partial z}{\partial u} = \frac{2x \cdot \tan \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} \frac{\partial z}{\partial u},$$

即

$$\frac{\partial^2 z}{\partial v^2} = \frac{2u}{u^2 + v^2} \frac{\partial z}{\partial u}.$$  

3499. \(x \frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} = 0 \quad (x \gg 0, \ y \gg 0), \) 设 \(x = (u + v)^2\) 及 \(y = (u - v)^2.\)

解 由 \(x = (u + v)^2\) 及 \(y = (u - v)^2\) 分别对 \(x\) 及对 \(y\) 求偏导函数，得

$$\begin{cases} 1 = 2 (u + v) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right), \\ 0 = 2 (u - v) \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right); \end{cases}$$

$$\begin{cases} 0 = 2 (u + v) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right), \\ 1 = 2 (u - v) \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right). \end{cases}$$

解得

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{1}{4(u + v)}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = -\frac{1}{4(u - v)}. $$

于是，
\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{1}{4(u+v)} \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right),
\]
\[
\frac{\partial z}{\partial y} = \frac{1}{4(u-v)} \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right),
\]
\[
\frac{\partial^2 z}{\partial x^2} = -\frac{1}{4(u+v)^2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)
+ \frac{1}{4(u+v)} \left( \frac{\partial^2 z}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2} \frac{\partial u}{\partial x} \right)
\]
\[
= -\frac{1}{8(u+v)^3} \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + \frac{1}{16(u+v)^2} \cdot \left( \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right).
\]

同法可求得
\[
\frac{\partial^2 z}{\partial y^2} = \frac{1}{8(u-v)^3} \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + \frac{1}{16(u-v)^2} \cdot \left( \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right).
\]

代入原方程，得
\[
x \frac{\partial^2 z}{\partial x^2} - y \frac{\partial^2 z}{\partial y^2} = -\frac{1}{8(u+v)} \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)
+ \frac{1}{16} \left( \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right)
\]
\[
\frac{1}{8(u-v)} \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \\
- \frac{1}{16} \left( \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \\
= \frac{1}{16} \left( \frac{4u}{u^2-v^2} \frac{\partial z}{\partial u} - \frac{4u^2}{u^2-v^2} \frac{\partial z}{\partial v} + 4 \frac{\partial^2 z}{\partial u \partial v} \right) = 0,
\]

即

\[
\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{u^2-v^2} \left( v \frac{\partial z}{\partial u} - u \frac{\partial z}{\partial v} \right) = 0.
\]

3500. \( \frac{\partial^2 z}{\partial x \partial y} = \left( 1 + \frac{\partial z}{\partial y} \right)^3 \)，设 \( u = x, v = y+z \)。

解 由 \( u = x, v = y+z \) 得
\[
du = dx, \quad dv = dy + dz,
\]

\[
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{\partial z}{\partial u} dx + \frac{\partial z}{\partial v} (dy + dz).
\]

于是，

\[
\left( 1 - \frac{\partial z}{\partial v} \right) dz = \frac{\partial z}{\partial u} dx + \frac{\partial z}{\partial v} dy,
\]

\[
\frac{\partial z}{\partial x} = \frac{\frac{\partial z}{\partial u}}{1 - \frac{\partial z}{\partial v}}, \quad \frac{\partial z}{\partial y} = \frac{\frac{\partial z}{\partial v}}{1 - \frac{\partial z}{\partial v}}.
\]

\[
1 + \frac{\partial z}{\partial y} = 1 + \frac{\frac{\partial z}{\partial v}}{1 - \frac{\partial z}{\partial v}} = \frac{1}{1 - \frac{\partial z}{\partial v}}.
\] (1)
又

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( 1 + \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{1}{1 - \frac{\partial z}{\partial v}} \right)
\]

\[
= \frac{1}{\left( 1 - \frac{\partial z}{\partial v} \right)^2} \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial v} \right)
\]

\[
= \frac{1}{\left( 1 - \frac{\partial z}{\partial v} \right)^2} \left( \frac{\partial^2 z}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2} \frac{\partial v}{\partial x} \right)
\]

\[
= \frac{1}{\left( 1 - \frac{\partial z}{\partial v} \right)^2} \left( \frac{\partial^2 z}{\partial u \partial v} \left( 1 - \frac{\partial z}{\partial v} \right) + \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial u} \right).
\] (2)

将（1）式和（2）式代入原方程，去分母即得

\[
\left( 1 - \frac{\partial z}{\partial v} \right) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} \frac{\partial^2 z}{\partial v^2} = 1.
\]

3501. 利用线性变换

\[
\xi = x + \lambda_1 y, \quad \eta = x + \lambda_2 y
\]

变换方程

\[
A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0,
\] (1)

（其中 \( A, B \) 和 \( C \) 为常数及 \( C \neq 0, \ AC-B^2 < 0 \)）

为下面的形状

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\[
\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.
\]

求满足方程（1）的函数的普遍形制。

解

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \quad \frac{\partial u}{\partial y} = \lambda_1 \frac{\partial u}{\partial \xi} + \lambda_2 \frac{\partial u}{\partial \eta},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \lambda_1 \frac{\partial^2 u}{\partial \xi^2} + (\lambda_1 + \lambda_2) \frac{\partial^2 u}{\partial \xi \partial \eta} + \lambda_2 \frac{\partial^2 u}{\partial \eta^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \lambda_1^2 \frac{\partial^2 u}{\partial \xi^2} + 2 \lambda_1 \lambda_2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \lambda_2^2 \frac{\partial^2 u}{\partial \eta^2}.
\]

将上述结果代入原方程，得

\[
(A + 2B \lambda_1 + C \lambda_1^2) \frac{\partial^2 u}{\partial \xi^2} + 2(A + B(\lambda_1 + \lambda_2)
\]

\[
+ C \lambda_1 \lambda_2) \frac{\partial^2 u}{\partial \xi \partial \eta} + (A + 2B \lambda_2 + C \lambda_2^2) \frac{\partial^2 u}{\partial \eta^2} = 0.
\]

当

\[
A + 2B \lambda_1 + C \lambda_1^2 = 0 \quad \text{及} \quad A + 2B \lambda_2 + C \lambda_2^2 = 0.
\]

即 \(\lambda_1\) 与 \(\lambda_2\) 为方程

\[
A + 2B \lambda + C \lambda^2 = 0
\]

的根时（注意，由假定 \(C \neq 0\), \(AC - B^2 < 0\)，故此方程恰有两个相异的实根），原方程变换为

\[
(A + B(\lambda_1 + \lambda_2) + C \lambda_1 \lambda_2) \frac{\partial^2 u}{\partial \xi \partial \eta} = 0.
\]

由根与系数的关系得：\(\lambda_1 + \lambda_2 = -\frac{2B}{C}\), \(\lambda_1 \lambda_2 = \frac{A}{C}\).  

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于是，

\[ A + B(\lambda_1 + \lambda_2) + C\lambda_1 \lambda_2 = \frac{2(AC - B^2)}{C} \neq 0. \]

从而必有

\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \]

此时，\[ \frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta} \left( \frac{\partial u}{\partial \xi} \right) = 0, \quad \text{故} \quad \frac{\partial u}{\partial \xi} = f(\xi) \]

\[ u = \int f(\xi) d\xi \psi(\eta) = \varphi(\xi) + \psi(\eta) \]

\[ = \varphi(x + \lambda_1 y) + \psi(x + \lambda_2 y). \]

3502. 证明拉普拉斯方程

\[ \Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \]

在满足条件\[ \frac{\partial \varphi}{\partial u} = \frac{\partial \psi}{\partial v}, \quad \frac{\partial \varphi}{\partial v} = -\frac{\partial \psi}{\partial u} \]

的非退化的变数代换

\[ x = \varphi(u, v), \quad y = \psi(u, v) \]

下形式不变。

证

\[ \begin{cases} 
  dx = \frac{\partial \varphi}{\partial u} du + \frac{\partial \varphi}{\partial v} dv, \\
  dy = \frac{\partial \psi}{\partial u} du + \frac{\partial \psi}{\partial v} dv = -\frac{\partial \varphi}{\partial v} du + \frac{\partial \varphi}{\partial u} dv.
\end{cases} \]

令\[ I = \left( \frac{\partial \varphi}{\partial u} \right)^2 + \left( \frac{\partial \varphi}{\partial v} \right)^2. \]由于变换是非退化的，故知
\[
\frac{D(x, y)}{D(u, v)} = \begin{vmatrix}
\frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\
\frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v}
\end{vmatrix} = \left(\frac{\partial \varphi}{\partial u}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2 \neq 0 .
\]

由上述方程组解得
\[
du = \frac{1}{l} \left( \frac{\partial \varphi}{\partial u} dx - \frac{\partial \varphi}{\partial v} dy \right),
\]
\[
dv = \frac{1}{l} \left( \frac{\partial \varphi}{\partial v} dx + \frac{\partial \varphi}{\partial u} dy \right).
\]

于是，
\[
\frac{\partial u}{\partial x} = \frac{1}{l} \frac{\partial \varphi}{\partial u} = \frac{1}{l} \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{1}{l} \frac{\partial \varphi}{\partial v} = -\frac{\partial v}{\partial x} .
\]

由3492题的证明及公式11，并考虑到
\[
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \frac{1}{l^2} \left[ \left( \frac{\partial \varphi}{\partial u} \right)^2 + \left( \frac{\partial \varphi}{\partial v} \right)^2 \right] = \frac{1}{l} ,
\]
即得
\[
\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}
\]
\[
= \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right]\left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)
\]
\[
= \frac{1}{l} \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) = 0 ,
\]
或

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\[
\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0 ,
\]

即形式是不变的。

3503. 假定 \( u = f(r) \)，其中 \( r = \sqrt{x^2 + y^2} \)，改变方程

(a) \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \);  (b) \( \Delta (\Delta u) = 0 \).

解 (a) \( \frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{x}{r} \), \( \frac{\partial u}{\partial y} = f'(r) \frac{y}{r} \). 于是，

\[
\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x} \left( f'(r) \frac{x}{r} \right) = \frac{f''(r) r - xf'(r)}{r^2}
\]

\[
+ \frac{x^2}{r^2} f''(r) + x f'(r) \cdot \left( -\frac{x}{r^3} \right)
\]

\[
= \frac{x^2 f''(r)}{r^2} + \frac{y^2}{r^2} f'(r).
\]

同法可得

\[
\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) + \frac{x^2}{r^2} f'(r).
\]

于是，

\[
\Delta u = f''(r) + \frac{1}{r} f'(r) = \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = 0 ,
\]

也可写成 \( \Delta u = \frac{1}{r} \frac{d}{dr} (r \frac{du}{dr}) = 0 \).
\[
\begin{align*}
\Delta (\Delta u) &= \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} (\Delta u) \right] \\
&= \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \left( \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right) \right] \\
&= \frac{1}{r} \frac{d}{dr} \left[ r \frac{d^3 u}{dr^3} + \frac{d^2 u}{dr^2} - \frac{1}{r} \frac{du}{dr} \right] \\
&= \frac{d^4 u}{dr^4} + \frac{2}{r} \frac{d^3 u}{dr^3} - \frac{1}{r^2} \frac{d^2 u}{dr^2} + \frac{1}{r^3} \frac{du}{dr} = 0. 
\end{align*}
\]

3504. 若令
\[ w = f(u), \]
其中
\[ u = (x-x_0)(y-y_0), \]
方程
\[ \frac{\partial^2 w}{\partial x \partial y} + cw = 0 \]
变成怎样的形状？

解
\[ \frac{\partial w}{\partial x} = (y-y_0) \frac{dw}{du}, \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{dw}{du} + u \frac{d^2 w}{du^2}. \]

于是，方程
\[ \frac{\partial^2 w}{\partial x \partial y} + cw = 0 \]
变换成
\[ u \frac{d^2 w}{du^2} + \frac{dw}{du} + cw = 0. \]

3505. 假定
\[ x + y = X, \quad y = XY, \]
变换式子
\[ A = x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}. \]
解  \( X = x + y, \ Y = \frac{y}{X} = \frac{y}{x + y} = 1 - \frac{x}{x + y} \). 于

是，\( \frac{\partial X}{\partial x} = 1, \ \frac{\partial X}{\partial y} = 1, \ \frac{\partial Y}{\partial x} = -\frac{y}{(x + y)^2}, \)

\[ \frac{\partial Y}{\partial y} = \frac{x}{(x + y)^2}, \]

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} - \frac{y}{(x + y)^2} \frac{\partial u}{\partial Y}, \]

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial X^2} - \frac{2y}{(x + y)^2} \frac{\partial^2 u}{\partial X \partial Y} \]

\[ + \frac{y^2}{(x + y)^4} \frac{\partial^2 u}{\partial Y^2} + \frac{2y}{(x + y)^4} \frac{\partial u}{\partial Y}, \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial X^2} + \frac{x - y}{(x + y)^2} \frac{\partial^2 u}{\partial X \partial Y} \]

\[ - \frac{xy}{(x + y)^4} \frac{\partial^2 u}{\partial Y^2} - \frac{x - y}{(x + y)^4} \frac{\partial u}{\partial Y}. \]

代入所给式子，得

\[ A = X \frac{\partial^2 u}{\partial X^2} - Y \frac{\partial^2 u}{\partial X \partial Y} + \frac{\partial u}{\partial X}. \]

3506. 证明：方程

\[ \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial z}{\partial x} + 2(y - y^2) \frac{\partial z}{\partial y} + x^2y^2z^2 = 0 \]

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在变换  \( x = uv \) 及  \( y = \frac{1}{v} \) 下形状不变。

证  \( v = \frac{1}{y} \), \( u = \frac{x}{v} = xy \). 于是，

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = y \frac{\partial z}{\partial u},
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = x \frac{\partial z}{\partial u} - \frac{1}{y^2} \frac{\partial z}{\partial v},
\]

\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( y \frac{\partial z}{\partial u} \right) = y^2 \frac{\partial^2 z}{\partial u^2}.
\]

代入原方程，得

\[
y^2 \frac{\partial^2 z}{\partial u^2} + 2xy \frac{\partial z}{\partial u} + 2x(y - y^3) \frac{\partial z}{\partial u} - 2(y - y^3)
\]

\[
\frac{1}{y^2} \frac{\partial z}{\partial v} + x^2 y^2 z^2 = 0,
\]

即

\[
\frac{\partial^2 z}{\partial u^2} + 2uv^2 \frac{\partial z}{\partial u} + 2(v - v^3) \frac{\partial z}{\partial v} + u^2 v^2 z^2 = 0,
\]

故其形状不变。

3507. 证明：方程

\[
\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0
\]
在变换 \( u = x + z \) 及 \( v = y + z \)

下形状不变。

证 将 \( u, v \) 作中间变量，\( x, y \) 作自变量，微分得

\[
du = dx + dz, \quad dv = dy + dz, \quad d^2u = d^2v = d^2z.
\]

于是，

\[
dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) dz
\]

\[
+ \frac{\partial z}{\partial u} dx + \frac{\partial z}{\partial v} dy.
\]

令 \( A = 1 - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \)，则有

\[
dz = \frac{1}{A} \frac{\partial z}{\partial u} dx + \frac{1}{A} \frac{\partial z}{\partial v} dy，且
\]

\[
\frac{\partial z}{\partial x} = \frac{1}{A} \frac{\partial z}{\partial u}，\quad \frac{\partial z}{\partial y} = \frac{1}{A} \frac{\partial z}{\partial v}.
\]

从而有

\[
du = dx + dz = \frac{1}{A} \frac{\partial z}{\partial u} dx + \frac{\partial z}{\partial v} dy,
\]

\[
dv = dy + dz = \frac{\partial z}{\partial u} dx + \frac{1}{A} \frac{\partial z}{\partial u} dy,
\]

\[
d^2z = \frac{\partial^2 z}{\partial u^2} du^2 + 2 \frac{\partial^2 z}{\partial u \partial v} du dv + \frac{\partial^2 z}{\partial v^2} dv^2
\]

\[
+ \frac{\partial z}{\partial u} d^2u + \frac{\partial z}{\partial v} d^2v.
\]

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上面最后一个等式即

\[
Ad^2 z = \frac{1}{A^2} \left[ \frac{\partial^2 z}{\partial u^2} \left[ \left( 1 - \frac{\partial z}{\partial v} \right) dx + \frac{\partial z}{\partial v} dy \right]^2 + 2 \frac{\partial^2 z}{\partial u \partial v} \left[ \left( 1 - \frac{\partial z}{\partial v} \right) dx + \frac{\partial z}{\partial v} dy \right] \cdot \left[ \frac{\partial z}{\partial u} dx + \left( 1 - \frac{\partial z}{\partial v} \right) dy \right] + \frac{\partial^2 z}{\partial v^2} \left[ \frac{\partial z}{\partial u} dx \right] \right. \\
+ \left. \left( 1 - \frac{\partial z}{\partial u} \right) dy \right]^2 \right] \}
\]

于是，

\[
\frac{\partial^2 z}{\partial x^2} = \frac{1}{A^3} \left[ \left( 1 - \frac{\partial z}{\partial v} \right)^2 \frac{\partial^2 z}{\partial u^2} + 2 \left( 1 - \frac{\partial z}{\partial v} \right) \right. \\
\cdot \frac{\partial z}{\partial u} \frac{\partial^2 z}{\partial u \partial v} + \left( \frac{\partial z}{\partial u} \right)^2 \frac{\partial^2 z}{\partial u^2} \bigg] 
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{A^3} \left[ \frac{\partial z}{\partial v} \left( 1 - \frac{\partial z}{\partial v} \right) \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \frac{\partial^2 z}{\partial u \partial v} \\
+ \left( 1 - \frac{\partial z}{\partial u} \right) \left( 1 - \frac{\partial z}{\partial v} \right) \frac{\partial^2 z}{\partial u \partial v} \\
+ \frac{\partial z}{\partial u} \left( 1 - \frac{\partial z}{\partial u} \right) \frac{\partial^2 z}{\partial v^2} \bigg] 
\]

\[
\frac{\partial^2 z}{\partial y^2} = \frac{1}{A^3} \left[ \left( \frac{\partial z}{\partial v} \right)^2 \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial z}{\partial v} \left( 1 - \frac{\partial z}{\partial u} \right) \right. \\
\cdot \frac{\partial^2 z}{\partial u \partial v} + \left( 1 - \frac{\partial z}{\partial u} \right)^2 \frac{\partial^2 z}{\partial v^2} \bigg] 
\]
代入原方程，化简整理即得
\[ \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = 0, \]
故其形状不变。

3508．假定
\[ x = \eta \zeta, \ y = \xi \zeta, \ z = \xi \eta, \]
变换方程
\[ xy \frac{\partial^2 u}{\partial x \partial y} + yz \frac{\partial^2 u}{\partial y \partial z} + xz \frac{\partial^2 u}{\partial x \partial z} = 0. \]
解 由于
\[
\begin{cases}
1 = \xi \frac{\partial \eta}{\partial x} + \eta \frac{\partial \xi}{\partial x}, \\
0 = \xi \frac{\partial \xi}{\partial x} + \xi \frac{\partial \eta}{\partial x}, \\
0 = \eta \frac{\partial \xi}{\partial x} + \xi \frac{\partial \eta}{\partial x},
\end{cases}
\]
故有
\[ \frac{\partial \xi}{\partial x} = -\frac{\xi}{2\eta \zeta}, \quad \frac{\partial \eta}{\partial x} = \frac{1}{2\zeta}, \quad \frac{\partial \xi}{\partial x} = \frac{1}{2\eta}. \]
同法求得
\[ \frac{\partial \xi}{\partial y} = \frac{1}{2\zeta}, \quad \frac{\partial \eta}{\partial y} = -\frac{\eta}{2\xi \zeta}, \quad \frac{\partial \xi}{\partial y} = \frac{1}{2\xi}. \]
于是，
\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x}. \]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = -\frac{\partial}{\partial y} \left( \frac{\xi}{2\eta \zeta} \right) \frac{\partial u}{\partial \zeta} \\
-\frac{\xi}{2n \zeta} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial \zeta} \right) + \frac{\partial}{\partial y} \left( \frac{1}{2 \zeta} \right) \frac{\partial u}{\partial \eta} \\
+ \frac{1}{2 \zeta} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial \eta} \right) + \frac{\partial}{\partial y} \left( \frac{1}{2 \eta} \right) \frac{\partial u}{\partial \zeta} + \frac{1}{2 \eta} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial \zeta} \right) \\
= -\frac{1}{4\eta \zeta^2} \frac{\partial u}{\partial \xi} - \frac{\xi}{4\eta \zeta^2} \frac{\partial^2 u}{\partial \xi^2} \\
-\frac{1}{4\xi \zeta^2} \frac{\partial u}{\partial \eta} - \frac{\eta}{4\xi \zeta^2} \frac{\partial^2 u}{\partial \eta^2} \\
+ \frac{1}{4\xi \eta \zeta} \frac{\partial u}{\partial \zeta} + \frac{1}{4\xi \eta} \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{2 \xi^2} \frac{\partial^2 u}{\partial \xi \partial \zeta}.
\]

(1)

同法可求得

\[
\frac{\partial^2 u}{\partial y \partial z} = \frac{1}{4\xi \eta \zeta} \frac{\partial u}{\partial \xi} + \frac{1}{4\eta \zeta} \frac{\partial^2 u}{\partial \xi^2} \\
-\frac{1}{4\xi^2 \zeta} \frac{\partial u}{\partial \eta} - \frac{\eta}{4\xi^2 \zeta} \frac{\partial^2 u}{\partial \eta^2} \\
-\frac{1}{4\xi^2 \eta} \frac{\partial u}{\partial \zeta} - \frac{\xi}{4\xi^2 \eta} \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{2 \xi^2} \frac{\partial^2 u}{\partial \eta \partial \zeta},
\]

(2)

\[
\frac{\partial^2 u}{\partial z \partial x} = \frac{1}{4\eta^2 \zeta} \frac{\partial u}{\partial \zeta} - \frac{\xi}{4\eta^2 \zeta} \frac{\partial^2 u}{\partial \zeta^2}
\]

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+ \frac{1}{4\xi \eta^2} \frac{\partial u}{\partial \eta} + \frac{1}{4\xi \zeta} \frac{\partial^2 u}{\partial \eta^2} \]

\[ - \frac{1}{4\eta^2 \xi} \frac{\partial u}{\partial \zeta} - \frac{\xi}{4\eta^2 \xi} \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{2\eta^2} \frac{\partial^2 u}{\partial \zeta \partial \xi}. \quad (3) \]

将 (1), (2), (3) 三式连同 x, y, z 一起代入原方程，化简整理得

\[ \xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta} + \xi^2 \frac{\partial^2 u}{\partial \xi^2} + \eta^2 \frac{\partial^2 u}{\partial \eta^2} + \zeta^2 \frac{\partial^2 u}{\partial \zeta^2} \]

\[ = 2 \left( \xi \eta \frac{\partial^2 u}{\partial \xi \partial \eta} + \eta \zeta \frac{\partial^2 u}{\partial \eta \partial \zeta} + \xi \zeta \frac{\partial^2 u}{\partial \xi \partial \zeta} \right), \]

即

\[ \xi \frac{\partial}{\partial \xi} \left( \xi \frac{\partial u}{\partial \xi} \right) + \eta \frac{\partial}{\partial \eta} \left( \eta \frac{\partial u}{\partial \eta} \right) + \zeta \frac{\partial}{\partial \zeta} \left( \zeta \frac{\partial u}{\partial \zeta} \right) \]

\[ = 2 \left( \xi \eta \frac{\partial^2 u}{\partial \xi \partial \eta} + \eta \zeta \frac{\partial^2 u}{\partial \eta \partial \zeta} + \xi \zeta \frac{\partial^2 u}{\partial \xi \partial \zeta} \right). \]

3509. 假定

\[ y_1 = x_2 + x_3 - x_1, \quad y_2 = x_1 + x_3 - x_2, \]
\[ y_3 = x_1 + x_2 - x_3, \]

变换方程

\[ \frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} + \frac{\partial^2 z}{\partial x_3^2} + \frac{\partial^2 z}{\partial x_1 \partial x_2} \]

\[ + \frac{\partial^2 z}{\partial x_1 \partial x_3} + \frac{\partial^2 z}{\partial x_2 \partial x_3} = 0. \]

解 不难看出

304.
\[
\frac{\partial z}{\partial x_1} = \left(-\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right)z,
\]

\[
\frac{\partial z}{\partial x_2} = \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right)z,
\]

\[
\frac{\partial z}{\partial x_3} = \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3}\right)z.
\]

把上述结果代入所给方程的左端，即得

\[
\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} + \frac{\partial^2 z}{\partial x_3^2} + \frac{\partial^2 z}{\partial x_1 \partial x_2} + \frac{\partial^2 z}{\partial x_1 \partial x_3} + \frac{\partial^2 z}{\partial x_2 \partial x_3}
\]

\[= \frac{\partial}{\partial x_1} \left(\frac{\partial z}{\partial x_1} + \frac{\partial z}{\partial x_2}\right) + \frac{\partial}{\partial x_2} \left(\frac{\partial z}{\partial x_2} + \frac{\partial z}{\partial x_3}\right) + \frac{\partial}{\partial x_3} \left(\frac{\partial z}{\partial x_3} + \frac{\partial z}{\partial x_1}\right)
\]

\[= \frac{\partial}{\partial x_1} \left(2 \frac{\partial z}{\partial y_3}\right) + \frac{\partial}{\partial x_2} \left(2 \frac{\partial z}{\partial y_1}\right) + \frac{\partial}{\partial x_3} \left(2 \frac{\partial z}{\partial y_2}\right)
\]

\[= 2 \left[\left(-\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right) \frac{\partial z}{\partial y_1} + \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\right) \frac{\partial z}{\partial y_3}\right]
\]

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\[
+ \left( \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} \right) \frac{\partial z}{\partial y_2} \right)
\]
\[
= 2 \left( \frac{\partial^2 z}{\partial y_1^2} + \frac{\partial^2 z}{\partial y_2^2} + \frac{\partial^2 z}{\partial y_3^2} \right).
\]
从而原方程变换为
\[
\frac{\partial^2 z'}{\partial y_1^2} + \frac{\partial^2 z}{\partial y_2^2} + \frac{\partial^2 z}{\partial y_3^2} = 0.
\]
3510. 假定
\[
\xi = \frac{y}{x}, \eta = \frac{z}{x}, \xi = y - z,
\]
变换方程
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2xy \frac{\partial^2 u}{\partial x \partial y}
\]
\[
+ 2xz \frac{\partial^2 u}{\partial x \partial z} + 2yz \frac{\partial^2 u}{\partial y \partial z} = 0.
\]
解 定义算子 \( A \):
\[
Au = \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) u.
\]
则有
\[
A^2 u = A(Au) = x \frac{\partial}{\partial x} (Au) + y \frac{\partial}{\partial y} (Au)
\]
\[
+ z \frac{\partial}{\partial z} (Au)
\]
\[-x\left(\frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial x \partial y} + z \frac{\partial^2}{\partial x \partial z} + \frac{\partial}{\partial x}\right)u\]

\[+ y\left(\frac{\partial^2}{\partial x \partial y} + y \frac{\partial^2}{\partial y^2} + z \frac{\partial^2}{\partial y \partial z} + \frac{\partial}{\partial y}\right)u\]

\[+ z\left(\frac{\partial^2}{\partial x \partial z} + y \frac{\partial^2}{\partial y \partial z} + z \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial z}\right)u,\]

\[= \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right)^2 u + Au.\]

于是，原方程可改写成

\[\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right)^2 u = 0 \quad \text{或} \quad A^2 u - Au = 0.\]

但是，

\[Au = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\]

\[= x\left(-\frac{y}{x^2} \frac{\partial u}{\partial \xi} - \frac{z}{x^2} \frac{\partial u}{\partial \eta}\right) + y \left(\frac{1}{x} \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \zeta}\right)\]

\[+ z\left(\frac{1}{x} \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \zeta}\right)\]

\[= (y - z) \frac{\partial u}{\partial \zeta} = \zeta \frac{\partial u}{\partial \zeta},\]

\[A^2 u = A(Au) = \left(\zeta \frac{\partial}{\partial \zeta}\right)Au = \zeta \frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial u}{\partial \zeta}\right)\]

\[= \zeta^2 \frac{\partial^2 u}{\partial \zeta^2} + \zeta \frac{\partial u}{\partial \zeta},\]

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从而 \( A^2u - Au = \xi^2 \frac{\partial^2 u}{\partial \xi^2} \). 由于 \( \xi \neq 0 \)，故原方程变换为

\[
\frac{\partial^2 u}{\partial \xi^2} = 0.
\]

3511. 假定

\( x = r \sin \theta \cos \varphi, \ y = r \sin \theta \sin \varphi, \ z = r \cos \theta, \)

变换式子

\[
A_1 u = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2
\]

及

\[
A_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

为球坐标所表的式子。

解 先作变换

\( x = R \cos \varphi, \ y = R \sin \varphi, \ z = z, \)

它相当于对 \( x, y \) 坐标作一次极坐标变换。

利用 3483 题及 3484 题的结果，对新变元 \( R, \varphi, \theta \) 有

\[
A_1 u = \left( \frac{\partial u}{\partial R} \right)^2 + \frac{1}{R^2} \left( \frac{\partial u}{\partial \varphi} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2,
\]

\[
A_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

\[
- \frac{\partial^2 u}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial z^2}.
\]
再作变换

\[ R = r \sin \theta, \quad \varphi = \phi, \quad z = r \cos \theta. \]

它相当于对 \( R, z \) 坐标又作一次极坐标变换，其中 \( R \) 相当于公式 9 中的 \( y, \theta \) 相当于公式 9 中的 \( \phi \)。于是，

\[
\frac{\partial u}{\partial R} = \frac{R}{r} \frac{\partial u}{\partial r} + \frac{z}{r^2} \frac{\partial u}{\partial \varphi} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}.
\]

再利用 3483 题及 3484 题的结果，得

\[
\Delta_1 u = \left( \frac{\partial u}{\partial R} \right)^2 + \frac{1}{R^2} \left( \frac{\partial u}{\partial \varphi} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2,
\]

\[
= \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial u}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial u}{\partial \varphi} \right)^2.
\]

\[
\Delta_2 u = \frac{\partial^2 u}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial z^2}
\]

\[
= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}
+ \frac{1}{r \sin \theta} \left( \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right)
\]

\[
= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}
+ \frac{1}{r^2 \tan \theta} \frac{\partial u}{\partial \theta}
\]

\[
= \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) \right].
\]
\[ + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} \]

注意到两次变换的乘积就是所给的变换，因此，最后得到的 \( \Delta_1 u \) 及 \( \Delta_2 u \) 的结果即为所求。

3512. 在方程

\[ z \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) = \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \]

中引入新函数 \( w \)，假定 \( w = z^2 \)。

解

\[ \frac{\partial w}{\partial x} = \frac{dz}{dw} \frac{\partial w}{\partial x} = \frac{1}{2z} \frac{\partial w}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{1}{2z} \frac{\partial w}{\partial y} \]

\[ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{1}{2z} \frac{\partial w}{\partial x} \right) \]

\[ = \frac{1}{2z} \frac{\partial^2 w}{\partial x^2} - \frac{1}{2z^2} \frac{\partial z}{\partial x} \frac{\partial w}{\partial x} \]

\[ = \frac{1}{2z} \frac{\partial^2 w}{\partial x^2} - \frac{1}{4z^3} \left( \frac{\partial w}{\partial x} \right)^2, \]

\[ \frac{\partial^2 z}{\partial y^2} = \frac{1}{2z} \frac{\partial^2 w}{\partial y^2} - \frac{1}{4z^3} \left( \frac{\partial w}{\partial y} \right)^2. \]

代入原方程，化简整理得

\[ w \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2, \]

即形式是不变的。

取 \( u \) 和 \( v \) 为新的自变量及 \( w = w(u, v) \) 为新函数，变
换下列方程：

3513. \( y \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial y} = \frac{2}{x} \), 设 \( u = \frac{x}{y} \), \( v = x \), \( w = xz - y \).

解 从 3513 题到 3522 题均属作变换

\( u = u(x, y) \), \( v = v(x, y) \), \( w = w(x, y, z) \)

的类型。我们来导出一般公式，顺便指出一般方法。

将 \( u, v \) 看作中间变量，\( x, y \) 看作自变量，则有

\[
\begin{align*}
    du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \\
    dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy, \\
    dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz.
\end{align*}
\]

\[
\begin{align*}
    d^2 u &= \frac{\partial^2 u}{\partial x^2} dx^2 + 2 \frac{\partial^2 u}{\partial x \partial y} dx dy + \frac{\partial^2 u}{\partial y^2} dy^2, \\
    d^2 v &= \frac{\partial^2 v}{\partial x^2} dx^2 + 2 \frac{\partial^2 v}{\partial x \partial y} dx dy + \frac{\partial^2 v}{\partial y^2} dy^2. \\
    d^2 w &= \frac{\partial^2 w}{\partial x^2} dx^2 + \frac{\partial^2 w}{\partial y^2} dy^2 + \frac{\partial^2 w}{\partial z^2} dz^2 \\
    &+ 2 \frac{\partial^2 w}{\partial x \partial y} dx dy + 2 \frac{\partial^2 w}{\partial y \partial z} dy dz \\
    &+ 2 \frac{\partial^2 w}{\partial z \partial x} dz dx + \frac{\partial w}{\partial z} d^2 z.
\end{align*}
\]

将 \( dw, du \) 及 \( dv \) 代入全微分式

\[ dw = \frac{\partial w}{\partial u} du + \frac{\partial w}{\partial v} dv, \]

化简整理得
\[
\frac{\partial w}{\partial z} \, dz = \left( \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} - \frac{\partial w}{\partial x} \right) \, dx \\
+ \left( \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} - \frac{\partial w}{\partial y} \right) \, dy.
\]

于是，

\[
\begin{aligned}
\frac{\partial z}{\partial x} &= \left( \frac{\partial w}{\partial z} \right)^{-1} \left( \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} - \frac{\partial w}{\partial x} \right), \\
\frac{\partial z}{\partial y} &= \left( \frac{\partial w}{\partial z} \right)^{-1} \left( \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} - \frac{\partial w}{\partial y} \right),
\end{aligned}
\]

公式12

其中，\( \frac{\partial z}{\partial x} \) 及 \( \frac{\partial z}{\partial y} \) 是原方程中旧变元间的偏导函数，而 \( \frac{\partial w}{\partial u} \) 及 \( \frac{\partial w}{\partial v} \) 是变换后新变元间的偏导函数，其它均为已给变换导出的已知关系式。

把上面求得的 \( d^2 w, \, du, \, dv, \, d^2 u, \, d^2 v \) 代入表示新变元关系的二阶全微分式：

\[
d^2 w = \frac{\partial^2 w}{\partial u^2} \, du^2 + 2 \frac{\partial^2 w}{\partial u \partial v} \, dudv + \frac{\partial^2 w}{\partial v^2} \, dv^2 \\
+ \frac{\partial w}{\partial u} \, d^2 u + \frac{\partial w}{\partial v} \, d^2 v,
\]

再把式中的 \( dz \) 表成已求得的 \( \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy \)，按 \( dx^2, \, dxdy \) 及 \( dy^2 \) 合并同类项，最后把所得的结果与表示旧变元关系的全微分式：

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\[ \frac{d^2 z}{dx^2} = \frac{\partial^2 z}{\partial x^2} \, dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \, dx \, dy + \frac{\partial^2 z}{\partial y^2} \, dy^2 \]

相比较，即得

\[ \frac{\partial^2 z}{\partial x^2} = \left( \frac{\partial w}{\partial z} \right)^{-1} \left[ \frac{\partial^2 w}{\partial u^2} \left( \frac{\partial u}{\partial x} \right)^2 \right. \]

\[ + 2 \frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial v^2} \left( \frac{\partial v}{\partial x} \right)^2 \]

\[ + \frac{\partial w}{\partial u} \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial v} \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} \]

\[ - \frac{\partial^2 w}{\partial z^2} \left( \frac{\partial z}{\partial x} \right)^2 - 2 \frac{\partial^2 w}{\partial x \partial z} \frac{\partial z}{\partial x} \right], \]

\[ \frac{\partial^2 z}{\partial x \partial y} = \left( \frac{\partial w}{\partial z} \right)^{-1} \left[ \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right. \]

\[ + \frac{\partial^2 w}{\partial u \partial v} \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \]

\[ + \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial u} \frac{\partial^2 u}{\partial x \partial y} \]

\[ + \frac{\partial w}{\partial v} \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial y} \]

\[ - \frac{\partial^2 w}{\partial z^2} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial x \partial z} \frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial y \partial z} \frac{\partial z}{\partial x} \right], \]

\[ \frac{\partial^2 z}{\partial y^2} = \left( \frac{\partial w}{\partial z} \right)^{-1} \left[ \frac{\partial^2 w}{\partial u^2} \left( \frac{\partial u}{\partial y} \right)^2 \right. \]
\[ + 2 \frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial^2 w}{\partial v^2} \left( \frac{\partial v}{\partial y} \right)^2 \]

\[ + \frac{\partial w}{\partial u} \frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial v} \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} \]

\[ - \frac{\partial^2 w}{\partial z^2} \left( \frac{\partial z}{\partial y} \right)^2 - 2 \frac{\partial^2 w}{\partial y \partial z} \frac{\partial z}{\partial y} \]. \text{ 公式13} \]

公式13太复杂，一般不直接应用。本题用求偏导数法较方便。由于

\[ \frac{\partial w}{\partial y} = x \frac{\partial z}{\partial y} - 1 \]

及

\[ \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = -x \frac{\partial w}{y^2 \frac{\partial u}{\partial y}} , \]

故得

\[ \frac{\partial z}{\partial y} = \frac{1}{x} - \frac{1}{y^2} \frac{\partial w}{\partial u} . \]

于是，

\[ y \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial y} = \frac{1}{y} \left( y^2 \frac{\partial^2 z}{\partial y^2} + 2y \frac{\partial z}{\partial y} \right) \]

\[ = y^{-1} \frac{\partial}{\partial y} \left( y^2 \frac{\partial z}{\partial y} \right) \]

\[ = y^{-1} \frac{\partial}{\partial y} \left[ y^2 \left( \frac{1}{x} - \frac{1}{y^2} \frac{\partial w}{\partial u} \right) \right] \]
\[ y^{-1} \frac{\partial}{\partial y} \left( \frac{y^2}{x} \right) - y^{-1} \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial u} \right) \]

\[ = \frac{2}{x} - y^{-1} \left[ \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial u} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial u} \right) \frac{\partial v}{\partial y} \right] \]

\[ = \frac{2}{x} + \frac{x}{y^8} \frac{\partial^2 w}{\partial u^2} = \frac{2}{x}. \]

由于 \( \frac{x}{y^8} \neq 0 \)，故原方程变换为

\[ \frac{\partial^2 w}{\partial u^2} = 0. \]

3514. \[ \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0, \quad \text{设} u = x + y, \quad v = \frac{y}{x}, \]

\[ w = \frac{z}{x}. \]

解

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}, \]

\[ \frac{\partial w}{\partial x} = -\frac{z}{x^2}, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = \frac{1}{x}. \]

代入公式12，得

\[ \frac{\partial z}{\partial x} = x \left( \frac{\partial w}{\partial u} - \frac{y}{x} \frac{\partial w}{\partial v} + \frac{z}{x^2} \right) \]

\[ = \frac{z}{x} \frac{\partial w}{\partial u} - \frac{y}{x} \frac{\partial w}{\partial v} + \frac{z}{x}, \]
\[
\frac{\partial z}{\partial y} = x \left( \frac{\partial w}{\partial u} + \frac{1}{x} \frac{\partial w}{\partial v} \right) = x \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}.
\]

令 \( R = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = -\frac{y}{x} \frac{\partial w}{\partial v} + \frac{z}{x} \frac{\partial w}{\partial v} = w - (1 + v) \)

\[
\frac{\partial w}{\partial v}.
\]

于是，

\[
\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \left( \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} \right)
\]

\[-\left( \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} \right)\]

\[= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right) = \frac{\partial R}{\partial x} - \frac{\partial R}{\partial y}
\]

\[= \frac{\partial R}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial x} - \frac{\partial R}{\partial u} \frac{\partial u}{\partial y} - \frac{\partial R}{\partial v} \frac{\partial v}{\partial y}
\]

\[= \frac{\partial R}{\partial u} (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}) + \frac{\partial R}{\partial v} (\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y})
\]

\[= \frac{\partial}{\partial v} \left[ w - (1 + v) \frac{\partial w}{\partial v} \right] \left( -\frac{y}{x^2} - \frac{1}{x} \right)
\]

\[= \left[ \frac{\partial w}{\partial v} - \frac{\partial w}{\partial u} - (1 + v) \frac{\partial^2 w}{\partial u^2} \right] \left[ -\frac{1}{x} (1 + v) \right]
\]

\[= \frac{1}{x} (1 + v)^2 \frac{\partial^2 w}{\partial u^2} = 0,
\]
由于 $x \neq 0$, $1 + v \neq 0$, 故原方程变为

$$\frac{\partial^2 w}{\partial v^2} = 0.$$  

3515. $\frac{\partial^2 x}{\partial x^2} + 2 \frac{\partial^2 x}{\partial x \partial y} + \frac{\partial^2 x}{\partial y^2} = 0$, 设 $u = x + y, v = x - y$, 代入式

$$w = xy - z.$$  

解

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -1, \quad \frac{\partial v}{\partial y} = 1,$$

$$\frac{\partial w}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x, \quad \frac{\partial w}{\partial z} = -1.$$  

代入公式12, 得

$$\frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}, \quad \frac{\partial z}{\partial y} = x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}.$$  

令 $R = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y - 2\frac{\partial w}{\partial u} = u - 2\frac{\partial w}{\partial u}$. 于是,

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \right)$$

$$+ \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \right)$$

$$= \frac{\partial R}{\partial x} + \frac{\partial R}{\partial y} = \frac{\partial R}{\partial u} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial R}{\partial v} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$= 2 \frac{\partial}{\partial u} \left( u - 2\frac{\partial w}{\partial u} \right) = 2 - 4 \frac{\partial^2 w}{\partial u^2} = 0,$$  

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原方程变换为
\[ \frac{\partial^2 w}{\partial u^2} = \frac{1}{2}. \]

3516. \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = z \), 设 \( u = \frac{x+y}{2}, \ v = \frac{x-y}{2} \),

\[ w = ze^v. \]

解
\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{1}{2} = -\frac{\partial v}{\partial y}, \]

\[ \frac{\partial w}{\partial x} = 0, \ \frac{\partial w}{\partial y} = ze^v, \ \frac{\partial w}{\partial z} = e^v. \]

代入公式12，得
\[ \frac{\partial z}{\partial x} = \frac{1}{2} e^{-v} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right), \]

\[ \frac{\partial z}{\partial y} = \frac{1}{2} e^{-v} \left( \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \right) - z. \]

于是，
\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z \right) \]
\[ = \frac{\partial}{\partial x} \left( e^{-v} \frac{\partial w}{\partial u} \right) \]
\[ = e^{-v} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial u} \right) = e^{-v} \left( \frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial u \partial v} \frac{\partial v}{\partial x} \right) \]
\[ = \frac{1}{2} e^{-z} (\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial u \partial v}) = z. \]

原方程变换为

\[ \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial u \partial v} = 2ze^z - 2w. \]

3517. \( \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \left( 1 + \frac{y}{x} \right) \frac{\partial^2 z}{\partial y^2} = 0 \)

设 \( u = x, \ v = x + y, \ w = x + y + z. \)

解 由公式12不难求出

\[ \frac{\partial z}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} - 1, \ \frac{\partial z}{\partial y} = \frac{\partial w}{\partial v} - 1. \]

于是，

\[ \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{\partial w}{\partial u}. \]

同3514题的方法可求得

\[ \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left( \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right) \]

\[ = -\frac{\partial}{\partial u} \left( \frac{\partial w}{\partial u} \right) \left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial u} \right) \]

\[ \cdot \left( \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right) = \frac{\partial^2 w}{\partial u^2}, \]

\[ \frac{y}{x} \frac{\partial^2 z}{\partial y^2} = \left( \frac{v}{u} - 1 \right) \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial v} - 1 \right) \]
\[
\begin{align*}
  &\left(\frac{v}{u} - 1\right) \left[ \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial v} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
  &= \left(\frac{v}{u} - 1\right) \frac{\partial^2 w}{\partial v^2} .
\end{align*}
\]

将上述结果代入原方程，即得

\[
\frac{\partial^2 w}{\partial u^2} + \left(\frac{v}{u} - 1\right) \frac{\partial^2 w}{\partial v^2} = 0 .
\]

3518. \((1-x^2) \frac{\partial^2 z}{\partial x^2} + (1-y^2) \frac{\partial^2 z}{\partial y^2} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}\)，设

\[x = \sin u, \quad y = \sin v, \quad z = e^w.\]

解

\[
\begin{align*}
  \frac{\partial z}{\partial x} &= \frac{dz}{dw} \frac{\partial w}{\partial u} \frac{du}{dx} = \frac{e^w}{\cos u} \frac{\partial w}{\partial u}, \\
  \frac{\partial z}{\partial y} &= \frac{e^w}{\cos v} \frac{\partial w}{\partial v}, \\
  \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{e^w}{\cos u} \frac{\partial w}{\partial u} \right) = \frac{\partial}{\partial u} \left( \frac{e^w}{\cos u} \frac{\partial w}{\partial u} \right) \frac{du}{dx} \\
  &= \frac{1}{\cos u} \left[ \frac{e^w}{\cos u} \left( \frac{\partial w}{\partial u} \right)^2 + \frac{e^w}{\cos u} \frac{\partial^2 w}{\partial u^2} + \frac{e^w \sin u}{\cos u} \frac{\partial w}{\partial u} \right] \\
  &= \frac{e^w}{\cos^2 u} \left[ \left( \frac{\partial w}{\partial u} \right)^2 + \frac{\partial^2 w}{\partial u^2} + i \sin u \cdot \frac{\partial w}{\partial u} \right].
\end{align*}
\]

将上述结果代入原方程，并注意到

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\[1 - x^2 = \cos^2 u, \quad 1 - y^2 = \cos^2 v,\]

化简整理即得

\[
\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} + \left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 0.
\]

3519. \((1 - x^2) \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 2x \frac{\partial z}{\partial x} - \frac{1}{4} z = 0\) \((|x| < 1)\), 设

\[u = \frac{1}{2} (y + \arccos x), \quad v = \frac{1}{2} (y - \arccos x), \quad w = z \sqrt{1 - x^2}.
\]

解 由公式12不难求出

\[
\frac{\partial z}{\partial x} = \frac{1}{2(1-x^2)^{\frac{3}{2}}} \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) + \frac{xz}{2(1-x^2)},
\]

\[
\frac{\partial z}{\partial y} = \frac{1}{2(1-x^2)^{\frac{1}{2}}} \left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right).
\]

于是，

\[\frac{\partial^2 z}{\partial x^2} - 2x \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left[ (1-x^2) \frac{\partial z}{\partial x} \right]
\]

\[= \frac{\partial}{\partial x} \left[ \frac{(1-x^2)^{\frac{1}{2}}}{2} \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) + \frac{xz}{2} \right]
\]

\[= -\frac{x}{4(1-x^2)^{\frac{3}{4}}} \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) + \frac{z}{2} + \frac{x}{2} \frac{\partial z}{\partial x} + \frac{(1-x^2)^{\frac{1}{2}}}{2} \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial u}\right)
\]
\[
\begin{align*}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = -\frac{1}{2(1-x^2)^{\frac{1}{4}}} \left[ \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) + \frac{\partial u}{\partial v} + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
&= -\frac{1}{4(1-x^2)^{\frac{1}{4}}} \left( \frac{\partial^2 w}{\partial u^2} + 2 \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2} \right).
\end{align*}
\]

将上述结果代入原方程，并注意到

\[
\arccos z = u - v, \quad x = \cos(u-v),
\]

\[
1 - x^2 = \sin^2(u-v),
\]

化简整理即得

\[
\frac{\partial^2 w}{\partial u \partial v} = -\frac{w}{4\sin^2(u-v)}.
\]

3520. \[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2 \frac{x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}}{x^2 - y^2} - \frac{3(x^2 + y^2)z}{(x^2 - y^2)^2} \cdot |x|.
\]
\[ y \geq 0, \text{设} \ u = x + y, v = x - y, \ w = \frac{z}{\sqrt{x^2 - y^2}}. \]

解 原方程可改写为

\[
\frac{1}{x^2 - y^2} \frac{\partial^2 z}{\partial x^2} + \frac{1}{x^2 - y^2} \frac{\partial^2 z}{\partial y^2} - \frac{2x}{(x^2 - y^2)^2} \\
\cdot \frac{\partial z}{\partial x} + \frac{2y}{(x^2 - y^2)^2} \frac{\partial z}{\partial y} = -\frac{3(x^2 + y^2)z}{(x^2 - y^2)^3}
\]

或

\[
\frac{\partial}{\partial x} \left( \frac{1}{x^2 - y^2} \frac{\partial z}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{x^2 - y^2} \frac{\partial z}{\partial y} \right)
= -\frac{3(x^2 + y^2)z}{(x^2 - y^2)^3} \quad (1)
\]

由公式不难求出

\[
\frac{\partial z}{\partial x} = \sqrt{x^2 - y^2} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) + \frac{xz}{x^2 - y^2},
\]

\[
\frac{\partial z}{\partial y} = \sqrt{x^2 - y^2} \left( \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \right) - \frac{yz}{x^2 - y^2}.
\]

于是，

\[
\frac{\partial}{\partial x} \left( \frac{1}{x^2 - y^2} \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{x^2 - y^2}} \right]
\cdot \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) + \frac{xz}{(x^2 - y^2)^2}
\]

\[
= -\frac{x}{(x^2 - y^2)^{\frac{3}{2}}} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) + \frac{x}{(x^2 - y^2)^{\frac{3}{2}}} \frac{\partial z}{\partial x}
\]

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\[
\begin{align*}
&- \frac{z}{(x^2 - y^2)^2} - \frac{4x^2 z}{(x^2 - y^2)^3} \\
&+ \frac{1}{\sqrt{x^2 - y^2}} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) \\
&= \frac{z}{(x^2 - y^2)^2} - \frac{3x^2 z}{(x^2 - y^2)^3} + \frac{1}{\sqrt{x^2 - y^2}} \\
&\cdot \left[ \frac{\partial}{\partial u} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left( \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\
&= \frac{z}{(x^2 - y^2)^2} - \frac{3x^2 z}{(x^2 - y^2)^3} \\
&+ \frac{1}{\sqrt{x^2 - y^2}} \left( \frac{\partial^2 w}{\partial u^2} + 2 \frac{\partial^2 w}{\partial udv} + \frac{\partial^2 w}{\partial v^2} \right).
\end{align*}
\]

同法可求得
\[
\begin{align*}
&\frac{\partial}{\partial y} \left( \frac{1}{x^2 - y^2} \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \frac{1}{\sqrt{x^2 - y^2}} \right] \\
&\cdot \left( \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \right) - \frac{yz}{(x^2 - y^2)^2} \\
&= \frac{z}{(x^2 - y^2)^2} - \frac{3y^2 z}{(x^2 - y^2)^3} \\
&+ \frac{1}{\sqrt{x^2 - y^2}} \left( \frac{\partial^2 w}{\partial u^2} - 2 \frac{\partial^2 w}{\partial udv} + \frac{\partial^2 w}{\partial v^2} \right).
\end{align*}
\]

把上述结果代入方程(1)，化简整理即得
\[ \frac{\partial^2 u}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0. \]

3521. 证明：任何方程

\[ \frac{\partial^2 z}{\partial x \partial y} + a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + cz = 0 \]

(a, b, c 为常数) 用代换

\[ z = u e^{x \alpha + y \beta} \]

其中 \(\alpha\) 与 \(\beta\) 为常数，\(u = u(x, y)\) 可以化为下面的形式

\[ \frac{\partial^2 u}{\partial x \partial y} + c_1 u = 0 \quad (c_1 = \text{常数}). \]

证

\[ \frac{\partial z}{\partial x} = e^{x \alpha + y \beta} (au + \frac{\partial u}{\partial x}), \quad \frac{\partial z}{\partial y} = e^{x \alpha + y \beta} (bu + \frac{\partial u}{\partial y}), \]

\[ \frac{\partial^2 z}{\partial x \partial y} = e^{x \alpha + y \beta} (a \beta u + \beta \frac{\partial u}{\partial x} + \alpha \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x \partial y}). \]

将上述结果代入所给方程，得

\[ \frac{\partial^2 u}{\partial x \partial y} + (\beta + a)\frac{\partial u}{\partial x} + (\alpha + b)\frac{\partial u}{\partial y} + (\alpha \beta + a \alpha + b \beta + c) u = 0. \]

按题意，需 \(\beta + a = 0\) 及 \(\alpha + b = 0\)，即 \(\beta = -a, \alpha = -b\)，

这是可能的。事实上，只需取代换

\[ z = u e^{-(x \alpha + y \beta)}, \]

原方程即变换为

\[ \frac{\partial^2 u}{\partial x \partial y} + c_1 u = 0 \quad (c_1 = \text{常数}). \]
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \]

对于变量变换

\[ x' = \frac{x}{y}, \quad y' = -\frac{1}{y}, \quad u = \frac{u'}{\sqrt{y}} e^{-\frac{x^2}{4y}} \]

（\( u' \) 为变量 \( x' \) 与 \( y' \) 的函数）其形状不变。

证

\[ dx' = \frac{dx}{y} - \frac{x}{y^2} dy, \quad dy' = \frac{1}{y^2} dy, \]

\[ \ln u' = \ln u + \frac{1}{2} \ln y + \frac{x^2}{4y}, \]

\[ du' = \frac{u'}{u} du + \frac{u'}{2y} dy + \frac{x u'}{2y} dx - \frac{x^2 u'}{4y^2} dy. \]

把上面三个微分式代入

\[ du' = \frac{\partial u'}{\partial x'} dx' + \frac{\partial u'}{\partial y'} dy' \]

得

\[ \frac{u'}{u} du + \frac{u'}{2y} dy + \frac{x u'}{2y} dx - \frac{x^2 u'}{4y^2} dy = \frac{\partial u'}{\partial x'} \left( \frac{1}{y} dx - \frac{x}{y^2} dy \right) + \frac{\partial u'}{\partial y'} \frac{dy}{y^2}, \]

整理得

\[ du = \left( \frac{u}{yu'} \frac{\partial u'}{\partial x'} - \frac{x u}{2y} \right) dx + \left( \frac{u}{y^2 u'} \frac{\partial u'}{\partial y'} \right) \]
于是，

\[
\frac{\partial u}{\partial x} = \frac{u}{yu'} \frac{\partial u'}{\partial x'} - \frac{xu}{2y},
\]

\[
\frac{\partial u}{\partial y} = \frac{u}{y^2u'} \frac{\partial u'}{\partial y'} - \frac{xu}{y^2u'} \frac{\partial u'}{\partial x'} + \frac{x^2u}{4y^2} - \frac{u}{2y},
\]

(1)

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{u}{yu'} \frac{\partial u'}{\partial x'} - \frac{xu}{2y} \right)
\]

\[
= \frac{u}{yu'} \frac{\partial^2 u'}{\partial x'^2} \frac{\partial x'}{\partial x} + \frac{1}{yu'} \frac{\partial u'}{\partial x'} \frac{\partial u}{\partial x} - \frac{u}{yu'^2}
\]

\[
\cdot \left( \frac{\partial u'}{\partial x'} \right)^2 \frac{\partial x'}{\partial x} - \frac{u}{2y} - \frac{x}{2y} \frac{\partial u}{\partial x}
\]

\[
= \frac{u}{yu'} \frac{\partial^2 u'}{\partial x'^2} + \left( \frac{1}{yu'} \frac{\partial u'}{\partial x'} - \frac{x}{2y} \right)
\]

\[
\cdot \left( \frac{u}{yu'} \frac{\partial u'}{\partial x'} - \frac{xu}{2y} \right) - \frac{u}{y^2u'^2} \left( \frac{\partial u'}{\partial x'} \right)^2 - \frac{u}{2y}
\]

\[
= \frac{u}{yu'} \frac{\partial^2 u'}{\partial x'^2} - \frac{xu}{y^2u'} \frac{\partial u'}{\partial x'}
\]

\[
+ \frac{x^2u}{4y^2} - \frac{u}{2y},
\]

(2)
将(1)式和(2)式代入原方程，得
\[
\frac{\partial^4 u'}{\partial x'^2} = \frac{\partial u'}{\partial y'},
\]
即方程的形式不变。

3523. 在方程
\[
q(1 + q) \frac{\partial^2 z}{\partial x^2} - (1 + p + q + 2pq) \frac{\partial^2 z}{\partial x \partial y} + p(1 - p) \frac{\partial^2 z}{\partial y^2} = 0
\]
（其中 \( p = \frac{\partial z}{\partial x} \)， \( q = \frac{\partial z}{\partial y} \)）中令 \( u = x + z, \ v = y + z \)，
\( w = x + y + z \)，假定 \( w = w(u, v) \)。
解 本题用全微分法解较好。由
\[
dz = p\, dx + q\, dy \text{及} u = x + z, v = y + z, w = x + y + z
\]
可得
\[
du = dx + dz = (1 + p)\, dx + q\, dy,
\]
\[
dv = dy + dz = p\, dx + (1 + q)\, dy,
\]
\[
d^2 u = d^2 v = d^2 w = d^2 z.
\]
把上述结果代入新变元的全微分式
\[
d^2 w = \frac{\partial^2 w}{\partial u^2} du^2 + 2 \frac{\partial^2 w}{\partial u \partial v} du dv + \frac{\partial^2 w}{\partial v^2} dv^2
\]
\[
+ \frac{\partial w}{\partial u} d^2 u + \frac{\partial w}{\partial v} d^2 v,
\]
并记 \( S = 1 - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \)，即得
\[ S d^2 z = \frac{\partial^2 w}{\partial u^2} \left[ (p + 1) d x + q d y \right]^2 + 2 \frac{\partial^2 w}{\partial u \partial v} \left[ (p + 1) d x + q d y \right] \left[ p d x + (q + 1) d y \right] + \frac{\partial^2 w}{\partial v^2} \left[ p d x + (q + 1) d y \right]^2. \]

将上式与
\[ d^2 z = \frac{\partial^2 z}{\partial x^2} d x^2 + 2 \frac{\partial^2 z}{\partial x \partial y} d x d y + \frac{\partial^2 z}{\partial y^2} d y^2 \]
比较，可得
\[ \frac{\partial^2 z}{\partial x^2} = \frac{1}{S} \left[ (1 + p)^2 \frac{\partial^2 w}{\partial u^2} + 2 p (1 + p) \right. \]
\[ \left. \cdot \frac{\partial^2 w}{\partial u \partial v} + p^2 \frac{\partial^2 w}{\partial v^2} \right], \]
\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{S} \left[ q (p + 1) \frac{\partial^2 w}{\partial u^2} + (1 + p + q + 2 p q) \right. \]
\[ \left. \cdot \frac{\partial^2 w}{\partial u \partial v} + p (q + 1) \frac{\partial^2 w}{\partial v^2} \right], \]
\[ \frac{\partial^2 z}{\partial y^2} = \frac{1}{S} \left[ q^2 \frac{\partial^2 w}{\partial u^2} + 2 q (q + 1) \frac{\partial^2 w}{\partial u \partial v} \right. \]
\[ \left. + (q + 1)^2 \frac{\partial^2 w}{\partial v^2} \right]. \]
代入原方程，并注意到
\[ q (1 + q) (1 + p)^2 - (1 + p + q + 2 p q) q \]
\[ \cdot (p + 1) + p (1 + p) q^2 \]
\[ = q (1 + p) \left[ (1 + p) (1 + q) - (1 + p) \right. \]
\[ \left. - (1 + p) (1 + q) + \frac{\partial^2 w}{\partial u^2} \right]. \]

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\[ + q + 2pq + pq = 0, \]
\[ p^2q(1+q) - (1 + p + q + 2pq)p(q+1) \]
\[ + p(1+p)(q+1)^2 = 0 \]

及
\[ 2p(1+p)q(1+q) - (1 + p + q + 2pq)^2 \]
\[ + 2q(q+1)p(1 + p) = -(1 + p + q)^2, \]

原方程变换为
\[ -\frac{(1 + p + q)^2}{S} \frac{\partial^2 w}{\partial u \partial v} = 0 \text{ 或 } \frac{\partial^2 w}{\partial u \partial v} = 0. \]

3524. 在方程
\[ x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} = \left( x \frac{\partial u}{\partial x} \right)^2 \]
\[ + \left( y \frac{\partial u}{\partial y} \right)^2 + \left( z \frac{\partial u}{\partial z} \right)^2 \]

中令 \( x = e^t, y = e^v, z = e^w, u = e^w \), 其中 \( w = w(\xi, \eta, \zeta) \).

解
\[ \frac{\partial u}{\partial x} = \frac{du}{d\xi} \frac{\partial w}{\partial \xi} \frac{d\xi}{dx} = x \frac{\partial w}{\partial \xi}, \]
\[ x \frac{\partial u}{\partial x} = e^w \frac{\partial w}{\partial \xi}, \]  \hspace{1cm} (1)
\[ y \frac{\partial u}{\partial y} = e^w \frac{\partial w}{\partial \eta}, \]
\[ z \frac{\partial u}{\partial z} = e^w \frac{\partial w}{\partial \zeta}. \]

(1) 式两端对 \( x \) 求偏导函数，得
\[ x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = e^w \left( \frac{\partial w}{\partial \xi} \right)^2 \frac{d\xi}{dx} + e^w \frac{\partial^2 w}{\partial \xi^2} \frac{d\xi}{dx}. \]

两端同乘 \( x \)，整理得
\[ x^2 \frac{\partial^2 u}{\partial x^2} = e^w \left( \frac{\partial w}{\partial \xi} \right)^2 + e^w \frac{\partial^2 w}{\partial \eta^2} - e^w \frac{\partial w}{\partial \eta}. \] (2)

同法可得

\[ y^2 \frac{\partial^2 u}{\partial y^2} = e^w \left( \frac{\partial w}{\partial \eta} \right)^2 + e^w \frac{\partial^2 w}{\partial \eta^2} - e^w \frac{\partial w}{\partial \eta}. \] (3)

\[ z^2 \frac{\partial^2 u}{\partial z^2} = e^w \left( \frac{\partial w}{\partial \xi} \right)^2 + e^w \frac{\partial^2 w}{\partial \xi^2} - e^w \frac{\partial w}{\partial \xi}. \] (4)

将(2), (3), (4)三式代入原方程，化简整理即得

\[
\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + \frac{\partial^2 w}{\partial \xi^2} = (e^w - 1) \left[ \left( \frac{\partial w}{\partial \xi} \right)^2 + \left( \frac{\partial w}{\partial \eta} \right)^2 + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} + \frac{\partial w}{\partial \xi}. \right]
\]

3525. 证明：方程

\[
\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0
\]

的形状与变量 \(x, y\) 和 \(z\) 所分别担任的角色无关。

证 令 \(p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}\)，则 \(dz = pdx + qdy\)。若以 \(x\)

作为新函数，则有

\[
d^2x = \frac{\partial^2 x}{\partial y^2} dy^2 + 2 \frac{\partial^2 x}{\partial y \partial z} dy dz + \frac{\partial^2 x}{\partial z^2} dz^2
\]

\[+ \frac{\partial x}{\partial y} d^2y + \frac{\partial x}{\partial z} d^2z.\]
今以作为旧变元的关系：
\[ d^2x = 0, \quad d^2y = 0, \quad dz = pdx + qdy \]
代入上式，可得
\[
d^2z = -\frac{1}{\frac{\partial x}{\partial z}} \left[ \frac{\partial^2 x}{\partial y^2} d^2y + 2 \frac{\partial^2 x}{\partial y \partial z} d y \right.
\]
\[
\left. \cdot (pdx + qdy) + \frac{\partial^2 x}{\partial z^2} (pdx + qdy)^2 \right] .
\]
于是，
\[
\frac{\partial^2 z}{\partial x^2} = -p \left( p^2 \frac{\partial^2 x}{\partial z^2} \right), \quad (1)
\]
\[
\frac{\partial^2 z}{\partial x \partial y} = -p \left( p \frac{\partial^2 x}{\partial y \partial z} + pq \frac{\partial^2 x}{\partial z^2} \right), \quad (2)
\]
\[
\frac{\partial^2 z}{\partial y^2} = -p \left( \frac{\partial^2 x}{\partial y^2} + 2q \frac{\partial^2 x}{\partial y \partial z} + q^2 \frac{\partial^2 x}{\partial z^2} \right). \quad (3)
\]
代入原方程，得
\[
\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = p^2 \left( p^2 \frac{\partial^2 x}{\partial z^2} \right)
\]
\[
\cdot \left( \frac{\partial^2 x}{\partial y^2} + 2q \frac{\partial^2 x}{\partial y \partial z} + q^2 \frac{\partial^2 x}{\partial z^2} \right)
\]
\[
- p^2 \left( p \frac{\partial^2 x}{\partial y \partial z} + pq \frac{\partial^2 x}{\partial z^2} \right)^2
\]
\[
= p^4 \left[ \frac{\partial^2 x}{\partial y^2} \frac{\partial^2 x}{\partial z^2} - \left( \frac{\partial^2 x}{\partial y \partial z} \right)^2 \right] = 0,
\]

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即
\[ \frac{\partial^2 x}{\partial y^2} \frac{\partial^2 x}{\partial z^2} - \left( \frac{\partial^2 x}{\partial y \partial z} \right)^2 = 0. \]

类似地，若以 y 作为函数，则也有
\[ \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 y}{\partial z^2} - \left( \frac{\partial^2 y}{\partial x \partial z} \right)^2 = 0, \]

即方程的形状与变量 x，y 和 z 所分别担任的角色无关。

3526. 取 x 作为变量 y 和 z 的函数，解方程
\[ \left( \frac{\partial z}{\partial y} \right)^2 \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial x \partial y} \]
\[ + \left( \frac{\partial z}{\partial x} \right)^2 \frac{\partial^2 z}{\partial y^2} = 0. \]

解 将 3525 题中的(1)，(2)，(3)三式及 \( p = \frac{\partial z}{\partial x} \)，
\[ q = \frac{\partial z}{\partial y} \] 代入，得
\[ q^2 \left( - p^3 \frac{\partial^2 x}{\partial z^2} \right) + 2pq \left( p^2 \frac{\partial^2 x}{\partial y \partial z} + p^2 q \frac{\partial^2 x}{\partial z^2} \right) \]
\[ - p^2 \left( p \frac{\partial^2 x}{\partial y^2} + 2pq \frac{\partial^2 x}{\partial y \partial z} + p^2 q \frac{\partial^2 x}{\partial z^2} \right) \]
\[ = - p^3 \frac{\partial^2 x}{\partial y^2} = 0, \]

即 \( \frac{\partial^2 x}{\partial y^2} = 0 \) 或 \( p = 0 \). 由
\[ \frac{\partial^2 x}{\partial y^2} = 0 \]

解之，得原方程的解为
\[ x = \varphi(x) + \psi(y) \]
其中 \( \varphi \), \( \psi \) 为任意函数；由 \( p = 0 \) 解之，得 \( z = f(y) \)
(\( f \) 为任意函数)，它也是原方程的解。

3527+。运用勒薛德变换

\[ X = \frac{\partial z}{\partial x}, \quad Y = \frac{\partial z}{\partial y}, \quad Z = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z, \]
其中 \( Z = Z(X, Y) \)，变换方程

\[ A \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \frac{\partial^2 z}{\partial x^2} + 2B \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \frac{\partial^2 z}{\partial x \partial y} \]
\[ + C \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \frac{\partial^2 z}{\partial y^2} = 0. \]

解 \( dZ = d\left( x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z \right) \)

\[ = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy - dz + xdX + ydY \]
\[ = xdX + ydY. \]

于是，

\[ \frac{\partial Z}{\partial X} = x, \quad \frac{\partial Z}{\partial Y} = y. \]

微分上式，得

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\[
\begin{aligned}
\begin{cases}
  dX = \frac{\partial^2 Z}{\partial X^2} dX + \frac{\partial^2 Z}{\partial X \partial Y} dY, \\
  dY = \frac{\partial^2 Z}{\partial X \partial Y} dX + \frac{\partial^2 Z}{\partial Y^2} dY.
\end{cases}
\end{aligned}
\tag{1}
\]

又由 \(X = \frac{\partial z}{\partial x}, \ Y = \frac{\partial z}{\partial y}\) 微分得

\[
\begin{aligned}
\begin{cases}
  dX = \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial x \partial y} dy, \\
  dY = \frac{\partial^2 z}{\partial x \partial y} dx + \frac{\partial^2 z}{\partial y^2} dy.
\end{cases}
\end{aligned}
\tag{2}
\]

由（1）式与（2）式，得

\[
\begin{pmatrix}
  dX \\
  dY
\end{pmatrix}
= \begin{pmatrix}
  \frac{\partial^2 Z}{\partial X^2} & \frac{\partial^2 Z}{\partial X \partial Y} \\
  \frac{\partial^2 Z}{\partial X \partial Y} & \frac{\partial^2 Z}{\partial Y^2}
\end{pmatrix}
\begin{pmatrix}
  dx \\
  dy
\end{pmatrix}
\]

由此可知

\[
\begin{pmatrix}
  \frac{\partial^2 Z}{\partial X^2} & \frac{\partial^2 Z}{\partial X \partial Y} \\
  \frac{\partial^2 Z}{\partial X \partial Y} & \frac{\partial^2 Z}{\partial Y^2}
\end{pmatrix}
\begin{pmatrix}
  \frac{\partial^2 Z}{\partial x^2} & \frac{\partial^2 Z}{\partial x \partial y} \\
  \frac{\partial^2 Z}{\partial x \partial y} & \frac{\partial^2 Z}{\partial y^2}
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix},
\]

从而

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\[
\frac{\partial^2 Z}{\partial X^2} \frac{\partial^2 Z}{\partial X \partial Y} - \frac{\partial^2 Z}{\partial X \partial Y} \frac{\partial^2 Z}{\partial Y^2} = 1,
\]

因此
\[
I = \begin{vmatrix}
\frac{\partial^2 Z}{\partial X^2} & \frac{\partial^2 Z}{\partial X \partial Y} \\
\frac{\partial^2 Z}{\partial X \partial Y} & \frac{\partial^2 Z}{\partial Y^2}
\end{vmatrix} \neq 0.
\]

于是，由（1）式解之，得
\[
\begin{cases}
\frac{dX}{dY} = I^{-1} \left( \frac{\partial^2 Z}{\partial Y^2} dx - \frac{\partial^2 Z}{\partial X \partial Y} dy \right), \\
\frac{dY}{dX} = I^{-1} \left( -\frac{\partial^2 Z}{\partial X \partial Y} dx + \frac{\partial^2 Z}{\partial X^2} dy \right).
\end{cases}
\tag{3}
\]

比较（2）式与（3）式，得
\[
\frac{\partial^2 Z}{\partial x^2} = I^{-1} \frac{\partial^2 Z}{\partial Y^2}, \quad \frac{\partial^2 Z}{\partial x \partial y} = -I^{-1} \frac{\partial^2 Z}{\partial X \partial Y},
\]
\[
\frac{\partial^2 Z}{\partial y^2} = I^{-1} \frac{\partial^2 Z}{\partial X^2}.
\]

代入原方程，即得
\[
A(X, Y) \frac{\partial^2 Z}{\partial Y^2} - 2B(X, Y) \frac{\partial^2 Z}{\partial X \partial Y} + C(X, Y) \frac{\partial^2 Z}{\partial X^2} = 0.
\]
§5. 几何上的应用

1° 切线和法平面 在曲线

\[ x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t) \]

上的一点 \( M(x, y, z) \) 的切线方程为

\[ \frac{X - x}{dx}{dt} = \frac{Y - y}{dy}{dt} = \frac{Z - z}{dz}{dt}. \]

在此点的法平面方程为

\[ \frac{dx}{dt}(X - x) + \frac{dy}{dt}(Y - y) + \frac{dz}{dt}(Z - z) = 0. \]

2° 切平面和法线 曲面 \( z = f(x, y) \) 上点 \( M(x, y, z) \) 处的切平面方程为

\[ Z - z = \frac{\partial z}{\partial x}(X - x) + \frac{\partial z}{\partial y}(Y - y). \]

在 \( M \) 点处的法线方程为

\[ \frac{X - x}{\partial z}{\partial x} = \frac{Y - y}{\partial z}{\partial y} = \frac{Z - z}{1}. \]

若曲面的方程给成隐函数的形状 \( F(x, y, z) = 0 \)，则切平面方程为

\[ \frac{\partial F}{\partial x}(X - x) + \frac{\partial F}{\partial y}(Y - y) + \frac{\partial F}{\partial z}(Z - z) = 0, \]

法线方程为

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\[
\frac{X-x}{\frac{\partial F}{\partial x}} = \frac{Y-y}{\frac{\partial F}{\partial y}} = \frac{Z-z}{\frac{\partial F}{\partial z}}.
\]

3° 平面曲线族的包线 含一个参数的曲线族 \( f(x, y, \alpha) = 0 \) (\( \alpha \)为参数) 的包线满足方程组:

\[
f(x, y, \alpha) = 0, \quad f_\alpha(x, y, \alpha) = 0.
\]

4° 曲面族的包面 含一个参数的曲面族 \( F(x, y, z, \alpha) = 0 \) 的包面满足方程组:

\[
F(x, y, z, \alpha) = 0, \quad F_\alpha(x, y, z, \alpha) = 0.
\]

在含两个参数的曲面族 \( \Phi(x, y, z, \alpha, \beta) = 0 \) 的情形，其包面满足下面的方程组:

\[
\Phi(x, y, z, \alpha, \beta) = 0, \quad \Phi_\alpha(x, y, z, \alpha, \beta) = 0,
\]

\[
\Phi_\beta(x, y, z, \alpha, \beta) = 0.
\]

对下列曲线写出在已知点的切线和法平面方程:

3528. \( x = a \cos \alpha \cos t, \ y = a \sin \alpha \cos t, \ z = a \sin t \); 在点 \( t = t_0 \).

解 曲线

\[
x = x(t), \quad y = y(t), \quad z = z(t)
\]

在点 \( t = t_0 \) 的切向量为

\[
\vec{v}(t_0) = \{x'(t_0), \ y'(t_0), \ z'(t_0)\}.
\]

本题中，当 \( t = t_0 \) 时曲线上点的坐标及曲线在该点的切向量分别为

\[
x_0 = x(t_0) = a \cos \alpha \cos t_0,
\]

\[
y_0 = y(t_0) = a \sin \alpha \cos t_0,
\]

\[
z_0 = z(t_0) = a \sin t_0,
\]

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\[ \mathbf{v}(t_0) = \{-a \cos \alpha \sin t_0, -a \sin \alpha \sin t_0, a \cos t_0\}. \]

于是，切线方程为
\[ \frac{x-x_0}{-a \cos \alpha \sin t_0} = \frac{y-y_0}{-a \sin \alpha \sin t_0} = \frac{z-z_0}{a \cos t_0}, \]
即
\[ \frac{x-x_0}{-\cos \alpha \sin t_0} = \frac{y-y_0}{-\sin \alpha \sin t_0} = \frac{z-z_0}{\cos t_0}; \]
法平面方程为
\[ (-a \cos \alpha \sin t_0)(x-x_0) + (-a \sin \alpha \sin t_0)(y-y_0) + (a \cos t_0)(z-z_0) = 0. \]
以 \( x_0, y_0, z_0 \) 的值代入上式，化简整理得
\[ x \cos \alpha \sin t_0 + y \sin \alpha \sin t_0 - z \cos t_0 = 0, \]
即法平面过原点。

3529. \( x = a \sin^2 t, y = b \sin t \cos t, z = c \cos^2 t; \) 在点 \( t = \frac{\pi}{4} \).

解 \( x_0 = a \sin^2 \frac{\pi}{4} = \frac{a}{2}, y_0 = \frac{b}{2}, z_0 = \frac{c}{2}; \)
\[ \mathbf{v}(\frac{\pi}{4}) = \{a, 0, -c\}. \]
于是，切线方程为
\[ \begin{cases} \frac{x-a}{\frac{a}{2}} = \frac{z-c}{-c}, \\ y = \frac{b}{2} \end{cases} \]
或 \( \begin{cases} \frac{x}{a} + \frac{z}{c} = 1, \\ y = \frac{b}{2}; \end{cases} \)
法平面方程为
\[ a(x - \frac{a}{2}) + (-c)(z - \frac{c}{2}) = 0, \]
即
\[ ax - cz = \frac{1}{2}(a^2 - c^2). \]

3530. \( y = x, z = x^2; \) 在点 \( M(1, 1, 1) \).
解 设 \( x = t, \) 则 \( y = t, z = t^2. \) 于是，
\[ \mathbf{v}(1) = \{1, 1, 2\}, \]
切线方程为
\[ \frac{x - 1}{1} = \frac{y - 1}{1} = \frac{z - 1}{2}; \]
法平面方程为
\[ (x - 1) + (y - 1) + 2(z - 1) = 0 \text{ 或 } x + y + 2z = 4. \]

3531. \( x^2 + z^2 = 10, y^2 + z^2 = 10; \) 在点 \( M(1, 1, 3) \).
解 当曲线以两个曲面方程
\[ F_1(x, y, z) = 0, F_2(x, y, z) = 0 \]
交线形式给出时，可先求出两曲面在交点处的法向量：
\[ \vec{n}_1 = \{F'_{1x}, F'_{1y}, F'_{1z}\}, \quad \vec{n}_2 = \{F'_{2x}, F'_{2y}, F'_{2z}\}, \]
则曲线在该点的切向量为
\[ \vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} F'_{1y} & F'_{1z} & F'_{1x} \\ F'_{2y} & F'_{2z} & F'_{2x} \\ F'_{2x} & F'_{2y} & F'_{2z} \end{vmatrix}. \]
本题中，
\[ \vec{n}_1 = \{2, 0, 6\}, \quad \vec{n}_2 = \{0, 2, 6\}, \]
\[ \vec{v} = \{1, 0, 3\} \times \{0, 1, 3\} = \{-3, -3, 1\}. \]

于是，切线方程为
\[ \frac{x - 1}{-3} = \frac{y - 1}{-3} = \frac{z - 3}{1} \]
或
\[ \frac{x - 1}{3} = \frac{y - 1}{3} = \frac{z - 3}{1}; \]

法平面方程为
\[ -3(x - 1) - 3(y - 1) + (z - 3) = 0, \]
即
\[ 3x + 3y - z = 3. \]

3532. \( x^2 + y^2 + z^2 = 6, \) \( x + y + z = 0; \) 在点 \( M(1, -2, 1). \)

解
\[ F_1 = x^2 + y^2 + z^2 - 6 = 0, \quad F_2 = x + y + z = 0. \]
\[ \vec{n}_1 = 2\{1, -2, 1\}, \quad \vec{n}_2 = \{1, 1, 1\}, \]
\[ \vec{v} = \{1, -2, 1\} \times \{1, 1, 1\} \]
\[ = -3\{1, 0, -1\}. \]

于是，切线方程为
\[ \left\{ \begin{array}{l}
\frac{x - 1}{1} = \frac{z - 1}{-1}, \\
y = -2
\end{array} \right. \]
或
\[ \left\{ \begin{array}{l}
x + z = 2, \\
y + 2 = 0
\end{array} \right. ; \]

法平面方程为
\[ (x - 1) - (z - 1) = 0 \quad \text{或} \quad x - z = 0. \]

3533. 在曲线 \( x = t, \quad y = t^2, \quad z = t^3 \) 上求出一点，此点的切线是平行于平面 \( x + 2y + z = 4 \) 的。

解
\[ \vec{v} = \{1, 2t, 3t^2\}, \quad \text{平面法向量} \vec{n} = \{1, 2, 1\}. \]
按题设，应有
\[ \mathbf{v} \cdot \mathbf{n} = 1 + 4t + 3t^2 = 0. \]

解之，得 \[ t = -1 \] 或 \[ t = -\frac{1}{3} \]. 于是，所求的点为 \[ M_1 \]
\[ (-1, 1, -1), M_2 \left(-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}\right). \]

3534. 证明：螺旋线 \( x = a \cos t, y = a \sin t, z = bt \) 的切线与
\( Oz \) 轴形成定角。

证 \[ \frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = a \cos t, \quad \frac{dz}{dt} = b. \] 于是，切
线与 \( Oz \) 轴形成之角 \( \gamma \) 的余弦

\[ \cos \gamma = \frac{\frac{dz}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{b}{\sqrt{a^2 + b^2}}. \]

由于 \( \cos \gamma \) 为常数，故知切线与 \( Oz \) 轴形成定角。

3535. 证明：曲线
\[ x = ae^t \cos t, \quad y = ae^t \sin t, \quad z = ae^t \]
与锥面 \( x^2 + y^2 = z^2 \) 的各母线相交的角相同。

证 顶锥 \( x^2 + y^2 = z^2 \) 的顶点在原点，过顶锥上任一点 \( P(x, y, z) \) 的母线也过原点。因此，母线的方向
向量为 \( \mathbf{v_1} = \{x, y, z\} \).

曲线在点 \( P \) 的切向量为 \( \mathbf{v_2} = \{x', y', z'\} = \{ae^t \cdot (\cos t - \sin t) = \{ae^t \sin t + \cos t), ae^t\} = \{x - y, x + y, \}

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注意到 \( x^2 + y^2 = z^2 \)，即得

\[
\cos(\mathbf{v}_1, \mathbf{v}_2) = \frac{-\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1|| ||\mathbf{v}_2||} = \frac{x(x-y) + y(x+y) + z^2}{\sqrt{x^2 + y^2 + z^2} \sqrt{(x-y)^2 + (x+y)^2 + z^2}}
\]

\[
= \frac{2z^2}{\sqrt{2z^2} \sqrt{3z^2}} = \frac{2}{\sqrt{6}}.
\]

于是，交角相同。

3536. 证明斜线

\[
tg\left(\frac{\pi}{4} + \frac{\psi}{2}\right) = e^{k\psi} \quad (k = \text{常数}),
\]

（其中\( \varphi \)——地球上点的经度，\( \psi \)——地球上点的纬度）

与地球的一切子午线相交成定角。

证 取直角坐标系如下：赤道平面为Oxy平面，球心为坐标原点，Ox轴正向过0°子午线，Oz轴正向过北极，并取Oxyz坐标系为右手系。

下面我们先确定斜线和子午线在直角坐标系中的方程。为此，假定讨论地球上的点的经度为\( \varphi (0 \leq \varphi \leq 2\pi) \)，纬度为\( \psi \left( -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2} \right) \)，则它在上述坐标系下的坐标为

\[
\begin{align*}
x &= R\cos\psi \cos\varphi, \\
y &= R\cos\psi \sin\varphi, \\
z &= R\sin\psi,
\end{align*}
\]

其中 \( R \) 为地球半径。
对 $\tan (\frac{\pi}{4} + \frac{\psi}{2}) = e^{i\psi}$ 的两端微分，得

$$\frac{d\psi}{2\cos^2 (\frac{\pi}{4} + \frac{\psi}{2})} = k e^{i\psi} d\phi = k \tan (\frac{\pi}{4} + \frac{\psi}{2}) d\phi.$$ 

于是，

$$\frac{d\phi}{d\psi} = \left[ 2\cos^2 (\frac{\pi}{4} + \frac{\psi}{2}) k \tan (\frac{\pi}{4} + \frac{\psi}{2}) \right]^{-1}$$ 

$$= \frac{1}{k\sin (\frac{\pi}{2} + \psi)} = \frac{1}{k\cos \psi}.$$ 

今将斜导线方程看作决定 $\phi$ 为 $\psi$ 的隐函数。因此，对斜导线来说，在 $(\phi_0, \psi_0)$ 点，有

$$\frac{dx}{d\psi} = -R \sin \psi_0 \cos \psi_0 - R \cos \psi_0 \sin \phi_0 \frac{d\phi}{d\psi}$$ 

$$= -R \left( \sin \psi_0 \cos \psi_0 + \frac{\sin \phi_0}{k} \right).$$ 

$$\frac{dy}{d\psi} = -R \sin \psi_0 \sin \phi_0 + R \cos \psi_0 \cos \phi_0 \frac{d\phi}{d\psi}$$ 

$$= -R \left( \sin \psi_0 \sin \phi_0 - \frac{\cos \phi_0}{k} \right),$$ 

$$\frac{dz}{d\psi} = R \cos \phi_0.$$ 

于是，可取斜导线切向量

$$\vec{v}_1 = \left\{ \sin \phi_0 \cos \phi_0 + \frac{\sin \phi_0}{k}, \sin \phi_0 \sin \phi_0 \right\}$$
\[-\frac{\cos \varphi_0}{k}, \ -\cos \psi_0\}\right\}.

当 \( \varphi \) 为常数时即得子午线，故其参数方程为

\[
\begin{aligned}
x &= R \cos \psi \cos \varphi_0, \\
y &= R \cos \psi \sin \varphi_0, \\
z &= R \sin \psi.
\end{aligned}
\]

于是，子午线在点 \((\varphi_0, \psi_0)\) 的切向量为

\[
\vec{u}_2 = \{\sin \varphi_0 \cos \varphi_0, \ \sin \varphi_0 \sin \varphi_0, \ -\cos \psi_0\},
\]

从而得

\[
\cos(\vec{u}_1, \vec{u}_2) = \frac{\vec{u}_1 \cdot \vec{u}_2}{|\vec{u}_1| |\vec{u}_2|} = \frac{1}{\sqrt{1 + \frac{1}{k_2^2}}} = \text{常数},
\]

即斜射线与子午线相交成定角。

3537. 已知曲线

\[
z = f(x, y), \quad \frac{x - x_0}{\cos \alpha} = \frac{y - y_0}{\sin \alpha},
\]

其中 \( f \) 为可微分函数。求曲线上 \( M_0(x_0, y_0) \) 点的切线与 \( Ox \) 与 \( Oy \) 平面所成角的正切。

解  解法一

将曲线看作由参数方程

\[
x = x, \quad y = \varphi(x) = y_0 + (x - x_0) \tan \alpha, \quad z = \psi(x)
\]

\( = f(x, \varphi(x)) \) 给出，则切向量为

\[
\vec{u} = \{1, \ \varphi'(x_0), \ \psi'(x_0)\}
\]

\[
= \{1, \ \tan \alpha, \ \frac{f'(x_0, \varphi(x_0))}{\psi'(x_0)} + f'(x_0, \varphi(x_0))\psi'(x_0)\}
\]

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\[
= \{ 1, \tan \alpha, f'_x(x_0, y_0) + \tan \alpha \cdot f'_y(x_0, y_0) \}.
\]
于是，曲线上点的切线与Oxy平面所成角\(\varphi\)的正切为
\[
\tan \varphi = \frac{\psi'(x_0)}{\sqrt{1 + \psi'^2(x_0)}} = \frac{f'_x(x_0, y_0) + \tan \alpha \cdot f'_y(x_0, y_0)}{\sqrt{1 + \tan^2 \alpha}}
\]
\[
= f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \sin \alpha.
\]
解法二
将曲线看作两条曲线的交线，则所给曲线在点的切线方程为
\[
\begin{vmatrix}
\frac{x-x_0}{f'_x(x_0, y_0) - 1} & 0 \\
0 & \frac{1}{\sin \alpha}
\end{vmatrix} = \begin{vmatrix}
y-y_0 \\ -1 \frac{f'_x(x_0, y_0)}{f'_y(x_0, y_0)}
\end{vmatrix}
\]
即
\[
\begin{align*}
\frac{x-x_0}{\cos \alpha} &= \frac{y-y_0}{\sin \alpha} = \frac{z-z_0}{f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \sin \alpha},
\end{align*}
\]
因此，切线与Oxy平面所成角\(\varphi\)的正切为
\[
\tan \varphi = \frac{f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \sin \alpha}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}}
\]
\[
= f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \sin \alpha.
\]
3538. 求函数

\[ u = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \]

在点 \( M(1, 2, -2) \)沿曲线

\[ x = t, \quad y = 2t^2, \quad z = -2t^4 \]

在此点的切线方向上的导函数。

解

\[ \frac{\partial u}{\partial x} = -\frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \]

\[ \frac{\partial u}{\partial y} = -\frac{x y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \]

\[ \frac{\partial u}{\partial z} = -\frac{x z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \]

在点 \( M(1, 2, -2) \)它们的值分别为 \( \frac{8}{27}, \frac{-2}{27}, \frac{2}{27} \)

又曲线在该点的切线的方向余弦为 \( \frac{1}{9}, \frac{4}{9}, -\frac{8}{9} \)

于是，所求的导数为

\[ \frac{\partial u}{\partial t} \bigg|_{t} = \frac{8}{27} \cdot \frac{1}{9} + (\frac{-2}{27}) \cdot \frac{4}{9} + \frac{2}{27} \cdot (\frac{-8}{9}) = -\frac{16}{243} \]

写出下列曲面上已知点的切面和法线方程:

3539. \( z = x^2 + y^2 \); 在点 \( M_{0}(1, 2, 5) \).

解 当曲面由方程 \( F(x, y, z) = 0 \) 给出时，法向量为

\[ \mathbf{n} = \left\{ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\} \]; 特别是曲面由显式方程

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$z = f(x, y)$ 给出时，法向量为 $\vec{n} = \{f_x, f_y, -1\}$. 本题中，$\vec{n} = \{2x, 2y, -1\}$, $\vec{u}_0 = \{2, 4, -1\}$. 于是，切面方程为

$$2(x - 1) + 4(y - 2) - (z - 5) = 0,$$

或

$$2x + 4y - z = 5,$$

法线方程为

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}.$$

3540. $x^2 + y^2 + z^2 = 169$; 在点 $M_0(3, 4, 12)$.

解 设 $F(x, y, z) = x^2 + y^2 + z^2 - 169 = 0$, 则在点 $M_0$ 处 $\vec{n} = \{2x, 2y, 2z\}$, $\vec{u}_0 = \{6, 8, 24\} = 2 \{3, 4, 12\}$. 于是，切面方程为

$$3(x - 3) + 4(y - 4) + 12(z - 12) = 0$$

或

$$3x + 4y + 12z = 169;$$

法线方程为

$$\frac{x-3}{3} = \frac{y-4}{4} = \frac{z-12}{12} \text{ 或 } \frac{x}{3} = \frac{y}{4} = \frac{z}{12}.$$

3541. $z = \arctg \frac{y}{x}$; 在点 $M_0(1, 1, \frac{\pi}{4})$.

解 $\vec{n} = \left\{ \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, -1 \right\}$, $M_0 = \left\{ -\frac{1}{2}, \frac{1}{2}, -1 \right\}$. 于是，切面方程为
\[ z - \frac{\pi}{4} = -\frac{1}{2} (x - 1) + \frac{1}{2} (y - 1) \]

或
\[ z = \frac{\pi}{4} - \frac{1}{2} (x - y); \]

法线方程为
\[ \frac{x - 1}{1} = \frac{y - 1}{-1} = \frac{z - \frac{\pi}{4}}{2}. \]

3542. \( ax^2 + by^2 + cz^2 = 1 \); 在点 \( M_0(x_0, y_0, z_0) \).
解 \( \vec{n} = 2\{ax_0, by_0, cz_0\} \). 于是，切面方程为
\[ ax_0(x - x_0) + by_0(y - y_0) + cz_0(z - z_0) = 0, \]
注意到 \( ax_0^2 + by_0^2 + cz_0^2 = 1 \)，上述方程即化为
\[ ax_0x + by_0y + cz_0z = 1; \]
法线方程为
\[ \frac{x - x_0}{ax_0} = \frac{y - y_0}{by_0} = \frac{z - z_0}{cz_0}. \]

3543. \( z = y + 1n \frac{X}{X} \); 在点 \( M_0(1, 1, 1) \).
解 \( F(x, y, z) = y + 1nx - 1nz - z = 0 \).
\[ \vec{n} = \left\{ \frac{1}{x}, 1, -\frac{1}{z} - 1 \right\}_{M_0} = \{1, 1, -2\}. \]
于是，切面方程为
\[ (x - 1) + (y - 1) - 2(z - 1) = 0 \text{ 或 } x + y - 2z = 0; \]
法线方程为
\[ \frac{x - 1}{1} = \frac{y - 1}{1} = \frac{z - 1}{-2}. \]
3544. $2^x + 2^y = 8$; 在点 $M_0(2, 2, 1)$。

解 $F(x, y, z) = 2^x + 2^y - 8,$

$$
\vec{n} = \left\{ \frac{1}{2} \ln 2, \frac{1}{2} \ln 2, \left( x \cdot 2^x 
+ y \cdot 2^y \right) \left( -\frac{1}{2} \ln 2 \right) \right\}_{M_0}
$$

$= 4 \ln 2 \{ 1, 1, -4 \}.$

于是，切面方程为

$(x - 2) + (y - 2) - 4(z - 1) = 0$ 或 $x + y - 4z = 0$；

法线方程为

$$
\frac{x - 2}{1} = \frac{y - 2}{1} = \frac{z - 1}{-4}.
$$

3545. $x = a \cos \varphi \cos \varphi, y = b \cos \varphi \sin \varphi, z = c \sin \varphi$; 在点 $M_0$ $(\varphi_0, \psi_0)$。

解 当曲面由参数方程

$x = x(u, v), y = y(u, v), z = z(u, v)$

给出时，曲面上分别令 $u = u_0, v = v_0$ 得到的两条曲线

的切向量分别为

$$
\vec{v}_1 = \left\{ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\},
$$

$$
\vec{v}_2 = \left\{ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\},
$$

则切面的法向量为

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$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \left\{ \begin{array}{c} \frac{\partial y}{\partial u} \frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial u} \frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \end{array} \right\} , \left\{ \begin{array}{c} \frac{\partial y}{\partial v} \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial v} \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \end{array} \right\} , \left\{ \begin{array}{c} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial u} \frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial u} \frac{\partial x}{\partial u} \end{array} \right\} .$$

本题中，

$$\mathbf{v}_i = \left\{ \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right\} \big|_{\varphi = \varphi_0}$$

$$= \{-a \cos \varphi_0 \sin \varphi_0, b \cos \varphi_0 \cos \varphi_0, 0\}$$

$$= \cos \varphi_0 \{-a \sin \varphi_0, b \cos \varphi_0, 0\},$$

$$\mathbf{v}_2 = \left\{ \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right\} \big|_{\varphi = \varphi_0}$$

$$= \{-a \sin \varphi_0 \cos \varphi_0, -b \sin \varphi_0 \sin \varphi_0, c \cos \varphi_0\},$$

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$$

$$= abc \left\{ a \cos \varphi_0 \cos \varphi_0, b \cos \varphi_0 \sin \varphi_0, c \sin \varphi_0 \right\} .$$

于是，切面方程为

$$\frac{\cos \varphi_0 \cos \varphi_0}{a} (x - a \cos \varphi_0 \cos \varphi_0) + \frac{\cos \varphi_0 \sin \varphi_0}{b}$$

$$\cdot (y - b \cos \varphi_0 \sin \varphi_0)$$

$$+ \frac{\sin \varphi_0}{c} (z - c \sin \varphi_0) = 0 ,$$

即

$$\frac{x}{a} \cos \varphi_0 \cos \varphi_0 + \frac{y}{b} \cos \varphi_0 \sin \varphi_0 + \frac{z}{c} \sin \varphi_0 = 1 ;$$

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法线方程为
\[
\frac{x - ac\cos\psi_0 \cos\phi_0}{\cos\psi_0 \cos\phi_0} = \frac{y - bc\cos\psi_0 \sin\phi_0}{\cos\psi_0 \sin\phi_0} = \frac{z - csin\psi_0}{\sin\psi_0},
\]
即
\[
\frac{x \sec\psi_0 \sec\phi_0 - a}{bc} = \frac{y \sec\psi_0 \sec\phi_0 - b}{ac} = \frac{z \sec\psi_0 - c}{ab}.
\]

3546. \(x = r \cos\psi, \ y = r \sin\psi, z = r \cot\alpha;\) 在点 \(M_0(\psi_0, \phi_0, r_0).\)

解
\[
\mathbf{v}_1 = \left\{\frac{\partial x}{\partial \psi}, \frac{\partial y}{\partial \psi}, \frac{\partial z}{\partial \psi}\right\}_{M_0}
\]
\[
= r_0 \{- \sin\psi_0, \cos\psi_0, 0\},
\]
\[
\mathbf{v}_2 = \left\{\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right\}_{M_0}
\]
\[
= \{\cos\psi_0, \sin\psi_0, \cot\alpha\},
\]
\[
\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = r_0 \{\cos\psi_0 \cot\alpha, \sin\psi_0 \cot\alpha, -1\}.
\]
于是，切面方程为
\[
\cos\psi_0 \cot\alpha (x - r_0 \cos\psi_0) + \sin\psi_0 \cot\alpha
\]
\[
(y - r_0 \sin\psi_0) - (z - r_0 \cot\alpha) = 0.
\]
即
\[
x \cos\psi_0 + y \sin\psi_0 - z \tan\alpha = 0;
\]
法线方程为
\[
\frac{x - r_0 \cos\psi_0}{\cos\psi_0 \cot\alpha} = \frac{y - r_0 \sin\psi_0}{\sin\psi_0 \cot\alpha} = \frac{z - r_0 \cot\alpha}{-1}
\]
或

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\[ \frac{x - r_0 \cos \varphi_0}{\cos \varphi_0} = \frac{y - r_0 \sin \varphi_0}{\sin \varphi_0} = \frac{z - r_0 \cot \alpha}{-\tan \alpha}. \]

3547. \( x = u \cos v, \quad y = u \sin v, \quad z = au; \) 在点 \( M_0(u_0, v_0). \)

解
\[ v_1 = \left\{ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\}_{M_0} = \{\cos v_0, \sin v_0, 0\}, \]
\[ v_2 = \left\{ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\}_{M_0} = \{-u_0 \sin v_0, u_0 \cos v_0, a\}, \]
\[ n = v_1 \times v_2 = \{a \sin v_0, -a \cos v_0, u_0\}. \]

于是，切面方程为
\[ a \sin v_0 (x - u_0 \cos v_0) - a \cos v_0 (y - u_0 \sin v_0) + u_0 (z - au_0) = 0, \]

即
\[ ax \sin v_0 - ay \cos v_0 + u_0 z = au_0 v_0; \]

法线方程为
\[ \frac{x - u_0 \cos v_0}{a \sin v_0} = \frac{y - u_0 \sin v_0}{-a \cos v_0} = \frac{z - au_0}{u_0}. \]

3548. 求曲面
\[ x = u + v, \quad y = u^2 + v^2, \quad z = u^3 + v^3 \]
的切平面当切点 \( M(u, v)(u \neq v) \) 无限接近于曲面的边界线 \( u = v \) 上的点 \( M_0(u_0, v_0) \) 时的极限位置。

解
\[ n(u, v) = \{1, 2u, 3u^2\} \times \{1, 2v, 3v^2\} = (v - u) \{6uv, -3(u + v), 2\}, \]

则 \( n \) 方向上的单位向量为
\[ \vec{n}(u, v) = \left\{ \frac{6uv}{l}, \ -\frac{3(u+v)}{l}, \ \frac{2}{l} \right\}, \]

其中 \( l = \sqrt{36u^2v^2 + 9(u+v)^2 + 4} \)。于是
\[ \lim_{u \to 1} \vec{n} = \left\{ \frac{6u_0^2}{l_0}, \ -\frac{6u_0}{l_0}, \ \frac{2}{l_0} \right\}, \]
其中 \( l_0 = \sqrt{36u_0^4 + 36u_0^2 + 4} \)。而 \( M_0(u_0, v_0) = (2u_0, 2u_0^2, 2u_0^3) \)，故知切面在 \( M_0 \) 点的极限位置为
\[ 3u_0^2x - 3u_0y + z \]
\[ = 3u_0^2(2u_0) - 3u_0(2u_0^2) + 2u_0^3 \]
\[ = 2u_0^3, \]
或
\[ \frac{3x}{u_0} - \frac{3y}{u_0^2} + \frac{z}{u_0^3} = 2. \]

3549. 在曲面 \( x^2 + 2y^2 + 3z^2 + 2xy + 2xz + 4yz = 8 \) 上求出切平面平行于坐标平面的诸切点。
解 \( \vec{n} = \{2(x+y+z), \ 2(x+2y+2z), \ 2(x+2y+3z)\} \)。当
\[ \begin{cases} \ x+y+z = 0, \\ \ x+2y+2z = 0, \\ \ x+2y+3z = \lambda \end{cases} \]
时，\( \vec{n} = \{0, \ 0, \ \lambda \} \)平行，即切面平行于 \( Oxy \) 平面。解之，得 \( x = 0, \ y = -\lambda, \ z = \lambda \)。将求得的 \( x, y, z \) 值代入所给的曲面方程，得 \( \lambda = \pm 2\sqrt{2} \)。于是，切面平行于 \( Oxy \) 坐标平面的切点为 \( (0, \ \pm 2\sqrt{2}, \ \lambda) \)。
\( \mp 2 \sqrt{2} \). 同法可求得切面平行于 Oxz 坐标平面及
Oxz 坐标平面的各切点分别为 \(( \pm 4, \mp 2, 0)\) 及
\(( \pm 2, \mp 4, \pm 2)\).

3550. 在椭球面

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

上怎样的点，椭球面的法线与坐标轴成等角？

解：\( n = 2 \left\{ \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\} \). 按题设，应有

\[
\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2} = \lambda
\]

即

\[
\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2} = \lambda
\]

将上式代入椭球面方程，得 \( \lambda = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}} \).

于是，所求的点为 \( x = \pm \frac{a^2}{d}, y = \pm \frac{b^2}{d}, z = \pm \frac{c^2}{d} \),

其中 \( d = \sqrt{a^2 + b^2 + c^2} \).

3551. 求曲面 \( x^2 + 2y^2 + 3z^2 = 21 \) 的平行于平面

\( x + 4y + 6z = 0 \)

的各切平面。

解：\( n = 2\{x, 2y, 3z\} \). 按题设，应有

\( x = \lambda, 2y = 4\lambda, 3z = 6\lambda, \)

解之，得 \( x = \lambda, y = 2\lambda, z = 2\lambda \). 将它们代入方程

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\[ x^2 + 2y^2 + 3z^2 = 21, \text{得 } \lambda = \pm 1, \text{故切点为 } (\pm 1, \pm 2, \pm 2). \] 于是，所求的切面方程为
\[ (x \mp 1) + 4(y \mp 2) + 6(z \mp 2) = 0, \]
即
\[ x + 4y + 6z = \pm 21. \]

3552. 证明：曲面 \( xyz = a^3 (a > 0) \) 的切平面与坐标面形成体积一定的四面体。
证 在曲面上任取一点 \( P_0(x_0, y_0, z_0) \)，则曲面在该点的切平面方程为
\[ y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0, \]
它与各坐标面的交点为 \( A(3x_0, 0, 0), B(0, 3y_0, 0), C(0, 0, 3z_0) \)。注意到各坐标轴的垂直关系，即知以 \( A, B, C, O \) 诸点为顶点的四面体的体积为
\[ V_{ABCO} = \frac{1}{3} OC \cdot \left( \frac{1}{2} OA \cdot OB \right) \]
\[ = \frac{1}{6} 3z_0 \cdot 3x_0 \cdot 3y_0 = \frac{9}{2} x_0 y_0 z_0 = \frac{9}{2} a^3, \]
它为一个常数，本题获证。

3553. 证明：曲面
\[ \sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a} \quad (a > 0) \]
的切平面在坐标轴上割下的诸线段，其和为常量。
证 在曲面上任取一点 \( P_0(x_0, y_0, z_0) \)，则曲面在该点的切平面方程为
\[ \frac{1}{2} \sqrt{x_0} (x - x_0) + \frac{1}{2} \sqrt{y_0} (y - y_0) + \frac{1}{2} \sqrt{z_0} (z - z_0) = 0, \]

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\[ + \frac{1}{2} \sqrt{z_0} (z-z_0) = 0, \]

即
\[ \sqrt{y_0 z_0} (x-x_0) + \sqrt{x_0 z_0} (y-y_0) + \sqrt{x_0 y_0} \cdot (z-z_0) = 0. \]

此切面在坐标轴上所割下的诸线段分别为
\[ \sqrt{ax_0}, \sqrt{ay_0}, \sqrt{az_0}, \]
其和为\[ \sqrt{a} (\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}) = \sqrt{a} \cdot \sqrt{a} = a, \]它是常数，本题获证。

3554. 证明：锥面
\[ z = x f \left( \frac{y}{x} \right) \]
的切平面经过其顶点。

证 \[ \frac{\partial z}{\partial x} = f \left( \frac{y}{x} \right) - \frac{y}{x} f' \left( \frac{y}{x} \right), \quad \frac{\partial z}{\partial y} = f' \left( \frac{y}{x} \right). \]于是，

锥面在任一点 \( P_0(x_0, y_0, z_0) \) 的切面方程为
\[ z-z_0 = \left[ f \left( \frac{y_0}{x_0} \right) - \frac{y_0}{x_0} f' \left( \frac{y_0}{x_0} \right) \right] (x-x_0) \]
\[ + f' \left( \frac{y_0}{x_0} \right) (y-y_0), \]

化简整理得
\[ z = \left[ f \left( \frac{y_0}{x_0} \right) - \frac{y_0}{x_0} f' \left( \frac{y_0}{x_0} \right) \right] x + f' \left( \frac{y_0}{x_0} \right) y, \]
它显然通过锥面 \( z = x f \left( \frac{y}{x} \right) \) 的顶点 \( (0, 0, 0) \).
3555. 证明：旋转面

\[ z = f(\sqrt{x^2 + y^2}) \quad (f' \neq 0) \]

的法线与旋转轴相交。
证 在旋转面上任取一点 \( P_o(x_0, y_0, z_0) \)，其中

\[ z_0 = f(\sqrt{x_0^2 + y_0^2}) \]

则曲面在该点的法向量为

\[ \vec{n} = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\}_{P_o} = \frac{1}{\sqrt{x_0^2 + y_0^2}} \cdot \left\{ x_0 f', y_0 f', -\sqrt{x_0^2 + y_0^2} \right\}. \]

于是，法线方程为

\[ \frac{x - x_0}{x_0 f'} = \frac{y - y_0}{y_0 f'} = \frac{z - z_0}{\sqrt{x_0^2 + y_0^2}}, \]

显然，法线通过 \( Oz \) 轴上的点

\[ (0, 0, f(\sqrt{x_0^2 + y_0^2}) + \frac{\sqrt{x_0^2 + y_0^2}}{f'(\sqrt{x_0^2 + y_0^2})}), \]

即法线和 \( Oz \) 轴相交。

3556. 求椭球面

\[ x^2 + y^2 + z^2 - xy = 1 \]

在坐标面上的射影。
解 先考虑椭球面 \( x^2 + y^2 + z^2 - xy = 1 \) 在 \( Ox'y' \) 平面上的射影。该射影即通过所给曲面上的每一点向 \( Ox'y' \) 平面作垂线所得到的垂足的全体，它是 \( Ox'y' \) 平面上的一个区域。这个区域的边界由曲面上这样的点的投影构成：这一点向 \( Ox'y' \) 平面所作的垂线在它的切面内（这里用到了椭球面的凸性），即该点的法线与 \( Ox'y' \)
平面平行，注意到该点的法向量为 \(2x - y, 2y - x, 2z\)。因此，该点的坐标满足
\[
\begin{cases}
2z = 0, \\
x^2 + y^2 + z^2 - xy = 1,
\end{cases}
\]
这些点的投影为
\[
\begin{cases}
z = 0, \\
x^2 + y^2 - xy = 1,
\end{cases}
\]
它即椭球面在 \(Ox'y'\)平面上射影的边界。

同法可考虑切面与 \(Oxz\)平面垂直，则有
\[
2y - x = 0.
\]
因此，对 \(Oxz\)平面投影为边界点的椭球面上的点应满足方程
\[
\begin{cases}
2y - x = 0, \\
x^2 + y^2 + z^2 - xy = 1.
\end{cases}
\]
这是椭球面与平面的交线，将它改写为柱面与平面的交线
\[
\begin{cases}
2y - x = 0, \\
\frac{3x^2}{4} + z^2 = 1.
\end{cases}
\]
于是，椭球面在 \(Oxz\)平面上射影的边界由方程
\[
\begin{cases}
y = 0, \\
\frac{3x^2}{4} + z^2 = 1
\end{cases}
\]
所确定。

同法可确定椭球面在 \(Oyz\)平面上射影的边界由
方程

\[ \begin{cases} x = 0, \\ \frac{3y^2}{4} + z^2 = 1 \end{cases} \]

所确定。

于是，椭球面 \( x^2 + y^2 + z^2 - xy = 1 \) 在 \( Oxy \) 平面上的射影为圆：\( x^2 + y^2 - xy \leq 1, \quad z = 0 \); 在 \( Oyz \) 平面上的射影为椭圆：\( \frac{3}{4} y^2 + z^2 \leq 1, \quad x = 0 \); 在 \( Oxz \) 平面上的射影为椭圆：\( \frac{3}{4} x^2 + z^2 \leq 1, \quad y = 0 \).

3557. 分正方形 \{ 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \} 为直径 \leq \delta 的有限个部分 \sigma。若曲面

\[ z = 1 - x^2 - y^2 \]

在属于同一部分 \sigma 的任何两点 \( P(x, y) \) 及 \( P_1(x_1, y_1) \) 的法线方向相差小于1°，求数 \( \delta \) 的上界。

解 记曲面在点 \( P(x, y) \) 及 \( P_1(x_1, y_1) \) 的法向量为 \( \vec{n} \) 及 \( \vec{n}_1 \)，则 \( \vec{n} = \{2x, 2y, 1\} \)，\( |\vec{n}| \geq 1 \)，\( \vec{n}_1 = \{2x_1, 2y_1, 1\} \)，\( |\vec{n}_1| \geq 1 \)，且有

\[ \vec{n} \times \vec{n}_1 = \{2(y - y_1), 2(x_1 - x), 4(xy_1 - x_1y)\}, \]

\[ \sin (\vec{n}, \vec{n}_1) = \frac{|\vec{n} \times \vec{n}_1|}{|\vec{n}| |\vec{n}_1|} \leq |\vec{n} \times \vec{n}_1| \]

\[ = 2 \sqrt{(y - y_1)^2 + (x - x_1)^2 + 4(x_1y - xy_1)^2}. \]

注意到 \( (xy_1 - x_1y)^2 = (x(y_1 - y) + y(x - x_1))^2 \)

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\[ \leq 2 (x^2 (y_1 - y)^2 + y^2 (x - x_1)^2) \leq 2 \left( (y - y_1)^2 + (x - x_1)^2 \right), \]

并记 \( \rho = \sqrt{(y - y_1)^2 + (x - x_1)^2} \)，即有

\[ \overrightarrow{\sin(n, n_1)} \leq 2 \sqrt{\rho^2 + 4 \cdot 2 \rho^2} = 6 \rho. \]

当 \( \varphi = (n, n_1) \leq 1^\circ \) 时，\( \varphi \approx \sin \overrightarrow{(n, n_1)} \)，于是，要

\[ \varphi \leq \frac{\pi}{180}, \]

只要 \( 6 \rho \leq \frac{\pi}{180} \)，即 \( \rho \leq \frac{\pi}{1080} \approx 0.003 \) 即可。

从而得

\[ \delta = 0.003. \]

3558. 设：

\[ z = f(x, y), \quad \text{其中} (x, y) \in D \]

为曲面的方程，\( \varphi(P, P) \) 为曲面 (1) 在点 \( P(x, y) \in D \) 及 \( P_1(x_1, y_1) \in D \) 二点的法线之间的夹角。

证明：若域 \( D \) 有界且为封闭的，函数 \( f(x, y) \) 在域 \( D \) 内有有界的二阶导函数，则李雅普诺夫不等式

\[ \varphi(P_1, P) \leq C \rho(P_1, P) \]  (2)

成立，其中 \( C \) 为常数，\( \rho(P_1, P) \) 为点 \( P \) 与 \( P_1 \) 之间的距离。

证 本题应加区域是凸的这个条件，否则结论不成立。例如，

\[ z = \begin{cases} 
0, & \text{当} y \leq 0, \; x^2 + y^2 \leq 1, \\
y^3, & \text{当} y > 0, \; x > y^4, \; x^2 + y^2 \leq 1, \\
-y^3, & \text{当} y > 0, \; x < -y^4, \; x^2 + y^2 \leq 1, 
\end{cases} \]

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如图6.30所示，函数在单位圆内缺一个角的闭区域内定义，且有连续的二
阶偏导函数，取 $P_n \left( \frac{1}{n^2}, \frac{1}{n} \right)$ 与 $P_n' \left( -\frac{1}{n^2}, \frac{1}{n} \right)$，则

$$n = n(P_n) = \{ 0, 3y^2, -1 \}$$

$$P_n = \{ 0, \frac{3}{n^2}, -1 \}$$

图 6.30

$$n' = n(P_n') = \{ 0, -3y^2, -1 \}$$

$$P_n' = \{ 0, -\frac{3}{n^2}, -1 \}$$

$$n \times n' = \{ -\frac{6}{n^2}, 0, 0 \}$$

$$\sin \varphi_n = \frac{|n \times n'|}{|n| |n'|} = \frac{6}{n^2} \rightarrow 0 \quad (n \to \infty)$$

又因

$$\rho_n(P_n, P_n') = \frac{2}{n^3}$$

$$\lim_{n \to \infty} \frac{\varphi_n}{\rho_n} = \lim_{n \to \infty} \left( \frac{\sin \varphi_n \cdot \varphi_n}{\rho_n \cdot \sin \varphi_n} \right) = \lim_{n \to \infty} \frac{\sin \varphi_n}{\rho_n}$$
\[
\frac{6}{n^2} \quad \frac{9}{n^4} 
\]

\[
\lim_{n \to \infty} \frac{2}{n^3} = +\infty,
\]

故不存在常数 \( C \), 使 \( \varphi_n \leq C \rho_n \).

下面证明当 \( D \) 为凸的有界闭域时，不等式 (2) 为真。

由 3255 题知：当 \( f(x, y) \) 在 \( D \) 内有二阶连续的偏导函数时，\( \frac{\partial f}{\partial x} \) 及 \( \frac{\partial f}{\partial y} \) 在 \( D \) 内是二元连续的. 又因 \( D \) 是有界闭域，故 \( \frac{\partial f}{\partial x} \) 及 \( \frac{\partial f}{\partial y} \) 在 \( D \) 上有界，记

\[
\left| \frac{\partial f}{\partial x} \right| \leq M, \quad \left| \frac{\partial f}{\partial y} \right| \leq M.
\]

又由 3254 题的证明过程可知：当 \( D \) 是凸域，\( f(x, y) \) 有有界二阶偏导函数时，对 \( D \) 中任意两点 \( P \) 及 \( P_1 \)，

\( \frac{\partial f}{\partial x} \) 及 \( \frac{\partial f}{\partial y} \) 满足里普什兹条件，即存在常数 \( L \)，使有

\[
\left| \frac{\partial f(P)}{\partial x} - \frac{\partial f(P_1)}{\partial x} \right| \leq L \rho(P_1, P),
\]

\[
\left| \frac{\partial f(P)}{\partial y} - \frac{\partial f(P_1)}{\partial y} \right| \leq L \rho(P_1, P).
\]

由

\[
\hat{n}(P_1) = \{ \frac{\partial f(P_1)}{\partial x}, \quad \frac{\partial f(P_1)}{\partial y}, \quad -1 \}
\]

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与 \( n(P) = \left\{ \frac{\partial f(P)}{\partial x}, \frac{\partial f(P)}{\partial y}, -1 \right\} \) 知，对于 \( \varphi = \varphi(P, P_1) \) 有下列不等式

\[
\sin^2 \varphi = \frac{|n(P_1) \times n(P)|^2}{|n(P_1)|^2 |n(P)|^2} \leq |n(P_1) \times n(P)|^2
\]

\[
= \left( \frac{\partial f(P)}{\partial y} - \frac{\partial f(P_1)}{\partial y} \right)^2 + \left( \frac{\partial f(P)}{\partial x} - \frac{\partial f(P_1)}{\partial x} \right)^2
\]

\[
+ \left( \frac{\partial f(P_1)}{\partial x} \frac{\partial f(P)}{\partial y} - \frac{\partial f(P_1)}{\partial y} \frac{\partial f(P)}{\partial x} \right)^2
\]

\[
= L^2 \rho^2 + L^2 \rho^2 + 2 \left[ \frac{\partial f(P_1)}{\partial x} \right]^2
\]

\[
\left[ \frac{\partial f(P)}{\partial y} - \frac{\partial f(P_1)}{\partial y} \right]^2
\]

\[
+ 2 \left[ \frac{\partial f(P_1)}{\partial y} \right]^2 \left[ \frac{\partial f(P_1)}{\partial x} - \frac{\partial f(P)}{\partial x} \right]^2
\]

\[
= 2L^2 \rho^2 + 2M^2 L^2 \rho^2 + 2M^2 L^2 \rho^2
= 2L^2 \rho^2 (1 + 2M^2).
\]

于是，

\[
\sin \varphi \leq C_1 \rho(P_1, P).
\]

其中 \( C_1^2 = 2L^2 (1 + 2M^2) \)，从而得

\[
\varphi(P_1, P) \leq \frac{\pi}{2} \sin \varphi^* \leq \frac{\pi}{2} C_1 \rho(P_1, P)
= C \rho(P_1, P),
\]
其中 $C = \frac{\pi}{2} C_1$ 为常数，本题获证。

*) 利用 1290 题的结果。

3559. 圆柱 $x^2 + y^2 = a^2$ 与曲面 $bz = x y$ 在公共点 $M_0 (x_0, y_0, z_0)$ 相交成怎样的角？
解：二曲面在 $M_0$ 点的法向量为
\[ n_1 = \{ y_0, x_0, -b \} \text{ 及 } n_2 = \{ 2x_0, 2y_0, 0 \}. \]
于是，交角 $\varphi$ 满足
\[ \cos \varphi = \frac{-\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2x_0 y_0 + 2x_0 y_0 + 0}{\sqrt{x_0^2 + y_0^2 + b^2} \sqrt{4x_0^2 + 4y_0^2}} = \frac{4b z_0}{\sqrt{a^2 + b^2} \cdot 2a} = \frac{2b z_0}{a \sqrt{a^2 + b^2}}. \]

3560. 证明：球坐标系的坐标曲面 $x^2 + y^2 + z^2 = r^2, y = x \tan \varphi, x^2 + y^2 = z^2 \tan^2 \theta$ 两两相交。
证：各曲面在其交点 $P (x, y, z)$ 处的法向量分别为
\[ n_1 = \{ 2x, 2y, 2z \}, \quad n_2 = \{ \tan \varphi, -1, 0 \}, \quad n_3 = \{ 2x, 2y, -2z \tan \theta \tan^2 \theta \}. \]
由于
\[ n_1 \cdot n_2 = 2x \tan \varphi - 2y - 2y = 0, \]
\[ n_1 \cdot n_3 = 4x^2 + 4y^2 - 4x^2 \tan^2 \varphi = 4z^2 \tan^2 \theta, \]
\[ -4x^2 \tan^2 \theta = 0, \]
\[ n_2 \cdot n_3 = 2x \tan \varphi - 2y = 0, \]
故知这些曲面在其交点处分别两两垂直。

3561. 证明：球 $x^2 + y^2 + z^2 = 2a x, x^2 + y^2 + z^2 = 2b y, x^2 + y^2 + z^2 = 2c z$ 形成三直交系。
证  设球 \( x^2 + y^2 + z^2 = 2ax \) 与球 \( x^2 + y^2 + z^2 = 2by \) 交于 \( P_0(x_0, y_0, z_0) \) 点，则它们在 \( P_0 \) 点的法向量为

\[
\begin{align*}
\mathbf{n}_1 &= \{2(x_0-a), 2y_0, 2z_0\}, \\
\mathbf{n}_2 &= \{2x_0, 2(y_0-b), 2z_0\}.
\end{align*}
\]

由于

\[
\mathbf{n}_1 \cdot \mathbf{n}_2 = 4[x_0(x_0-a) + y_0(y_0-b) + z_0^2]
= 2[2x_0^2 + 2y_0^2 + 2z_0^2 - 2ax_0 - 2by_0]
= 2[(x_0^2 + y_0^2 + z_0^2 - 2ax_0) + (x_0^2 + y_0^2 + z_0^2 - 2by_0)]
= 0,
\]

故此二球在其交点处相交，同法可知其它球的两两相交性。

3562. 当 \( \lambda = \lambda_1, \lambda = \lambda_2, \lambda = \lambda_3 \) 时，经过每一点 \( M(x, y, z) \) 有三个二次曲面：

\[
\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} + \frac{z^2}{c^2 - \lambda^2} = -1 \quad (a > b > c > 0).
\]

证明这些曲面是相交的。

证 先证 \( \lambda_i \ (i=1, 2, 3) \) 的存在性。考虑 \( \lambda^2 \) 的多项式

\[
\begin{align*}
F(\lambda^2) &= x^2(b^2 - \lambda^2)(c^2 - \lambda^2) + y^2(a^2 - \lambda^2) \\
&\cdot (c^2 - \lambda^2) + z^2(a^2 - \lambda^2)(b^2 - \lambda^2) \\
&+ (a^2 - \lambda^2)(b^2 - \lambda^2)(c^2 - \lambda^2).
\end{align*}
\]

显然有

\[
\begin{align*}
F(a^2) &= x^2(b^2 - a^2)(c^2 - a^2) \geq 0, \\
F(b^2) &= y^2(a^2 - b^2)(a^2 - b^2) \leq 0, \\
F(c^2) &= z^2(a^2 - c^2)(b^2 - c^2) \geq 0, \\
\lim_{{\lambda \to \infty}} F(\lambda^2) &= -\infty.
\end{align*}
\]

因此，\( F(\lambda^2) = 0 \) 在 \( (a^2, +\infty), (b^2, a^2) \) 及 \( (c^2, \)
内各有一根，记为 $\lambda_1^2$, $\lambda_2^2$, $\lambda_3^2$. 但 $F(\lambda^2)$ 是关于 $\lambda$ 的三次多项式，因此，也仅有三个实根 $\lambda_i^2$ ($i=1$, 2, 3), 且知 $\lambda_i \neq \lambda_j$ ($i \neq j; i, j=1, 2, 3$). 由 $F(\lambda_i^2)=0$ 不难推得

$$\frac{x^2}{a^2-\lambda_i^2} + \frac{y^2}{b^2-\lambda_i^2} + \frac{z^2}{c^2-\lambda_i^2} = -1 \quad (i=1, 2, 3).$$

下面再证明这三个二次面是两两正交的，由于

$$\vec{n}_i = \left\{ \frac{2x}{a^2-\lambda_i^2}, \frac{2y}{b^2-\lambda_i^2}, \frac{2z}{c^2-\lambda_i^2} \right\} \quad (i=1, 2, 3),$$

及当 $i \neq j$ 时,

$$\vec{n}_i \cdot \vec{n}_j = \frac{4x^2}{(a^2-\lambda_i^2)(a^2-\lambda_j^2)} + \frac{4y^2}{(b^2-\lambda_i^2)(b^2-\lambda_j^2)} + \frac{4z^2}{(c^2-\lambda_i^2)(c^2-\lambda_j^2)}$$

$$= \frac{4}{\lambda_i^2-\lambda_j^2} \left[ \left( \frac{x^2}{a^2-\lambda_i^2} + \frac{y^2}{b^2-\lambda_i^2} + \frac{z^2}{c^2-\lambda_i^2} \right) \right.$$

$$\left. - \left( \frac{x^2}{a^2-\lambda_j^2} + \frac{y^2}{b^2-\lambda_j^2} + \frac{z^2}{c^2-\lambda_j^2} \right) \right]$$

$$= \frac{4}{\lambda_i^2-\lambda_j^2} \left( (-1) - (-1) \right) = 0,$$

故本题获证.

3563. 求函数 $u=x+y+z$ 在给定面上 $x^2+y^2+z^2=1$ 上 $M_0(x_0, y_0, z_0)$ 点的外法线方向上的导函数.
在球面上怎样的点使得上述的导函数有：（a）最大值，（b）最小值，（c）等于零？

解  \( r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2} = 1 \)，则在 \( M_0 \) 点处球面的外法线单位向量为

\[
\left\{ \frac{x_0}{r_0}, \frac{y_0}{r_0}, \frac{z_0}{r_0} \right\} = \{x_0, y_0, z_0\}.
\]

于是，

\[
\frac{\partial u}{\partial n} = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\} \cdot \{x_0, y_0, z_0\} = \{1, 1, 1\} \cdot \{x_0, y_0, z_0\} = x_0 + y_0 + z_0.
\]

（a）利用 1294 题的结果，得

\[
x_0 + y_0 + z_0 = 1 \cdot x_0 + 1 \cdot y_0 + 1 \cdot z_0 \\
\leq \sqrt{3} \cdot \sqrt{x_0^2 + y_0^2 + z_0^2} = \sqrt{3}.
\]

当 \( x_0 = y_0 = z_0 = \frac{1}{\sqrt{3}} \) 时，上述等式成立，此点恰好在球面上。因此，在 \( \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\} \) 点 \( \frac{\partial u}{\partial n} \) 取得最大值。

（b）同法可得

\[
-(x_0 + y_0 + z_0) = (-1) x_0 + (-1) y_0 + (-1) z_0 \\
+(-1) z_0 \leq -\sqrt{3},
\]

或

\[
x_0 + y_0 + z_0 \geq -\sqrt{3}.
\]

故在点 \( \left\{ -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\} \)，\( \frac{\partial u}{\partial n} \) 取得最小值。
（a）当 \( x + y + z = 0 \) 及 \( x^2 + y^2 + z^2 = 1 \) 时 \( \frac{\partial u}{\partial n} = 0 \)。因此，所求的点为由方程

\[
\begin{cases}
  x + y + z = 0, \\
  x^2 + y^2 + z^2 = 1
\end{cases}
\]

所确定的解 \((x, y, z)\)，它在单位球面与过圆心的平面 \( x + y + z = 0 \) 的交线——圆上面。

3564. 求函数 \( u = x^2 + y^2 + z^2 \) 在椭球面 \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) \( M_0(x_0, y_0, z_0) \) 点的外法线方向上的导函数。

解

\[
\vec{n} = \left\{ \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\},
\]

此法向量的单位向量

\[
\vec{n}^* = \left\{ \frac{x_0}{a^2 \mathcal{A}}, \frac{y_0}{b^2 \mathcal{A}}, \frac{z_0}{c^2 \mathcal{A}} \right\},
\]

其中

\[
\mathcal{A} = \sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}}.
\]

于是，

\[
\frac{\partial u}{\partial n} \bigg|_{M_0} = \frac{x_0}{a^2 \mathcal{A}} \cdot 2x_0 + \frac{y_0}{b^2 \mathcal{A}} \cdot 2y_0 + \frac{z_0}{c^2 \mathcal{A}} \cdot 2z_0
\]

\[
= \frac{2}{\mathcal{A}} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) = \frac{2}{\mathcal{A}}
\]

\[
= \frac{2}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}}}
\]

3565. 设 \( \frac{\partial n}{\partial n} \) 和 \( \frac{\partial u}{\partial n} \) 为函数 \( u \) 和 \( v \) 在沿曲面 \( F(x, y, z) = 0 \)
上的点的法线方向上的导函数，证明：

\[ \frac{\partial}{\partial n}(uv) = u \frac{\partial v}{\partial n} + v \frac{\partial u}{\partial n}. \]

证 \( \frac{\partial}{\partial n}(uv) = \frac{\partial}{\partial x}(uv) \cos \alpha \)

\[ + \frac{\partial}{\partial y}(uv) \cos \beta + \frac{\partial}{\partial z}(uv) \cos \gamma \]

\[ = u \left( \frac{\partial v}{\partial x} \cos \alpha + \frac{\partial v}{\partial y} \cos \beta + \frac{\partial v}{\partial z} \cos \gamma \right) \]

\[ + v \left( \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma \right) \]

\[ = u \frac{\partial v}{\partial n} + v \frac{\partial u}{\partial n}. \]

求含一个参变量的平面曲线族的包线：

3566. \( x \cos \alpha + y \sin \alpha = p \) （\( p = \)常数）。

解

\[ \begin{cases} f(x, y, \alpha) = x \cos \alpha + y \sin \alpha = p = 0, \\ f'(x, y, \alpha) = -x \sin \alpha + y \cos \alpha = 0. \end{cases} \]

消去\( \alpha \)，得

\[ x^2 + y^2 = p^2. \] （1）

由于原曲线族没有奇点，且（1）也不是原曲线族中的某一支，故（1）为原曲线族的包线方程。

3567. \( (x-a)^2 + y^2 = \frac{a^2}{2} \).
解 \[
\begin{aligned}
(x-a)^2 + y^2 = \frac{a^2}{2} = 0, \\
2(x-a) + a = 0.
\end{aligned}
\]
消去a，得 \(y = \pm x\)，同3566题的理由可知，它是包线方程。

3568. \(y = kx + \frac{a}{k} (a = \text{常数})\)。

解 \[
\begin{aligned}
kx - y + \frac{a}{k} = 0, \\
x - \frac{a}{k^2} = 0.
\end{aligned}
\]
消去k，得 \(y^2 = 4ax\)，同3566题的理由可知，它是包线方程。

3569. \(y^2 = 2px + p^2\)。

解 \[
\begin{aligned}
2px - y^2 + p^2 = 0, \\
x + p = 0.
\end{aligned}
\]
消去p，得 \(x^2 + y^2 = 0\)，它仅为一点 \((0, 0)\)。于是，原曲线族无包线。

3570. 设有一长为l的线段，其两端点沿坐标轴滑动，求由此产生的线段族的包线。

解 如图6·31所示，直线方程为
$$\frac{x}{a} + \frac{y}{b} = 1.$$  

但是 $a=1\sin\theta$, $b=1\cos\theta$, 所以,

$$\frac{x}{\sin\theta} + \frac{y}{\cos\theta} = 1. \tag{1}$$

对 $\theta$ 求导数，得

$$-\frac{x}{\sin^2\theta}\cos\theta + \frac{y}{\cos^2\theta}\sin\theta = 0$$

或

$$\frac{x}{\sin^3\theta} = -\frac{y}{\cos^3\theta}. \tag{2}$$

由(1)，(2)消去 $\theta$，得 $x^2 + y^3 = l^3$，同 3566 题的理由可知，它是包线方程。

3571. 求椭圆族 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的包线，已知此族中椭圆的面积 $S$ 为常数。

解 由题设 $\pi ab = S$，得 $a = \frac{S}{\pi b}$，且

$$\frac{\pi^2 b^2 x^2}{S^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

对 $b$ 求导数，得

$$\frac{2\pi^2 b x^2}{S^2} - \frac{2y^2}{b^3} = 0. \tag{2}$$

由(2)式 $b^4 = \frac{\pi^4 S^2}{\pi^2 x^2}$ 或 $b^2 = \pm \frac{y S}{\pi x}$. 再代入(1)，得

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\[ \pm \frac{\pi x y}{S} \pm \frac{\pi x y}{S} = 1, \text{ 即} \]
\[ |x y| = \frac{S}{2\pi}, \]

同 3566 题的理由可知，它是包线方程。

3572. 炮弹在真空中以初速度 \( v_0 \) 射出，当投射角 \( \alpha \) 在铅垂平面中变化下，求炮弹轨迹的包线。

解 炮弹轨迹方程为
\[ y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}. \tag{1} \]

对 \( \alpha \) 求导数，得
\[ 0 = \frac{x}{\cos^3 \alpha} - \frac{gx^2 \sin \alpha}{v_0^2 \cos^3 \alpha}. \tag{2} \]

由 (2) 式得 \( \tan \alpha = \frac{v_0^2}{xg} \). 代入 (1) 式，得
\[ y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} = x \frac{v_0^2}{xg} - \frac{gx^2}{2v_0^2} \left( 1 + \frac{v_0^4}{x^2 g^2} \right) \]
\[ = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}, \]

同 3566 题的理由可知，它是包线方程。

3573. 证明：平面曲线的法线的包线是此曲线的渐屈线。

证 这里我们仅就由显式 \( y = f(x) \) 所给出的平面曲线情形加以证明。

曲线 \( y = f(x) \) 在点 \( P(x, y) \) 的法线方程为
\[ (X - x) + y'(Y - y) = 0. \tag{1} \]
对 \( x \) 求导数，得

\[ -1 + y''(Y - y) - y'^2 = 0 \]

或

\[ y''(Y - y) = 1 + y'^2 . \]  (2)

由 (1)，(2) 解得

\[
\begin{align*}
X &= x - \frac{y'(1 + y'^2)}{y''}, \\
Y &= y + \frac{1 + y'^2}{y''}.
\end{align*}
\]

此即 \( y = f(x) \) 的相屈线方程（参看第二章 2.4 节 3°）。

同 3566 题的理由可知，它是平面曲线的法线的 公 线

方程。

3574. 研究下列曲线族的判别曲线的性质 (c——参变数)：
(a) 立方抛物线 \( y = (x - c)^3 \);
(b) 半立方抛物线 \( y^2 = (x - c)^3 \);
(c) 奈尔半立方抛物线 \( y^3 = (x - c)^2 \);
(d) 环周线 \( (y - c)^2 = x^2 \frac{a - x}{a + c} \).

解 (a) \( f(x, y, c) = y - (x - c)^3 = 0 \),
\[ f'(x, y, c) = 3(x - c)^2 = 0 . \]

消去 \( c \)，得 \( y = 0 \)，它为判别曲线的方程。

原曲线无奇点，且 \( y = 0 \) 也不是原曲线族的某一

支，因此，它是公线。此公线与原曲线在 \((c, 0)\) 点

相切，且 \((c, 0)\) 点是曲线的拐点，即它又是原曲线

族拐点的轨迹。如图 6.32(1) 所示。
消去 $c$，得判别曲线 $y = 0$.

原曲线的奇点为 $(c, 0)$，因此它与奇点的轨迹。要判断是否为包线，还要看在奇点的两支是否与判别曲线相切。事实上，两支分别为 $y_1 = (x-c)^{3\over 2}$，$y_2 = -(x-c)^{3\over 2}$，均有 $y_1'(c) = 0$，$y_2'(c) = 0$。因此，$y = 0$ 为原曲线族的包线。如图6·32(2)所示。

消去 $c$，得判别曲线 $y = 0$.

原曲线的奇点为 $(c, 0)$。由于 $y = (x-c)^{3\over 2}$ 在 $x = c$ 处的导数为无穷，因此，它与 $y = 0$ 不相切，从而它无包线。奇点 $(c, 0)$ 为尖点，如图6·32(3)所示。

消去 $c$，得 $x^2 (a-x) = 0$，即判别曲线为直线 $x = 0$ 及 $x = a$。

显然 $x = 0$ 为原曲线族奇点的轨迹，用 §6 的方法可判别出它是二重点的轨迹。事实上，

$A = f_{xx}''(0, c) = 2$，$B = f_{xy}''(0, c) = 0$，

$C = F_{yy}''(0, c) = -2$，$AC - B^2 = -4 < 0$。

从而知 $x = 0$ 不是包线。
但是，在$x = a$ 处，$f_x'(a, y) \neq 0$ 与 $a \neq 0$，因此 $x = a$ 不是原曲线族奇点的轨迹，同时它又不是原曲线族的某一支。因此，$x = a$ 是原曲线族的包线，如图 6.32 (4) 所示。

图 6.32

3575. 求半径为 $r$，中心在圆周 $x = R \cos t, y = R \sin t, z = 0$ ($t$ 为参数)，$R \gg r$ 上的球族的包面。

解

\[
\begin{align*}
(X - R \cos t)^2 + (Y - R \sin t)^2 + Z^2 &= r^2, \\
2R \sin t(X - R \cos t) - 2R \cos t(Y - R \sin t) &= 0.
\end{align*}
\]
(2)式化简得 \(X \sin t - Y \cos t = 0\). 于是，

\[
tg t = \frac{Y}{X}, \quad \cos t = \pm \frac{X}{\sqrt{X^2 + Y^2}},
\]

\[
\sin t = \pm \frac{Y}{\sqrt{X^2 + Y^2}}, \tag{3}
\]

将(3)式代入(1)式，得

\[
(X^2 + Y^2) \left( 1 \pm \frac{R}{\sqrt{X^2 + Y^2}} \right)^2 + Z^2 = r^2.
\]

当取“+”号时，由于 \(R^2 \neq r^2\)，故它不代表任何点（不是虚的）的轨迹。

当取“−”号时，由于原曲面族无奇点，且 \((\sqrt{X^2 + Y^2} - R)^2 + Z^2 = r^2\) 不是原曲面族的某一个，因此，它是原曲面族的包面（圆环）。

3576. 求球簇

\[
(x - t \cos \alpha)^2 + (y - t \cos \beta)^2 + (z - t \cos \gamma)^2 = 1
\]

（其中 \(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1\) 及 \(t\) — 参变量）的包面。

解

\[
\begin{align*}
(x - t \cos \alpha)^2 + (y - t \cos \beta)^2 \\
+ (z - t \cos \gamma)^2 - 1 &= 0, \\
-2 \cos \alpha (x - t \cos \alpha) - 2 \cos \beta (y - t \cos \beta) \\
-2 \cos \gamma (z - t \cos \gamma) &= 0. 
\end{align*}
\]

由(2)得 \(t = x \cos \alpha + y \cos \beta + z \cos \gamma\). \tag{3}

将(3)式代入(1)式，化简整理得

\[
x^2 + y^2 + z^2 - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2 = 1. \tag{4}
\]

由于原曲面族的奇点均不在此方程所表示的曲面上，并且曲面(4)也不是原曲面族中的某一个，因此，曲面(4)为原曲面族的包面。

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3577. 求椭球面族 \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) 的包面，这些椭球的体积 \( V \) 是常数。
解 引入辅助函数
\[
F(x, y, z, a, b, c) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \lambda abc,
\]
则包面的方程由方程组
\[
\begin{align*}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1, \\
abc &= \frac{3V}{4\pi}, \\
F'_a &= -\frac{2x^2}{a^3} + \lambda bc = 0, \\
F'_b &= -\frac{2y^2}{b^3} + \lambda ac = 0, \\
F'_c &= -\frac{2z^2}{c^3} + \lambda ab = 0
\end{align*}
\]
确定。
由(3)、(4)、(5)可解得
\[
\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{\lambda abc}{2} = \mu. \tag{6}
\]
将(6)式代入(1)式，得
\[
\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \mu = \frac{1}{3}.
\]
于是,
\[ a = \sqrt{3} |x|, \quad b = \sqrt{3} |y|, \quad c = \sqrt{3} |z|. \quad (7) \]

将(7)式代入(2)式，得

\[ |xyz| = \frac{V}{4\pi \sqrt{3}}. \quad (8) \]

由于原曲面族无奇点，且曲面(8)也不是原曲面族中的某一个，故知曲面(8)为原曲面族的包面。

3578. 求半径为 \( \rho \)，中心在圆锥面 \( x^2 + y^2 = z^2 \) 上的球族的包面。

解 设球心为(\( a, b, c \))，则球的方程为

\[ (x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2, \]

其中 \( a^2 + b^2 = c^2. \)

引入辅助函数

\[ F(x, y, z, a, b, c) = (x-a)^2 + (y-b)^2 + (z-c)^2 + \lambda (a^2 + b^2 - c^2), \]

则包面方程由方程组

\[
\begin{align*}
(x-a)^2 + (y-b)^2 + (z-c)^2 &= \rho^2, \\
a^2 + b^2 &= c^2, \\
F' &= -2(x-a) + 2\lambda a = 0, \\
F' &= -2(y-b) + 2\lambda b = 0, \\
F' &= -2(z-c) + 2\lambda c = 0
\end{align*}
\]

确定。

由(3)、(4)、(5)可得

\[ \frac{x}{a} - 1 = \frac{y}{b} - 1 = -\frac{z}{c} + 1 = \lambda. \]

引入记号 \( \frac{1}{\rho} = \frac{x}{a} - \frac{y}{b} = 2 - \frac{z}{c} \)，则有
\[ a = \mu x, \; b = \mu y, \; c = \frac{\mu z}{2\mu - 1}. \] \hfill (6)

将(6)式代入(1), (2)两式，得

\[
\begin{cases}
  x^2 + y^2 + \frac{z^2}{(2\mu - 1)^2} = \frac{\rho^2}{(\mu - 1)^2}, \\
  x^2 + y^2 - \frac{z^2}{(2\mu - 1)^2} = 0.
\end{cases}
\] \hfill (7)

(7)+(8)得

\[ 2(x^2 + y^2) = \frac{\rho^2}{(\mu - 1)^2} \]

或

\[ \sqrt{2}\rho = \sqrt{x^2 + y^2} \pm 2|\mu - 2|. \] \hfill (9)

由(8)得

\[ 2\mu - 1 = \pm \frac{z}{\sqrt{x^2 + y^2}}. \] \hfill (10)

将(10)代入(9)，整理得

\[ \sqrt{2}\rho = |\sqrt{x^2 + y^2} \pm z|. \] \hfill (11)

由于原曲面族无奇点，且曲面(11)也不是原曲面族的某一个。因此，曲面(11)为原曲面族的包面。

3579. 有一发光点位于坐标原点。若 \( x_0^2 + y_0^2 + z_0^2 = R^2 \)，求由球

\[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq R^2 \]

投影所生成的阴影圆锥。

解

解法一。

所求的阴影圆锥的表面，可看作是一个过原点的平面族的包面。此平面族的方程为

\[ ax + by + cz = 0. \]
其中 \(a, b, c\) 满足约束条件

\[
\begin{align*}
ax_0 + by_0 + cz_0 &= \pm R, \\
a^2 + b^2 + c^2 &= 1.
\end{align*}
\]

引进辅助函数

\[
F(x, y, z, a, b, c) = ax + by + cz + \lambda(ax_0 + by_0 + cz_0) + \mu(a^2 + b^2 + c^2),
\]

则包面方程由方程组

\[
\begin{align*}
ax + by + cz &= 0, \\
a^2 + b^2 + c^2 &= 1, \\
ax_0 + by_0 + cz_0 &= \pm R, \\
F'_1 = x + \lambda x_0 + 2\mu a &= 0, \\
F'_2 = y + \lambda y_0 + 2\mu b &= 0, \\
F'_3 = z + \lambda z_0 + 2\mu c &= 0
\end{align*}
\]

确定。

方程 (4)、(5)、(6) 要能解出 \(\lambda, \mu\)，其中 \(a, b, c\) 必须满足关系式

\[
\begin{vmatrix}
  x & x_0 & a \\
  y & y_0 & b \\
  z & z_0 & c \\
\end{vmatrix} = 0.
\]

记 

\[
\begin{align*}
  r_1 &= \begin{vmatrix} y & y_0 \\ z & z_0 \end{vmatrix}, \\
r_2 &= \begin{vmatrix} z & z_0 \\ x & x_0 \end{vmatrix}, \\
r_3 &= \begin{vmatrix} x & x_0 \\ y & y_0 \end{vmatrix},
\end{align*}
\]

则上述关系式可记为 \(ar_1 + br_2 + cr_3 = 0\)。

由 (1)、(3)、(8) 可解得

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\[
\begin{vmatrix}
0 & y & z \\
\pm R & y_0 & z_0 \\
0 & r_2 & r_3 \\
\times & y & z \\
x_0 & y_0 & z_0 \\
r_1 & r_2 & r_3
\end{vmatrix} = \frac{\pm R(zr_2 - yr_0)}{(r_1^2 + r_2^2 + r_3^2)}
\]

或
\[
a^2 = \frac{R^2(zr_2 - yr_0)^2}{(r_1^2 + r_2^2 + r_3^2)^2}.
\]

\[
b^2 = \frac{R^2(xr_3 - zr_1)^2}{(r_1^2 + r_2^2 + r_3^2)^2}, \quad c^2 = \frac{R^2(xr_2 - yr_1)^2}{(r_1^2 + r_2^2 + r_3^2)^2}.
\]

将(9)式代入(2)式，即得
\[
(r_1^2 + r_2^2 + r_3^2)^2 = R^2[(yr_3 - zr_2)^2
+(xr_3 - zr_1)^2 + (xr_2 - yr_1)^2]
= R^2[(r_1^2 + r_2^2 + r_3^2)(x^2 + y^2 + z^2)
-(xr_1 + yr_2 + zr_3)^2]
= R^2(r_1^2 + r_2^2 + r_3^2)(x^2 + y^2 + z^2).
\]
(其中利用了 xr_1 + yr_2 + zr_3 = 0，这是不难验证的。)

于是，有
\[
r_1^2 + r_2^2 + r_3^2 = R^2(x^2 + y^2 + z^2). \quad (10)
\]

由于原平面无奇点，且曲面(10)不是平面族的某一个，因此，曲面(10)即为包面。所求的阴影圆锥为此锥面的内部，即满足不等式
\[
r_1^2 + r_2^2 + r_3^2 \leq R^2(x^2 + y^2 + z^2)
\]
的空间区域（严格说来，还要除去球前部的区域）。

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解法二

显然，阴影圆锥是由通过坐标原点的球面 \((x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2\) 的全体切线构成的。由解析几何知，如果点 \(P_1(x_1, y_1, z_1)\) 不在二次曲面

\[
F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz
+ 2gxz + 2hxy + 2px + 2qy + 2rz + d
= q(x, y, z) + 2gx + 2qy + 2rz + d = 0
\]

上，则通过点 \(P_1\) 与和二次曲面(1)相切的全体切线所构成的锥面方程为

\[
((x-x_1)F'(x_1, y_1, z_1) + (y-y_1)
\cdot F'(x_1, y_1, z_1) + (z-z_1)F'(x_1, y_1, z_1)) = 0
= 4\varphi(x-x_1, y-y_1, z-z_1)
\cdot F(x_1, y_1, z_1) = 0. \tag{2}
\]

今有 \(F(x, y, z) = (x-x_0)^2+(y-y_0)^2
+ (z-z_0)^2-R^2
= x^2+y^2+z^2-2(x_0x+y_0y+z_0z)
+ (x_0^2+y_0^2+z_0^2-R^2).\)

由于

\[
F'_x(0, 0, 0) = -2x_0, \quad F'_y(0, 0, 0) = -2y_0,
F'_z(0, 0, 0) = -2z_0.
\]

故由(2)即得阴影圆锥面的方程为

\[
-(2x_0x-2y_0y-2z_0z)^2 - 4(x^2+y^2+z^2)
\cdot (x_0^2+y_0^2+z_0^2-R^2) = 0
\]

或

\[
(y_0^2+z_0^2)x^2 + (x_0^2+z_0^2)y^2 + (x_0^2+y_0^2)z^2
\]
\[-2x_0y_0yz - 2y_0z_0yz - 2x_0z_0z_0 \]
\[-R^2(x^2 + y^2 + z^2) = 0.\]

由于
\[(x^2 + z^2)x_0^2 + (y^2 + z^2)y_0^2 + (z^2 + y_0^2)z_0^2\]
\[-2x_0^2y_0^2 - 2y_0^2z_0^2 - 2z_0^2x_0^2\]
\[-R^2(x_0^2 + y_0^2 + z_0^2) = -R^2(x_0^2 + y_0^2 + z_0^2) \leq 0,\]

故所求的阴影圆锥为此锥面的内部，即满足不等式
\[(x^2 + z^2)x^2 + (y_0^2 + x_0^2)y^2\]
\[+(x_0^2 + y_0^2)z^2 - 2x_0y_0xy - 2y_0z_0yz\]
\[-2z_0x_0z_0 - R^2(x^2 + y^2 + z^2) \leq 0\]

或
\[
\begin{vmatrix}
 x & y \\
 x_0 & y_0 \\
\end{vmatrix}^2 + \begin{vmatrix}
 y & z \\
 y_0 & z_0 \\
\end{vmatrix}^2 + \begin{vmatrix}
 z & x \\
 z_0 & x_0 \\
\end{vmatrix}^2 \\
\leq R^2(x^2 + y^2 + z^2) \\
\]

的空间区域（严格说来，还要除去球前部的区域）。

解法三

如图 6.33 所示，由三角形的面积公式
\[
\frac{1}{2} |\vec{r}| \cdot |\vec{t}_0| \sin \alpha
\]

的图 6.33
得到

\[ |\vec{r} \times \vec{l}_0| = |\vec{r}| \cdot |\vec{l}_0| \cdot \frac{R}{|\vec{l}_0|}, \]

其中 \( \vec{l}_0 = \{x_0, y_0, z_0\}, \) \( \vec{r} = \{x, y, z\}, \) 而 \( P(x, y, z) \)
为锥面上的任意一点。平方之，即得圆锥曲面的方程为

\[ |\vec{r} \times \vec{l}_0|^2 = R^2 |\vec{r}|^2. \]

于是，所求的阴影圆锥为适合不等式

\[ |\vec{r} \times \vec{l}_0|^2 \leq R^2 |\vec{r}|^2, \]

即

\[ \begin{vmatrix} x & y \cr x_0 & y_0 \end{vmatrix}^2 + \begin{vmatrix} y & z \cr y_0 & z_0 \end{vmatrix}^2 + \begin{vmatrix} z & x \cr z_0 & x_0 \end{vmatrix}^2 \leq R^2(x^2 + y^2 + z^2) \]

的空间区域（严格说来，还要除掉球前部的区域）。

3580. 若参数变量 \( p \) 和 \( q \) 受方程

\[ p^2 + q^2 = 1 \]

的限制，求平面族

\[ z - z_0 = p(x - x_0) + q(y - y_0) \]

的包面。

解  解法一

引进辅助函数

\[ F(x, y, z, p, q) = z - z_0 - p(x - x_0) - q(y - y_0) + \lambda(p^2 + q^2), \]

则包面方程由方程组

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\[
\begin{align*}
z - z_0 = p(x - x_0) + q(y - y_0), \\
p^2 + q^2 = 1, \\
F'_x = -(x - x_0) + 2\lambda p = 0, \\
F'_y = -(y - y_0) + 2\lambda q = 0
\end{align*}
\]
(1) (2) (3) (4)

确定。

\[(3) \times p + (4) \times q, \text{ 得 } 2\lambda = z - z_0. \text{ 于是，由(3)，(4)得} \]

\[
p = \frac{x - x_0}{z - z_0}, \quad q = \frac{y - y_0}{z - z_0}.
\]
(5)

将(5)式代入(1)式，得

\[(z - z_0)^2 = (x - x_0)^2 + (y - y_0)^2.\]

由于原平面族无奇点，且显见上述曲面不是平面，故

上述曲面即为包面。

解法二

引入新参数 \(\theta\); 令 \(p = \sin \theta, \quad q = \cos \theta.\)

\[
\begin{align*}
z - z_0 &= \cos \theta \cdot (x - x_0) + \sin \theta \cdot (y - y_0), \\
sin \theta \cdot (x - x_0) &= \cos \theta \cdot (y - y_0).
\end{align*}
\]
(1) (2)

于是，

\[
\sin \theta = \frac{\pm (y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},
\]

\[
\cos \theta = \frac{\pm (x - x_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.
\]

代入(1)式，得

\[(z - z_0)^2 = (x - x_0)^2 + (y - y_0)^2.\]

由于原平面族无奇点，且上述曲面不是平面，故上述

曲面即为包面。

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§6. 台劳公式

1° 台劳公式 若函数 $f(x, y)$ 在点 $(a, b)$ 的某邻域内有直到 $n+1$ 阶 (连 $n+1$ 阶的在内）的一切连续偏导函数，则在此区域内下面的公式成立

$$f(x, y) = f(a, b) + \sum_{i=1}^{n} \frac{1}{i!} (x-a)^{\frac{\partial}{\partial x}}
+ (y-b)^{\frac{\partial}{\partial y}} \int f(a, b) + R_n(x, y), \quad (1)$$

其中

$$R_n(x, y) = \frac{1}{(n+1)!} \left[ (x-a)^{\frac{\partial}{\partial x}}
+ (y-b)^{\frac{\partial}{\partial y}} \right]^{n+1} \left[ a + \theta (x-a),
b + \theta (y-b) \right] \quad (0 \leq \theta \leq 1).$$

2° 台劳级数 若函数 $f(x, y)$ 可以无穷次地微分及

$$\lim_{n \to \infty} R_n(x, y) = 0,$$
则此函数可表成幂级数的形式

$$f(x, y) = f(a, b) + \sum_{i+j=1}^{\infty} \frac{1}{i!j!} f^{(i+j)}(a, b)(x-a)^i(y-b)^j. \quad (2)$$

特别情形，当 $a = b = 0$ 时公式 (1) 和 (2) 分别名为马克斯林公式和马克斯林级数。

对于多于两个变量的函数有类似的公式。
3° 平面曲线的奇点 设在某点 \( M_0(x_0, y_0) \) 可微分两次的曲线 \( F(x, y) = 0 \) 满足下列条件
\[
F(x_0, y_0) = 0, \quad F_x(x_0, y_0) = 0, \quad F_y(x_0, y_0) = 0
\]
及数
\[
A = F''_x(x_0, y_0), \quad B = F''_y(x_0, y_0), \quad C = F''_{xy}(x_0, y_0)
\]
不全为零。于是，若
(1) \( AC - B^2 \neq 0 \)，则 \( M_0 \) 为孤立点；
(2) \( AC - B^2 = 0 \)，则 \( M_0 \) 为一圈点（节）；
(3) \( AC - B^2 = 0 \)，则 \( M_0 \) 为上升点或孤立点。

在 \( A = B = C = 0 \) 的情形，奇点的种类可能更复杂。至于
不属于光滑的曲线类 \( C^{(2)} \) 的曲线，奇点还可能会有更复杂的
类型：中断点，角点等等。

3581. 在点 \( A(1, -2) \) 的邻域内根据泰勒公式展开函数
\[
f(x, y) = 2x^2 - xy - y^2 - 6x - 3y + 5.
\]
解
\[
\frac{\partial f}{\partial x} = 4x - y - 6, \quad \frac{\partial f}{\partial y} = -x - 2y - 3;
\]
\[
\frac{\partial^2 f}{\partial x^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = -1, \quad \frac{\partial^2 f}{\partial y^2} = -2.
\]
所有三阶偏导函数均为零，因此，有 \( R_2(x, y) = 0 \)。
在点 \( A(1, -2) \) 处，
\[
f(1, -2) = 5, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,
\]
\[
\frac{\partial^2 f}{\partial x^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = -1, \quad \frac{\partial^2 f}{\partial y^2} = -2.
\]
于是，
\[ f(x, y) = 5 + 2(x - 1)^2 - (x - 1) \cdot (y + 2) - (y + 2)^2. \]

3582. 在点 \( A(1, 1, 1) \) 的邻域内根据台劳公式展开函数
\[ f(x, y, z) = x^3 + y^3 + z^3 - 3xyz. \]

解 \[ \frac{\partial f}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial f}{\partial y} = 3y^2 - 3xz, \]
\[ \frac{\partial f}{\partial z} = 3z^2 - 3xy; \]
\[ \frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial z^2} = 6z, \]
\[ \frac{\partial^2 f}{\partial x \partial y} = -3z, \quad \frac{\partial^2 f}{\partial y \partial z} = -3x, \]
\[ \frac{\partial^2 f}{\partial x \partial z} = -3y; \]
\[ \frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial z^3} = 6, \quad \frac{\partial^3 f}{\partial x \partial y \partial z} = -3, \] 其余的三阶混合偏导函数均为零；

所有的四阶偏导函数均为零，因此，\( R, (x, y, z) = 0 \)。在点 \( A(1, 1, 1) \) 处，
\[ f(1, 1, 1) = 0, \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 6, \quad \frac{\partial^2 f}{\partial x \partial y} \]
\[ = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial x \partial z} = -3, \quad \frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial z^3} = 6, \]
\[ \frac{\partial^3 f}{\partial x \partial y \partial z} = -3, \quad \frac{\partial^3 f}{\partial x^2 \partial y} = \cdots = \frac{\partial^3 f}{\partial z^2 \partial x} = 0. \]

于是，

\[ f(x, y, z) = f(1, 1, 1) + \sum_{i=1}^{3} \frac{1}{i!} \left[ (x-1) \frac{\partial}{\partial x} ight] \]

\[ + (y-1) \frac{\partial}{\partial y} + (z-1) \frac{\partial}{\partial z} \right] f(1, 1, 1) \]

\[ = 3 \left[ (x-1)^2 + (y-1)^2 + (z-1)^2 \right. \]
\[ - (x-1)(y-1) - (x-1)(z-1) \]
\[ - (y-1)(z-1) \left. \right] + (x-1)^3 + (y-1)^3 \]
\[ + (z-1)^3 - 3(x-1)(y-1)(z-1). \]

3583. 当从 \( x = 1, y = -1 \) 变到 \( x_1 = 1 + h, y_1 = -1 + k \) 时，求函数 \( f(x, y) = x^2 y + xy^2 - 2xy \) 的增量。

解 ；记 \( A(1, -1) \) 及 \( P(1 + h, -1 + k) \)，则

\[ \frac{\partial f}{\partial x} \bigg|_A = (2xy + y^2 - 2y) \bigg|_A = 1; \]

\[ \frac{\partial f}{\partial y} \bigg|_A = (x^2 + 2xy - 2x) \bigg|_A = -3; \]

\[ \frac{\partial^2 f}{\partial x^2} \bigg|_A = 2y \bigg|_A = -2, \quad \frac{\partial^2 f}{\partial y^2} \bigg|_A = 2x \bigg|_A = 2, \]

\[ \frac{\partial^2 f}{\partial x \partial y} \bigg|_A = (2x + 2y - 2) \bigg|_A = -2; \]

\[ \frac{\partial^3 f}{\partial x \partial y^2} \bigg|_A = \frac{\partial^3 f}{\partial y^3} \bigg|_A = 0, \quad \frac{\partial^3 f}{\partial x^2 \partial y} \bigg|_A = \frac{\partial^3 f}{\partial x \partial y^2} \bigg|_A = -2; \]

所有四阶偏导函数均为零，因此，\( R_4(x, y) = 0 \)。

于是，按泰勒公式即得
\[ \Delta f = f(P) - f(A) = \sum_{i=1}^{3} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(A) \\
= (h - 3k) + (-h^2 - 2hk + k^2) + hk(h + k). \]

3584. 设：
\[ f(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz, \]
按数 \( h, k \) 和 \( l \) 的正整数幂展开 \( f(x+h, y+k, z+l) \).

解：
\[ \frac{\partial f}{\partial x} = 2(Ax + Dy + Ez), \quad \frac{\partial^2 f}{\partial x^2} = 2A, \quad \frac{\partial^2 f}{\partial x \partial y} = 2D, \]
\[ \frac{\partial f}{\partial y} = 2(By + Dx + Fz), \quad \frac{\partial^2 f}{\partial y^2} = 2B, \]
\[ \frac{\partial^2 f}{\partial y \partial z} = 2F, \]
\[ \frac{\partial f}{\partial z} = 2(Cz + Ex + Fy), \quad \frac{\partial^2 f}{\partial z^2} = 2C, \quad \frac{\partial^2 f}{\partial z \partial x} = 2E. \]

所有三阶偏导函数均为零，因此，\( R_2(x, y) = 0 \).
于是，按台劳公式即得
\[ f(x+h, y+k, z+l) = f(x, y, z) \]
\[ + \sum_{i=1}^{3} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right) f(x, y, z) \]
\[ = f(x, y, z) + 2\left[ h(Ax + Dy + Ez) \right] \]
\[ + k(By + Dx + Fz) + l(Cz + Ex + Fy)] \]
\[ + \left[ Ah^2 + Bh^2 + Cl^2 + 2Dhkl + 2Ehl + 2Fkl \right] \]
\[ = f(x, y, z) + 2\left[ h(Ax + Dy + Ez) \right] + k(Dx \]
+ B y + F z) + l (E x + F y + C z)] + f (h, k, l).

3585. 写出函数

\[ f(x, y) = x^y \]

在点 \( A(1, 1) \) 的邻域内的展开式，到二次项为止。

解

\[ \frac{\partial f}{\partial x} = y x^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \ln x, \]

\[ \frac{\partial^2 f}{\partial x^2} = y (y-1) x^{y-2}, \quad \frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + y x^{y-1} \ln x, \]

\[ \frac{\partial^2 f}{\partial y^2} = x^y \ln^2 x, \quad \frac{\partial^8 f}{\partial x^4} = y (y-1) (y-2) x^{y-3}, \]

\[ \frac{\partial^8 f}{\partial x^2 \partial y} = (2y-1) x^{y-2} + y (y-1) x^{y-2} \ln x, \]

\[ \frac{\partial^8 f}{\partial x \partial y^2} = y x^{y-1} \ln^2 x + 2 x^{y-1} \ln x. \]

于是，按泰勒公式在点(1, 1)附近展开到二次项，得

\[ x^y = 1 + (x - 1) + (y - 1) + R_2[1 + \theta(x - 1), 1 + \theta(y - 1)], \quad 0 < \theta < 1, \]

其中余项

\[ R_2(x, y) = \frac{1}{3} \left\{ y (y-1) (y-2) x^{y-3} dx^3 \right. \]

\[ + 3((2y-1) x^{y-2} + y (y-1) x^{y-2} \ln x) dx^2 dy \]

\[ + 3(yx^{y-1} \ln^2 x + 2 x^{y-1} \ln x) dx dy + x^y ln^2 x \}

\[ = \frac{1}{6} x^y \left( \left( \frac{y}{x} dx + \ln x dy \right)^3 + 3 \left( \frac{y}{x} dx + \ln x \cdot dy \right) \right) \]
\[ \left( -\frac{y}{x^2} \, dx^2 + \frac{2}{x} \, dx \, dy \right) - \left( \frac{2y}{x^3} \, dx^3 - \frac{3}{x^2} \, dx \, dy \right) \]

\[ d \tau = x - 1, \quad d \gamma = y - 1. \]

3586. 根据马尔可夫公式展开函数
\[ f(x, y) = \sqrt{1 - x^2 - y^2} \]
到四次项为止。
解 由于
\[
(1 + x)^{\frac{1}{3}} = 1 + \frac{1}{2} \, x + \frac{(\frac{1}{2})(\frac{1}{2} - 1)}{2!} \, x^2 \\
+ \frac{(\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!} x^3 + \ldots
\]
\[
\approx 1 + \frac{1}{2} \, x - \frac{1}{8} \, x^2 + \frac{1}{16} \, x^3,
\]
故得
\[ f(x, y) = \sqrt{1 - x^2 - y^2} = [1 + ((-x^2 - y^2))]^{\frac{1}{2}} \]
\[
\approx 1 - \frac{1}{2} (x^2 + y^2) - \frac{1}{8} (x^2 + y^2)^2.
\]

3587. 若 \(|x|\) 和 \(|y|\) 同 1 比较为很小的量，对于下列二式
(a) \(\frac{\cos x}{\cos y}\);  (b) \(\arctg \frac{1 + x + y}{1 - x + y}\)
推出准确到二次项的近似公式。
解 (a) \(\frac{\cos x}{\cos y} = \cos x \cdot (1 - \sin^2 y)^{-\frac{1}{2}}\)
\[
= (1 - \frac{x^2}{2} + \cdots) \cdot (1 + \frac{1}{2} \sin^2 y + \cdots)
\]
\[
\approx (1 - \frac{x^2}{2})(1 + \frac{1}{2} \sin^2 y).
\]
\[
\approx (1 - \frac{x^2}{2})(1 + \frac{1}{2} y^2) \approx 1 - \frac{1}{2} (x^2 - y^2).
\]

\[
(6) \arctan \frac{1 + x + y}{1 - x + y} = \arctan \frac{1 + \frac{x}{1 + y}}{1 - \frac{x}{1 + y}}
\]
\[
= \frac{\pi}{4} + \arctan \frac{x}{1 + y}
\]
\[
= \frac{\pi}{4} + \left( \frac{x}{1 + y} \right) - \frac{1}{3} \left( \frac{x}{1 + y} \right)^3 + \cdots
\]
\[
\approx \frac{\pi}{4} + x(1 - y + y^2) \approx \frac{\pi}{4} + x - xy.
\]

358. 假定 \(x, y, z\) 的绝对值是很小的量，简化下式

\[
\cos(x + y + z) - \cos x \cos y \cos z.
\]

解  我们简化上式到二次项

\[
\cos(x + y + z) - \cos x \cos y \cos z
\]
\[
\approx 1 - \frac{1}{2} (x + y + z)^2 - \left( 1 - \frac{1}{2} x^2 \right)
\]
\[
\cdot \left( 1 - \frac{1}{2} y^2 \right) \left( 1 - \frac{1}{2} z^2 \right).
\]
\[
\approx 1 - \frac{1}{2} (x^2 + y^2 + z^2) - (xy + yz + zx)
\]
\[-(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2)\]
\[= - (xy + yz + zx).\]

3589. 依 h 的乘幂把函数

\[F(x, y) = \frac{1}{4} \left\{ f(x + h, y) + f(x, y + h) \right\} + f(x - h, y) + f(x, y - h) \big] - f(x, y)\]

展开，准确到 h^4.

解  记 \(\frac{\partial f(x, y)}{\partial x} = \frac{df}{dx}\) 及 \(\frac{\partial f(x, y)}{\partial y} = \frac{df}{dy}\)，...余类似，

即得

\[F(x, y) = \frac{1}{4} \left\{ \left[ f(x + h, y) - f(x, y) \right] \right\} + \left[ f(x, y + h) - f(x, y) \right] + \left[ f(x - h, y) - f(x, y) \right] + \left[ f(x, y - h) - f(x, y) \right] \]

\[\approx \frac{1}{4} \left\{ \left[ \frac{h}{\partial x} \frac{\partial f}{\partial x} + \frac{1}{2}h^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{6}h^3 \frac{\partial^3 f}{\partial x^3} + \frac{1}{24}h^4 \frac{\partial^4 f}{\partial x^4} \right] \right\} + \left[ \frac{h}{\partial y} \frac{\partial f}{\partial y} + \frac{1}{2}h^2 \frac{\partial^2 f}{\partial y^2} + \frac{1}{6}h^3 \frac{\partial^3 f}{\partial y^3} + \frac{1}{24}h^4 \frac{\partial^4 f}{\partial y^4} \right] \right\} + \left[ \frac{h}{\partial x} \frac{\partial f}{\partial x} + \frac{1}{2}h^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{6}h^3 \frac{\partial^3 f}{\partial x^3} + \frac{1}{24}h^4 \frac{\partial^4 f}{\partial x^4} \right] \right\} + \left[ \frac{h}{\partial y} \frac{\partial f}{\partial y} + \frac{1}{2}h^2 \frac{\partial^2 f}{\partial y^2} - \frac{1}{6}h^3 \frac{\partial^3 f}{\partial y^3} + \frac{1}{24}h^4 \frac{\partial^4 f}{\partial y^4} \right] \right\}

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\[\begin{align*}
&\left[ -h \frac{\partial f}{\partial y} + \frac{1}{2} h^2 \frac{\partial^2 f}{\partial y^2} - \frac{1}{6} h^3 \frac{\partial^3 f}{\partial y^3} \\
&+ \frac{1}{24} h^4 \frac{\partial^4 f}{\partial y^4} \right]
= \frac{h^2}{4} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + \frac{h^4}{48} \left( \frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} \right).
\end{align*}\]

3590. 已知中心在点 \( P(x, y) \) 半径为 \( \rho \) 的圆周，设 \( f(P) = f(x, y) \) 及 \( P_i(x_i, y_i) \) (\( i = 1, 2, 3 \)) 为已知圆周之内接正三角形的顶点，并且 \( x_1 = x + \rho, y_1 = y \)。依 \( \rho \) 的正整数幂把函数

\[ F(\rho) = \frac{1}{3} [ f(P_1) + f(P_2) + f(P_3) ] \]

展开准确到 \( \rho^2 \)。

解 如图6・34所示。\( \triangle P_1P_2P_3 \) 之三顶点分别为

\[ P_1(x + \rho, y), \]

\[ P_2(x - \frac{\rho}{2}, y + \frac{\sqrt{3}}{2} \rho), \]

\[ P_3(x - \frac{\rho}{2}, y - \frac{\sqrt{3}}{2} \rho). \]

于是，
\[
F(\rho) = \frac{1}{3} [f(P_1) + f(P_2) + f(P_3)]
\]

\[
\approx \frac{1}{3} \left\{ \left[ f(P) + \rho \frac{\partial f}{\partial x} + \frac{\rho^2}{2} \frac{\partial^2 f}{\partial x^2} \right] + \left[ f(P) + \left( -\frac{\rho}{2} \right) \frac{\partial f}{\partial x} + \frac{\sqrt{3}}{2} \rho \frac{\partial f}{\partial y} + \frac{\rho^2}{8} \frac{\partial^2 f}{\partial x^2} \right] + \left[ f(P) + \left( -\frac{\rho}{2} \right) \frac{\partial f}{\partial x} + \left( -\frac{\sqrt{3}}{2} \right) \rho \frac{\partial f}{\partial y} + \frac{\rho^2}{8} \frac{\partial^2 f}{\partial x^2} \right] \right\}
\]

\[
= f(P) + \frac{\rho^2}{4} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).
\]

397. 依 \( h \) 与 \( k \) 的乘幂把函数

\[
\Delta_{xy} f(x, y) = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y)
\]

展开。

解

\[
\Delta_{xy} f(x, y) = \left[ f(x, y) + h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{h^n k^{n-m}}{m!(n-m)!} \frac{\partial^n f}{\partial x^n \partial y^{n-m}} \right] - \left[ f(x, y) + \sum_{n=1}^{\infty} \frac{h^n}{n!} \frac{\partial^n f}{\partial x^n} \right]
\]

\[
= \left[ f(x, y) + \sum_{n=1}^{\infty} \frac{k^n}{n!} \frac{\partial^n f}{\partial y^n} \right] + f(x, y)
\]

397.
\[ F(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(x + \rho \cos \varphi, y + \rho \sin \varphi) d\varphi \]

展开。

解  \[ F(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \left[ f(x, y) + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{\rho^n}{m_1(n-m)!} \frac{\partial^n f(x, y)}{\partial x^n \partial y^{n-m}} \right] d\varphi \]

\[ = f(x, y) + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{\rho^n}{m_1(n-m)!} \frac{\partial^n f(x, y)}{\partial x^n \partial y^{n-m}} \]

\[ + \frac{1}{2\pi} \int_0^{2\pi} \cos^n \varphi \sin^{n-n} \varphi d\varphi. \]

下面计算上式中的积分。

\[ \frac{1}{2\pi} \int_0^{2\pi} \cos^n \varphi \sin^{n-n} \varphi d\varphi = \frac{1}{2\pi} \int_0^{\pi} \cos^n \varphi \sin^{n-n} \varphi d\varphi \]

\[ + \frac{1}{2\pi} \int_0^{\pi} \cos^n (\pi - \varphi) \sin^{n-n} (\pi - \varphi) d\varphi \]

\[ + \frac{1}{2\pi} \int_0^{\pi} \cos^n (\pi + \varphi) \sin^{n-n} (\pi + \varphi) d\varphi \]
\[ + \frac{1}{2\pi} \int_0^\frac{x}{2} \cos^m(2\pi - \varphi) \sin^{n-m}(2\pi - \varphi) \, d\varphi \]
\[ = \frac{1}{2\pi} \left[ 1 + (-1)^n + (-1)^n + (-1)^{n-m} \right] \cdot \int_0^\frac{x}{2} \cos^m \varphi \sin^{n-m} \varphi \, d\varphi. \]

当 \( m, n \) 中至少有一个为奇数时，显见上述积分为零。

当 \( m, n \) 均为偶数时，由 2290 题的结果知：
\[ \frac{1}{2\pi} \int_0^\frac{x}{2} \cos^m \varphi \sin^{n-m} \varphi \, d\varphi = \frac{4}{\pi} \int_0^\frac{x}{2} \cos^m \varphi \sin^{n-m} \varphi \, d\varphi \]
\[ = \frac{2 \cdot \frac{\pi (2m)! (2n-2m)!}{2^{2m+1} m_1 n_1 (n-m)!} \cdot (2m)! (2n-2m)!}{2^{2m+1} m_1 n_1 (n-m)!} \cdot \frac{2^{2m+1} m_1 n_1 (n-m)!}{2^{2m+1} m_1 n_1 (n-m)!}. \]

代入原式，并注意到其中的 \( m, n \) 只能为偶数，适当改变一下指标的编号，即得
\[ F(\rho) = f(x, y) + \sum_{i=1}^n \sum_{m=0}^\infty \frac{\rho^{2s}}{(2m)! (2n-2m)!} \left( \frac{x}{2} \right)^{m} \left( \frac{y}{2} \right)^{n-m} \]
\[ \cdot \frac{\partial^{2s} f(x, y)}{\partial x^{2m} \partial y^{2n-2m}} \cdot \frac{(2m)! (2n-2m)!}{2^{2m+1} m_1 n_1 (n-m)!} \cdot \frac{(2m)! (2n-2m)!}{2^{2m+1} m_1 n_1 (n-m)!} \]
\[ = f(x, y) + \sum_{i=1}^n \frac{1}{(n_1)!^2} \left( \frac{\rho}{2} \right)^{2s} \left( \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} \right)^s f(x, y). \]
将下列函数展开成马克劳林级数:

3593. \( f(x, y) = (1 + x)^n(1 + y)^m \).

解 \( f(x, y) = (1 + x)^n(1 + y)^m - [1 + n\, x + \frac{m(m - 1)}{2!}n(n - 1) y^2 + \cdots] \cdot \frac{1}{2!} \frac{(m(m - 1))}{2!} x^2 \) + \( 2mn\, x\, y + n(n - 1) y^2 \) + \( \cdots \)  

(\( |x| < 1, |y| < 1 \)).

3594. \( f(x, y) = \ln(1 + x + y) \).

解 \( f(x, y) = \ln(1 + (x + y)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x + y)^k \) 

\[ = \sum_{k=1}^{\infty} \left[ \sum_{m=0}^{k} \frac{(-1)^{k-1}}{m} \frac{1}{m!} \frac{(k-1)!}{(k-m)!} x^m y^{k-m} \right] \]  

\[ = \sum_{k=1}^{\infty} \sum_{m=0}^{k} \frac{(-1)^{k-1}(k-1)!}{m!} x^m y^{k-m} \]  \hspace{1cm} (1) 

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^{m+n-1}(m+n-1)!}{m!n!} x^m y^n. \]  \hspace{1cm} (2)

当 \( m = 0, n = 0 \) 时，分子出现 \(-1)^1\), 规定该项为零。下面讨论一下收敛区间。(1)成立，只要求 \( |x + y| < 1 \) 即可。但从(1)式到(2)式，必须要求(1)式绝对收敛，这样才能将各项重新排列。不难看出(1)式级数各项取绝对值后即函数 \(-\ln[1 - (|x| + |y|)]\) 的展开式，它的收敛性要求 \( |x| + |y| < 1 \)。这就是 \( f(x, y) \) 的展
开式的收敛区域。

3595. \( f(x, y) = e^x \sin y \).

解

\[ f(x, y) = \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} \right) \left( \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} \right) \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^m y^{2n+1}}{m! (2n+1)!} \]

\(|x| < +\infty, \quad |y| < +\infty\).

3596. \( f(x, y) = e^x \cos y \).

解

\[ f(x, y) = \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} \right) \left( \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} \right) \]

\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^m y^{2n}}{m! (2n)!} \]

\(|x| < +\infty, \quad |y| < +\infty\).

3597. \( f(x, y) = \sin x \sinh y \).

解

\[ \sinh y = \frac{e^y - e^{-y}}{2} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{y^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{y^n}{n!} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} \quad (|y| < +\infty). \]

于是,

\[ f(x, y) = \left( \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \right) \cdot \left( \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} \right) \]

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\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1} y^{2n+1}}{(2m+1)_1(2n+1)_1} \]

\((|x| \leq +\infty, |y| \leq +\infty)\).

3598. \( f(x, y) = \cos x \ c h y \).

解 \( c h y = \frac{e^y + e^{-y}}{2} = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} \) \((|y| \leq +\infty)\).

于是，

\[
f(x, y) = \left[ \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} \right] \left[ \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} \right]
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2m} y^{2n}}{(2m)! (2n)!} \]

\((|x| \leq +\infty, |y| \leq +\infty)\).

3599. \( f(x, y) = \sin(x^2 + y^2) \).

解 \( f(x, y) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2 + y^2)^{2n+1}}{(2n+1)_1} \)

\[ = \sum_{m=0}^{\infty} \sum_{k=0}^{2n+1} (-1)^m \frac{x^{2k} y^{2(2n+1-k)}}{(2n+1-k)_1} \]

\[ = \sum_{m, n=0}^{\infty} \frac{\sin \frac{n+m}{2} \pi}{2} \frac{x^{2n} y^{2m}}{n! m!} \]

\((x^2 + y^2 \leq +\infty)\).

3600. \( f(x, y) = \ln(1+x) \ln(1+y) \).

解 \( f(x, y) = \left[ \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} \right] \left[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n} \right] \)

\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{x^m y^n}{m! n!} \] \((|x| < 1, |y| < 1)\).
3601. 写出函数

\[ f(x, y) = \int_0^1 (1 + x) t^2 y \, dt \]

的麦克劳林级数前面不为零的三项。

解 \((1 + x) t^2 y = e^{t^2 y \ln(1 + x)} \approx 1 + t^2 y \ln (1 + x) + \frac{1}{2!} [t^2 y \ln (1 + x)]^2 \)

\[ \approx 1 + t^2 y (x - \frac{x^2}{2}) = 1 + t^2 xy - \frac{t^2}{2} x^2 y. \]

于是，

\[ f(x, y) \approx \int_0^1 (1 + t^2 xy - \frac{t^2}{2} x^2 y) \, dt \]

\[ = 1 + \frac{1}{3} y (x - \frac{x^2}{2}). \]

3602. 按二项式 \((x - 1)\) 和 \((y + 1)\) 的正整数幂将函数 \(e^{x+y}\) 展开成幂级数。

解 \(e^{x+y} = e^{(x-1)+(y+1)} = e^{x-1} \cdot e^{y+1} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{(x-1)^n (y+1)^i}{n!i!}\)

\((|x| < +\infty, |y| < +\infty)\).

3603. 写出函数 \(f(x, y) = \frac{x}{y}\) 在点 \(M(1, 1)\) 的邻域内的台劳级数展开式。

解 令 \(x = 1 + h, y = 1 + k\)，则得

\[ \frac{x}{y} = \frac{1 + h}{1 + k} = (1 + h) \sum_{n=0}^{\infty} (-1)^n h^n \]
\[ = \sum_{n=0}^{\infty} (-1)^n (1 + (x-1)) (y-1)^n \]

\[ (|x| \leq +\infty, \ 0 \leq y \leq 2) \]

3604. 设 \( z \) 为由方程 \( z^3 - 2xz + y = 0 \) 所定义的 \( x \) 和 \( y \) 的隐函数，当 \( x = 1 \) 和 \( y = 1 \) 时它的值为 \( z = 1 \)。

写出函数 \( z \) 按二项式 \( x-1 \) 和 \( y-1 \) 的升幂排列的展开式中的若干项。

解  对原方程微分一次，得
\[ 3z^2 \frac{dz}{dx} - 2xdz - 2zd - dy = 0. \]  

再微分一次，得
\[ (3z^2 - 2x) \frac{d^2 z}{dx^2} + 6zd - 4xdz = 0. \]

以 \( x = 1, y = 1, z = 1 \) 代入(1), (2)两式，得
\[ \frac{dz}{dx} = 2d - dy, \]
\[ \frac{d^2 z}{(4dx - 6dz) dz} = \frac{4dx - 12dx + 6dy}{(2dx - dy)} \]
\[ = -16dx^2 + 20dxdy - 6dy^2, \]

于是，可求得在 \( x = 1, y = 1 \) 处，
\[ \frac{dz}{dx} = 2, \frac{dz}{dy} = -1; \]
\[ \frac{d^2 z}{dx^2} = -16, \frac{d^2 z}{dx dy} = 10, \frac{d^2 z}{dy^2} = -6; \]

从而有
\[ z = 1 + 2(x-1) - (y-1) - \left[ 8(x-1)^2 \right] \]
研究下列曲线的奇点的种类并大略地绘出这些曲线：

3605. $y^2 = ax^2 + x^3$.

解 解方程组

$$
\begin{align*}
F(x,y) &= ax^2 + x^3 - y^2 = 0, \\
F'_x(x,y) &= 2ax + 3x^2 = 0, \\
F'_y(x,y) &= -2y = 0
\end{align*}
$$

得 $x = 0, y = 0$，故点 $(0, 0)$ 为奇点。

其次，由于

$$
A = F''_x(0,0) = 2a, B = F''_y(0,0) = 0, \\
C = F''_{xy}(0,0) = -2, AC - B^2 = -4a
$$

故当 $a > 0$ 时，点 $(0, 0)$ 为二重点；当 $a < 0$ 时，点 $(0, 0)$ 为孤立点；当 $a = 0$ 时，原方程化为 $y^2 = x^3$，由 3574(6) 的讨论知点 $(0, 0)$ 为尖点。

如图 6·35 所示，点 $A_1$ 为 $(-a, 0)$。

3606. $x^3 + y^3 - 3xy = 0$.

解 解方程组
\[
\begin{aligned}
F(x, y) &= x^3 + y^3 - 3xy = 0,
F_x(x, y) &= 3x^2 - 3y = 0,
F_y(x, y) &= 3y^2 - 3x = 0
\end{aligned}
\]
得 \( x = 0, y = 0 \)，故点 \((0, 0)\) 为奇点。
又因 \( A = F''_{xx}(0, 0) = 0, \quad B = F''_{xy}(0, 0) = -3, \quad C = F''_{yy}(0, 0) = 0, \quad AC - B^2 = -9 < 0 \)，故点 \((0, 0)\) 为二重点，图象参看 370 题 (6)。

3607. \( x^2 + y^2 = x^4 + y^4 \)。
解 方程组
\[
\begin{aligned}
F(x, y) &= x^2 + y^2 - x^4 - y^4 = 0,
F_x(x, y) &= 2x - 4x^3 = 0,
F_y(x, y) &= 2y - 4y^3 = 0
\end{aligned}
\]
得 \( x = 0, y = 0 \)，故点 \((0, 0)\) 为奇点。
又因 \( A = F''_{xx}(0, 0) = 2, \quad B = F''_{xy}(0, 0) = 0, \quad C = F''_{yy}(0, 0) = 2, \quad AC - B^2 = 4 > 0 \)，故点 \((0, 0)\) 为孤立点，图象参看 1542 题。

3608. \( x^2 + y^4 = x^6 \)。
解 解方程组
\[
\begin{aligned}
F(x, y) &= x^2 + y^4 - x^6 = 0,
F_x(x, y) &= 2x - 6x^5 = 0,
F_y(x, y) &= 4y^3 = 0
\end{aligned}
\]
得 \( x = 0, y = 0 \)，故点 \((0, 0)\) 为奇点。
又因 \( A = F''_{xx}(0, 0) = 2, \quad B = F''_{xy}(0, 0) = 0, \quad C = F''_{yy}(0, 0) = 0, \quad AC - B^2 = 0 \)，故点 \((0, 0)\) 为上升点或孤立点，本题中，点 \((0, 0)\) 为孤立点 (图 6.36)。事
实际上，将原方程改写为 $y^4 = x^5 - x^2$，对 $(0, 0)$
点的很小的邻域内的点 $(|x| < 1, |y| < 1)$，
左端 $y^4 \geq 0$，右端 $x^5 - x^2 = x^2 (x^4 - 1) \leq 0$，
除点 $(0, 0)$ 外没有适合方程的点，故点 $(0, 0)$
为孤立点。

3609. $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$。

解  方程组

$$
\begin{align*}
F(x, y) &= (x^2 + y^2)^2 - a^2 (x^2 - y^2) = 0, \\
F'_x(x, y) &= 4x(x^2 + y^2) - 2a^2 x = 0, \\
F'_y(x, y) &= 4y(x^2 + y^2) + 2a^2 y = 0
\end{align*}
$$

得 $x = 0, y = 0$，故点 $(0, 0)$ 为奇点。

又因 $A = F''_x(0, 0) = -2a^2, B = F''_y(0, 0) = 0, C = F'''_x(0, 0) = 2a^2$, 且 $AC - B^2 = -4a^4 \leq 0 (a \neq 0)$，
故点 $(0, 0)$ 为二重点。图象参看 3367 题，只须将该题中的 $1$ 换成 $a$。

3610. $(y - x^2)^2 = x^5$。

解  方程组

$$
\begin{align*}
F(x, y) &= (y - x^2)^2 - x^5 = 0, \\
F'_x(x, y) &= -4x(y - x^2) - 5x^4 = 0, \\
F'_y(x, y) &= 2(y - x^2) = 0
\end{align*}
$$

得 $x = 0, y = 0$，故点 $(0, 0)$ 为奇点。
又因 \( A = F''_{xx}(0, 0) = 0 \), \( B = F''_{xy}(0, 0) = 0 \), \( C = F''_{yy}(0, 0) = 2 \), 且 \( AC - B^2 = 0 \), 故对点 \((0, 0)\) 还需要再讨论一下。由原方程可解出 \( y = x^2 \pm x^2 \), 右边只允许 \( x \geq 0 \), 当 \( 0 < x < 1 \) 时不论取“+”号还是“−”号均有 \( y \geq 0 \), 且均有
\[
\lim_{x \to 0} \frac{dy}{dx} = 0 ,
\]
故点 \((0, 0)\) 为尖点，如图 6.37 所示。

3671. \((a + x)y^2 = (a - x)x^2\)。图 6.37

解  解方程组
\[
\begin{align*}
F(x, y) &= (a + x)y^2 - (a - x)x^2 = 0, \\
F'_x(x, y) &= y^2 - 2ax + 3ax^2 = 0, \\
F'_y(x, y) &= 2(a + x)y = 0.
\end{align*}
\]
由(3)得 \( x = -a \) 或 \( y = 0 \).
将 \( y = 0 \) 代入(1), (2), 得 \( x = 0 \).
将 \( x = -a \) 代入(1)式，得 \((a - x)x^2 = 0\)。若 \( a \neq 0 \), 则得出矛盾的结果。若 \( a = 0 \), 则也得到 \( x = 0 \), \( y = 0 \), 故点 \((0, 0)\) 为奇点。

又因 \( A = F''_{xx}(0, 0) = -2a \), \( B = F''_{xy}(0, 0) = 0 \), \( C = F''_{yy}(0, 0) = 2a \), 且 \( AC - B^2 = -4a^2 \), 故当 \( a \neq 0 \) 时，点 \((0, 0)\) 为二重点；当 \( a = 0 \) 时，方程转化为 \( xy = -x^2 \), 从而曲线为 \( x = 0 \), 点 \((0, 0)\) 为上升点。

如图 6.38 所示，图中点 \( A_1 \) 为 \((a, 0)\)。
3.612. 研究参数变量 \( a, b, c \) (\( a \leq b \leq c \))的值与曲线 \( y^2 = (x-a) \cdot (x-b) \cdot (x-c) \)的形状之关系。

解 　解方程组
\[
\begin{align*}
F(x, y) &= y^2 - (x-a)(x-b)(x-c) = 0, \quad (1) \\
F_x'(x, y) &= -(x-a)(x-b) - (x-a) \\
& \quad \cdot (x-c) - (x-b)(x-c) = 0, \quad (2) \\
F_y'(x, y) &= 2y = 0. \quad (3)
\end{align*}
\]

由(3)得 \( y = 0 \)，代入(1)，联立(1)，(2)求解。

当 \( a < b < c \) 时，(1)，(2)无解。因此无奇点，此时曲线如图6·39(1)所示。

当 \( a = b < c \) 时，显然(1)，(2)有解 \( x = a, y = 0 \)。
由于 \( A = F_x''(a, 0) = -2(a-c), B = F_y''(a, 0) = 0, \)
\( C = F_y''(a, 0) = 2, \) 且 \( AC - B^2 = -4(a-c) > 0, \) 故点 \( A_1(a, 0) \) 为孤立点，如图6·39(2)所示。

当 \( a < b = c \) 时，显然(1)，(2)有解 \( x = b, y = 0 \)。
由于 \( A = F_x''(b, 0) = -2(c-a), B = F_y''(b, 0) = 0, \)
\( C = F_y''(b, 0) = 2, \) 且 \( AC - B^2 = -4(c-a) < 0, \) 故点 \( A_2(b, 0) \) 为二重点，如图6·39(3)所示。
当 \(a = b = c\) 时，显然有解 \(x = a, y = 0\)。由于 \(AC - B^2 = 0\)，此时原方程为 \(y^2 = (x-a)^2\)，且由3574题(6)的结果知，点 \(A_1(a, 0)\) 为尖点，如图6·39(4)所示。

![图6·39](image)

研究超越曲线的奇点：

3613. \(y^2 = 1 - e^{-x^2}\)。

解  解方程组

\[
\begin{cases}
F(x, y) = y^2 - 1 + e^{-x^2} = 0, \\
F'(x, y) = -2xe^{-x^2} = 0, \\
F''(x, y) = 2y = 0
\end{cases}
\]

得 \(x = 0, y = 0\)，故点 \((0, 0)\) 为奇点。
又 \( A = F''_{xx}(0, 0) = -2 \), \( B = F''_{xy}(0, 0) = 0 \), \( C = F''_{yy}(0, 0) = 2 \), 且 \( AC - B^2 = -4 < 0 \), 故点 \((0, 0)\) 为二重点。

3614. \( y^2 = 1 - e^{-x^2} \).

解  方程组

\[
\begin{align*}
F(x, y) &= y^2 - 1 + e^{-x^2} = 0, \\
F'_x(x, y) &= -2xy^2 e^{-x^2} = 0, \\
F'_y(x, y) &= 2y = 0
\end{align*}
\]

得 \( x = 0 \), \( y = 0 \), 故点 \((0, 0)\) 为奇点。

又因 \( A = F''_{xx}(0, 0) = 0 \), \( B = F''_{xy}(0, 0) = 0 \), \( C = F''_{yy}(0, 0) = 2 \), 且 \( AC - B^2 = 0 \), 故对点 \((0, 0)\) 还需再讨论一下。原式可解为 \( x = -\sqrt{\ln(1 - y^2)} \geq 0 \), 在 \((0, 0)\) 附近，第一及第四象限各有原曲线的一支，因此，点 \((0, 0)\) 为尖点。

3615. \( y = x \ln x \).

解  \( F(x, y) = x \ln x - y, \)
\( F'_x(x, y) = 1 + \ln x, F'_y(x, y) = -1 \neq 0 \), 故无奇点。如图6-40所示。

3616. \( y = \frac{x}{1 + e^x} \).

解  在 \( x = 0 \) 处，由于

\[
\lim_{x \to 0} y = \lim_{x \to 0} y = 0,
\]

故 \( x = 0 \) 为“可移去”的第一类不连续点，补充函数在该点的值为零后，即得知函数
\[ y = \begin{cases} \frac{x}{1 + e^x}, & x \neq 0, \\ 0, & x = 0 \end{cases} \]

在点 \( x = 0 \) 连续，由于 \( F'_x(x, y) = 1 \neq 0 \)，故无奇点。当 \( x \neq 0 \) 时，由于，

\[
y' = \frac{(1 + \frac{1}{x})e^{\frac{1}{x}} + 1}{(1 + e^x)^2},
\]

\[
\lim_{x \to +0} y' = \lim_{x \to +\infty} \frac{(1 + \frac{1}{x})e^{\frac{1}{x}} + 1}{(1 + e^x)^2} = \lim_{x \to +\infty} \frac{e^{x}(x+2)}{2e^{x}(1+e^x)} = 0,
\]

\[
\lim_{x \to 0} y' = \lim_{x \to +\infty} \frac{(1-x)e^{-x} + 1}{(1 + e^{-x})^2} = 1,
\]

故点 \((0, 0)\) 为角点，如图 6-41 所示。

3617. \( y = \arctg\left(\frac{1}{\sin x}\right) \)。图 6-41

解 \( x = k\pi \) \( (k = 0, \pm 1, \pm 2, \ldots) \) 点为不连续点。由于

\[
\lim_{x \to k\pi \pm 0} y = (-1)^{k-1} \frac{\pi}{2}, \quad \lim_{x \to k\pi \pm 0} \frac{\pi}{2},
\]

故点 \( x = k\pi \) 为函数的第一类不连续点。

3618. \( y^2 = \sin \frac{\pi}{x} \)。
解 \[ y = \pm \sqrt{\sin \frac{\pi}{x}} \], 它在 \((\frac{1}{2k}, \frac{1}{2k+1}) \) \((k = \pm 1, \pm 2, \cdots)\)内无定义。

在边界点 \(x = \frac{1}{2k}\) 及 \(x = \frac{1}{2k+1}\), \(y = 0\)。

函数图象有上下两支。

设 \(F(x,y) = y^2 - \sin \frac{\pi}{x}\), 则在边界点, 由于 \(F_x = 0, F_y = 0\), 故也无奇点。

在 \((0,0)\) 点的任何邻域内，自无穷多个曲线的封闭分支，这些分支设有某过 \((0,0)\) 点，它不属于任何一种类型。

3619. \(y^2 = \sin x^2\)。

解 \(F(x,y) = y^2 - \sin x^2 = 0\)。

\[\begin{align*}
F_x(x,y) &= -2\cos x^2 = 0, \\
F_y(x,y) &= 2y = 0
\end{align*}\]

得 \(x = 0, y = 0\), 故点 \((0,0)\) 为奇点。

又因 \(A = F_{xx}(0,0) = -2, B = F_{xy}(0,0) = 0, C = F_{yy}(0,0) = 2\), 且 \(AC - B^2 = -4 < 0\), 故点 \((0,0)\)为二重点。

3620. \(y^2 = \sin^3 x\)。

解 显见，函数 \(y\) 的周期为 \(2\pi\)，在 \((2k\pi, (2k+1)\pi)\) 内函数有定义，而在 \((2k - 1)\pi, 2k\pi) (k = 0, \pm 1, \pm 2, \cdots)\) 内无定义。
解方程组

\[
\begin{align*}
F(x, y) &= y^2 - \sin^3 x = 0, \\
F'_x(x, y) &= -3\sin^2 x \cos x = 0, \\
F'_y(x, y) &= 2y = 0
\end{align*}
\]

得 \( x = 0, \ y = 0 \)，故点 \((0, 0)\) 为奇点。

在点 \((0, 0)\) 的左侧（指充分小的范围，下同，不再说明）无曲线的点，而在右侧的第一、第四象限分别有曲线的两枝，因此，点 \((0, 0)\) 为尖点，如图6.42所示。

由周期性可知，
点 \((k\pi, 0)\) \((k = \pm 1, \pm 2, \cdots)\) 也为尖点。只是当 \(k\) 是偶数时，
右侧才有曲线的两枝；当 \(k\) 是奇数时，
左侧才有曲线的两枝。
§7. 多变量函数的极值

1° 极值的定义 若函数 \( f(P) = f(x_1, \ldots, x_n) \) 于点 \( P_0 \) 的
邻域内有定义并且当 \( 0 < r(P_0, P) < \delta \) 时，\( f(P_0) > f(P) \) 或
\( f(P_0) < f(P) \)，则说函数 \( f(P) \) 在点 \( P_0 \) 有极值（相应地为
极大值或极小值）。

2° 极值的必要条件 可微分的函数 \( f(P) \) 仅在静止点 \( P_0 \)，
即是说在 \( df(P_0) = 0 \) 的点 \( P_0 \) 能达到极值，所以，函数 \( f(P) \)
的极值点应当满足方程组 \( f'_i(x_1, \ldots, x_n) = 0 \) (i = 1, \ldots, n)。

3° 极值的充分条件 函数 \( f(P) \) 于点 \( P_0 \) 有：
（a）极大值，若 \( df(P_0) = 0 \)，\( d^2f(P_0) < 0 \)；
（b）极小值，若 \( df(P_0) = 0 \)，\( d^2f(P_0) > 0 \)。

研究二次微分 \( d^2f(P_0) \) 的符号可用化相应的二次式成典
式的方法来进行。

特别是，对于两个自变量 \( x \) 和 \( y \) 的函数 \( f(x, y) \) 在静止
点 \((x_0, y_0) \) \( df(x_0, y_0) = 0 \) \( D = AC - B^2 \neq 0 \) [其中
\( A = f'_{xx}(x_0, y_0) \)，\( B = f'_{xy}(x_0, y_0) \)，\( C = f'_{yy}(x_0, y_0) \)] 成立时，有：
（1）极小值，若 \( D > 0 \)，\( A > 0 \) （\( C > 0 \)）；
（2）极大值，若 \( D > 0 \)，\( A < 0 \) （\( C < 0 \)）；
（3）极值不存在，若 \( D < 0 \)。

4° 条件极值 在关系式 \( g_i(P) = 0 \) (i = 1, \ldots, m; m < n)

\( \star \) 线者注：若将不等式 \( f(P_0) > f(P) \)（或 \( f(P_0) < f(P) \)）
换为不等式 \( f(P_0) > f(P) \) （或 \( f(P_0) < f(P) \)），则称 \( f(P) \) 在点 \( P_0 \) 有局部极大值
（或局部极小值）。

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存在的条件下，求函数 \( f(P_0) = f(x_1, x_2, \ldots, x_n) \) 的极值的问题，可归结为对于拉格朗日函数

\[
L(P) = f(P) + \sum_{i=1}^{n} \lambda_i \varphi_i(P)
\]

（其中 \( \lambda_i (i = 1, \ldots, m) \) 为常数因子）求普通极值的问题。关于条件极值的存在和性质的问题，在最简单的情况下，根据研究函数 \( L(P) \) 于静止点 \( P_0 \) 的二次微分 \( d^2 L(P_0) \) 的符号，并在变量 \( dx_1, dx_2, \ldots, dx_n \) 由下面的关系式

\[
\sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_j} dx_j = 0 \quad (i = 1, \ldots, m)
\]

所限制的条件下，得到解决。

5° 绝对极值 于界且闭合的区域内可微分的函数 \( f(P) \) 在此域内或于静止点，或于域的边界点达到自己的最大值和最小值。

研究下列多变量函数的极值：

3621. \( z = x^2 + (y-1)^2 \)。

解  解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x = 0, \\
\frac{\partial z}{\partial y} &= 2(y-1) = 0
\end{align*}
\]

得静止点 \( P_0 (0, 1) \)。显然 \( z(0, 1) = 0 \)，且当 \((x, y) \neq (0, 1) \) 时 \( z > 0 \)，故函数 \( z \) 在点 \( P_0 \) 取得极小值 \( z(P_0) = 0 \)（实际是极小值）。

3622. \( z = x^2 - (y-1)^2 \)。

解  解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x = 0, \\
\frac{\partial z}{\partial y} &= -2(y-1) = 0
\end{align*}
\]
得驻点 \( P_0(0, 1) \)。由于
\[
A = z_{xx}(0, 1) = 2, \quad B = z_{xy}(0, 1) = 0, \quad C = z_{yy}(0, 1) = -2, \quad \text{且} \quad AC - B^2 = -4 < 0, \quad \text{故极值不存在(或用该点附近的} z \text{值可正可负说明)}.
\]
3623. \( z = (x - y + 1)^2 \).
解  解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2(x - y + 1) = 0, \\
\frac{\partial z}{\partial y} &= -2(x - y + 1) = 0
\end{align*}
\]
得驻点分布在直线 \( x - y + 1 = 0 \) 上。对于此直线上的点均有 \( z = 0 \)，但是 \( z \geq 0 \) 恒成立。因此，函数 \( z \) 在直线 \( x - y + 1 = 0 \) 上的各点取得弱极小值 \( z = 0 \)。
3624. \( z = x^2 - xy + y^2 - 2x + y \).
解  解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x - y - 2 = 0, \\
\frac{\partial z}{\partial y} &= -x + 2y + 1 = 0
\end{align*}
\]
得驻点 \( P_0(1, 0) \)。由于
\[
A = z_{xx}(1, 0) = 2, \quad B = z_{xy}(1, 0) = -1, \quad C = z_{yy}(1, 0) = 2, \quad \text{且} \quad AC - B^2 = 3 > 0, \quad \text{故函数} \ z \text{在点} \ 417
\( P_0 \) 取得极小值 \( x(P_0) = -1 \).

3525. \( z = x^2 y^3 (6 - x - y) \).

解  方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= x y^3 (12 - 3x - 2y) = 0, \\
\frac{\partial z}{\partial y} &= x^2 y^2 (18 - 3x - 4y) = 0
\end{align*}
\]

得静止点 \( P_0 (2, 3) \)，并且直线 \( x = 0 \) 及直线 \( y = 0 \) 上的点都是静止点。

不难断定在 \( P_0 \) 点，\( A = -162, B = -108, C = -144, AC - B^2 = 0 \)，故函数 \( z \) 在点 \( P_0 \) 取得极大值 \( z(P_0) = 108 \)。

在直线 \( x = 0 \) 及 \( y = 0 \) 上的各点均有 \( z = 0 \)。先分析直线 \( y = 0 \) 的情况。在直线上 \( x \neq 0 \) 及 \( x = 6 \) 处，

\( x^2 (6 - x - y) \neq 0 \)，在确定点的足够小的邻域内也不变号，但是 \( y^3 \) 可正可负，因此函数 \( z \) 变号，即在上述情况下没有极值。当 \( x = 0 \) 及 \( x = 6 \) 类似地可判断也无极值。

其次分析直线 \( x = 0 \) 的情况。在直线上 \( y = 0 \) 及 \( y = 6 \) 的点的情况类似地可判断无极值。但当 \( 0 < y < 6 \) 时，\( y^3 (6 - x - y) \geq 0 \)，且在所讨论点的足够小的

邻域内保持正号。因此，在足够小的邻域内，

\( z = x^2 y^3 (6 - x - y) \geq 0 \)也成立，但邻域内任意近处总有 \( z = 0 \) 的点。于是，对于 \( x = 0 \)，\( 0 < y < 6 \) 的点函数 \( z \) 取得弱极小值 \( z = 0 \)。同法可判定，对于直线 \( x = 0 \) 上

\( y < 0 \) 及 \( y > 6 \) 的各点处，函数 \( z \) 取得弱极大值 \( z = 0 \)。
3626. \( z = x^3 + y^3 - 3xy \).

解  
解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 3x^2 - 3y = 0, \\
\frac{\partial z}{\partial y} &= 3y^2 - 3x = 0
\end{align*}
\]

得静止点 \( P_0(0,0) \) 及 \( P_1(1,1) \)。

不难断定，若点 \( P_0 \) 有 \( A = 0, B = -3, C = 0 \) 及 \( AC - B^2 = -9 < 0 \)，故无极值；而在点 \( P_1 \) 有 \( A = 6, B = -3, C = 6 \) 及 \( AC - B^2 = 27 > 0 \)，故函数 \( z \) 在该点取得极小值 \( z(P_1) = -1 \)。

3627. \( z = x^4 + y^4 - x^2 - 2xy - y^2 \).

解  
解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 4x^3 - 2x - 2y = 0, \\
\frac{\partial z}{\partial y} &= 4y^3 - 2x - 2y = 0
\end{align*}
\]

得静止点 \( P_0(0,0), P_1(1,1) \) 及 \( P_2(-1,1) \)。

在点 \( P_0 \) 附近，当 \( x = y \) 且足够小时，有 \( z = 2x^4 - 4x^2 < 0 \)；当 \( x = -y \) 时，\( z = 2x^4 \geq 0 \)，因此，在点 \( P_0 \) 无极值。

不难断定，在点 \( P_1 \) 及 \( P_2 \) 均有 \( A = 10, B = -2, C = 10 \) 及 \( AC - B^2 = 96 > 0 \)，故函数 \( z \) 在点 \( P_1 \) 及 \( P_2 \) 取得极小值 \( z = -2 \)。

3628. \( z = xy + \frac{50}{x} + \frac{20}{y} \) \( (x > 0, y > 0) \)。
解

解方程组

\[
\begin{cases}
\frac{\partial z}{\partial x} = y - \frac{50}{x^2} = 0, \\
\frac{\partial z}{\partial y} = x - \frac{50}{y^2} = 0
\end{cases}
\]

得静止点 \( P_0(5, 2) \)。不难断定，在该点有 \( A = \frac{4}{5} \), \( B = 1 \), \( C = 5 \) 及 \( AC - B^2 = 3 \geq 0 \)，故函数 \( z \) 在该点取得极小值 \( z(P_0) = 30 \)。

3629. \( z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \ (a \rightarrow 0, \ b \rightarrow 0) \)。

解

考虑函数 \( u = z^2 = x^2y^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \), \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \)。

显然 \( z \) 的极值均为 \( u \) 的极值，且 \( u \) 在点 \((x, y)\) 取得的极值不为零时，\( z \) 也在点 \((x, y)\) 取得极值，\( u \) 在点 \((x, y)\) 取得的极值为零时，情况复杂一些，但对 \( z \) 也不难讨论。

解方程组

\[
\begin{cases}
\frac{\partial u}{\partial x} = 2xy^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \frac{2}{a^2}x^3y^2 = 0, \\
\frac{\partial u}{\partial y} = 2x^2y\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \frac{2}{b^2}x^2y^3 = 0
\end{cases}
\]

得静止点 \( P_0(0, 0), P_1\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right), P_2\left(-\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right), \)

\( -\frac{b}{\sqrt{3}} \), \( P_3\left(\frac{a}{\sqrt{3}}, -\frac{b}{\sqrt{3}}\right)及 P_4\left(-\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)。\)

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由于 $z$ 在点 $P_0$ 附近变号，所以 $z(P_0)$ 不是极值。

$$
\frac{\partial^2 u}{\partial x^2} = 2y^2 \left( 1 - \frac{6x^2}{a^2} - \frac{y^2}{b^2} \right),
$$

$$
\frac{\partial^2 u}{\partial y^2} = 2x^2 \left( 1 - \frac{x^2}{a^2} - \frac{6y^2}{b^2} \right),
$$

$$
\frac{\partial^2 u}{\partial x \partial y} = 4xy \left( 1 - \frac{2x^2}{a^2} - \frac{2y^2}{b^2} \right).
$$

在 $P_1, P_2, P_3, P_4$ 各点，得

$$
A = -\frac{8}{9} b^2, \quad B = \pm \frac{4}{9} ab, \quad C = -\frac{8}{9} a^2,
$$

$$
AC - B^2 = \left( \frac{64}{81} - \frac{16}{81} \right) a^2 b^2 > 0,
$$

故函数 $u$ 取得正的极大值。于是，相应地函数 $z$ 在点 $P_1$ 及 $P_2$ 取得极大值 $z(P_1) = z(P_2) = \frac{ab}{3 \sqrt{3}}$，而在点 $P_3$ 及 $P_4$ 取得极小值 $z(P_3) = z(P_4) = -\frac{ab}{3 \sqrt{3}}$。

3630. $z = \frac{ax + by + c}{\sqrt{x^2 + y^2 + 1}}$ $(a^2 + b^2 + c^2 \neq 0)$。

解 计 $x = r \cos \varphi, \quad y = r \sin \varphi$，则

$$
z(x, y) = z(r \cos \varphi, \ r \sin \varphi) = \frac{ar \cos \varphi + br \sin \varphi + c}{\sqrt{r^2 + 1}}.
$$

解方程组

$$
\begin{aligned}
\frac{\partial z}{\partial r} = \frac{a \cos \varphi + b \sin \varphi - cr}{(1 + r^2)^{3/2}} = 0 , \\
\frac{\partial z}{\partial \varphi} = \frac{-ar \sin \varphi + br \cos \varphi}{(1 + r^2)^{1/2}} = 0.
\end{aligned}
$$

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先设 \(a, b\) 不同时为零。由 (2) 考虑到 \(r = 0\) 不是解

\((r = 0, \varphi\) 为任意值不满足(1)式)，故有 \(a \sin \varphi = b \cos \varphi\)。于是，

\[
\cos \varphi = \frac{\pm a}{\sqrt{a^2 + b^2}}, \quad \sin \varphi = \frac{\pm b}{\sqrt{a^2 + b^2}}. \tag{3}
\]

显见当 \(c = 0\) 时无解(因由(1)有 \(a \cos \varphi + b \sin \varphi = 0\)，
再由 (3) 得 \(a = b = 0\) 与 \(a, b\) 不同时为零之假定矛盾)。当 \(c \neq 0\) 时，

\[
r = \frac{a \cos \varphi + b \sin \varphi}{c} = \pm \frac{\sqrt{a^2 + b^2}}{c}.
\]

为保证 \(r > 0\)，在 \(\cos \varphi\) 及 \(\sin \varphi\) 前取与 \(c\) 一致的符号，此时，有

\[
x = \frac{a}{c}, \quad y = \frac{b}{c}.
\]

由于这时 \(z''_{rr} = -\frac{c(1 + 3r^2)}{(1 + r^2)^2}, \)

\(z''_{\varphi\varphi} = -\frac{cr^2}{(1 + r^2)^{\frac{3}{2}}}, \quad z''_{r\varphi} = 0\)

及 \(z''_{rr} z''_{\varphi\varphi} - (z''_{r\varphi})^2 > 0\)，故当 \(c > 0\) 时 \(z''_{rr} < 0\)。函数 \(z\) 在点 \((\frac{a}{c}, \frac{b}{c})\) 取得极大值 \(z = \sqrt{a^2 + b^2 + c^2}\)；当
\(c < 0\) 时 \(z''_{rr} > 0\)，函数 \(z\) 在点 \((\frac{a}{c}, \frac{b}{c})\) 取得极小值
\(z = -\sqrt{a^2 + b^2 + c^2}\)。

下设 \(a = b = 0\)。由假定 \(a^2 + b^2 + c^2 \neq 0\) 知 \(c \neq 0\)。
此时解方程组（1），（2）得 \( r = 0, \phi \) 任意，即 \( x = 0, y = 0 \)。由于这时 \( z = \frac{c}{\sqrt{x^2 + y^2} + 1} \)，故显然知：当 \( c > 0 \) 时 \( z \) 在点 \((0, 0)\) 取极大值 \( z = c \)；当 \( c < 0 \) 时，\( z \) 在点 \((0, 0)\) 取极小值 \( z = c \)。

综合上述结果，得结论：若 \( c > 0 \)，则 \( z \) 在点 \((\frac{a}{c}, \frac{b}{c})\) 取极大值 \( z_{\text{极大}} = \sqrt{a^2 + b^2 + c^2} \)；若 \( c < 0 \)，则 \( z \) 在点 \((\frac{a}{c}, \frac{b}{c})\) 取极小值 \( z_{\text{极小}} = -\sqrt{a^2 + b^2 + c^2} \)；若 \( c = 0 \)（由假定，这时 \( a^2 + b^2 \neq 0 \)），则 \( z \) 无极值。

注：此题也可不作变量代换 \( x = r \cos \phi, y = r \sin \phi \)（极坐标），而直接在直角坐标 \( x, y \) 下进行讨论，即解方程组 \( \frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0 \) 并计算 \( \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2} \) 之值。但此法计算较繁琐，没有用极坐标简单。

3531. \( z = 1 - \sqrt{x^2 + y^2} \)。

解：\( \frac{\partial z}{\partial x} = -\frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{x^2 + y^2}} \)。

点 \((0, 0)\) 为偏导函数无意义的点。当 \((x, y) \neq (0, 0)\) 时，\( z < 1 \)，故 \( z(0, 0) = 1 \) 为极大值。

3532. \( z = a^2x^3 + 5y(8x^2 - 6xy + 3y^2) \)。

解：解方程组
\[
\frac{\partial z}{\partial x} = 2e^{2x-3y}(8x^2 - 6xy + 3y^2 + 8x - 3y) = 0,
\]
\[
\frac{\partial z}{\partial y} = 3e^{2x-3y}(8x^2 - 6xy + 3y^2 - 2x + 2y) = 0
\]

得静止点 \( P_0 (0, 0) \) 及 \( P_1 \left( -\frac{1}{4}, -\frac{1}{2} \right) \).

\[
\frac{\partial^2 z}{\partial x^2} = 4e^{2x-3y}(8x^2 - 6xy + 3y^2 + 16x - 6y + 4),
\]
\[
\frac{\partial^2 z}{\partial y^2} = 9e^{2x-3y}(8x^2 - 6xy + 3y^2 - 4x + 4y + \frac{2}{3}),
\]
\[
\frac{\partial^2 z}{\partial x \partial y} = 6e^{2x-3y}(8x^2 - 6xy + 3y^2 + 6x - y - 1).
\]

在点 \( P_0 \), \( A = 16, B = -6, C = 6 \) 及 \( AC - B^2 = 60 > 0 \),
故函数 \( z \) 取得极小值 \( \dot{z}(P_0) = 0 \);
在点 \( P_1 \), \( A = 14e^{-2}, B = -9e^{-2}, C = \frac{3}{2}e^{-2} \) 及 \( AC - B = -60e^{-4} < 0 \), 故

无极值.

3633. \( z = e^{x^2 - y(5 - 2x + y)} \).

解
解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2e^{x^2 - y(5x - 2x^2 + xy - 1)} = 0, \\
\frac{\partial z}{\partial y} &= e^{x^2 - y(2x - y - 4)} = 0
\end{align*}
\]

得静止点 \( P_0 (1, -2) \).

\[
\frac{\partial^2 z}{\partial x^2} = 2e^{x^2 - y(10x^2 - 4x^3 + 2x^2 y - 6x + y + 5)},
\]

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\[
\frac{\partial^2 z}{\partial y^2} = e^{x^2 - y} (3 - 2x + y),
\]
\[
\frac{\partial^2 z}{\partial x \partial y} = 2e^{x^2 - y} (2x^2 - xy - 4x + 1).
\]

在点 \( P_0 \), \( A = -2e^b \), \( B = 2e^b \), \( C = -e^b \) 及 \( AC - B^2 = -2e^b < 0 \), 故无极值。

3634. \( z = (5x + 7y - 25)e^{-(x^2 + xy + y^2)} \).

解
解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 5e^{-(x^2 + xy + y^2)} - (5x + 7y - 25) \\
&- (2x + y)e^{-(x^2 + xy + y^2)} = 0, \quad (1) \\
\frac{\partial z}{\partial y} &= 7e^{-(x^2 + xy + y^2)} - (5x + 7y - 25) \\
&- (x + 2y)e^{-(x^2 + xy + y^2)} = 0. \quad (2)
\end{align*}
\]

(1) \times 7 - (2) \times 5, 消去因子 \( e^{-(x^2 + xy + y^2)} \), 得

\[3(5x + 7y - 25)(3x - y) = 0.\]

以 \( 5x + 7y - 25 = 0 \) 代入 \( (1) \), \( (2) \), 显然矛盾, 故必有 \( 5x + 7y - 25 \neq 0 \), 从而 \( y = 3x \). 代入 \( (1) \), 得

\[26x^2 - 25x - 1 = 0.\]

解得静止点 \( P_0 (1, 3) \) 及 \( P_1 \left(-\frac{1}{26}, -\frac{3}{26}\right) \). 在点 \( P_0 \),

\[
A = z_{xx}(P_0) = \left[z'_x(x, 3)'\right]_{x=1}
\]
\[
= \left[ e^{-(x^2 + 3x + 9)} \cdot [5 - (5x - 4)(2x + 3)] \right]_{x=1}
\]
\[
= \left[ e^{-(x^2 + 3x + 9)} \right]_{x=1} \cdot [5 - (5x - 4)(2x + 3)]_{x=1}
\]
\[
+ \left[ e^{-(x^2 + 3x + 9)} \right]_{x=1} \cdot [5 - (5x - 4)(2x + 3)]_{x=1}
\]
\[
\cdot (2x + 3)_{x=1}
\]
\[
= -27e^{-13}.
\]

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同法可求得
\[ B = z_{xx}(P_0) = -36e^{-18} \quad C = z_{yy}(P_0) = -51e^{-18}. \]
于是，\( AC - B^2 = 81e^{-36} > 0 \)，故函数z在点\( P_0 \)取得极大值z(\( P_0 \)) = \( e^{-18} \approx 2.26 \times 10^{-6} \)。

同法可得函数z在点\( P_1 \)取得极小值z(\( P_1 \)) = \( -26e^{-\frac{1}{2}} \approx -25.51 \)。

3635. \( z = x^2 + xy + y^2 - 4\ln x - 10\ln y \)。
解
解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x + y - \frac{4}{x} = 0, \\
\frac{\partial z}{\partial y} &= x + 2y - \frac{10}{y} = 0
\end{align*}
\]
得驻点\( P_0(1,2) \)。在点\( P_0 \)，
\[ A = 6, \quad B = 1, \quad C = \frac{9}{2}, \quad AC - B^2 = 26 > 0, \]
故函数z在点\( P_0 \)取得极小值z(\( P_0 \)) = \( 7 - 10\ln 2 \approx 0.0685 \)。

3636. \( z = \sin x + \cos y + \cos(x-y) \) (\( 0 \leq x \leq \frac{\pi}{2}; \quad 0 \leq y \leq \frac{\pi}{2} \) )。
解
解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= \cos x - \sin(x-y) = 0, \quad (1) \\
\frac{\partial z}{\partial y} &= -\sin y + \sin(x-y) = 0. \quad (2)
\end{align*}
\]
(1) + (2)，\( \cos x = \sin y \)。由于\( x, y \)均为锐角，故有
\[ y = \frac{\pi}{2} - x. \] 代入 (1)，得
\[ \cos x - \sin \left(2x - \frac{\pi}{2}\right) = \cos x + \cos 2x \]
\[ = 2 \cos \frac{x}{2} \cos \frac{3x}{2} = 0. \]

但是 \( \cos \frac{x}{2} \neq 0 \)，故 \( \cos \frac{3x}{2} = 0 \)。从而得静止点 \( P_0 \left( \frac{\pi}{3}, \frac{\pi}{6} \right) \)。由于
\[ \frac{\partial^2 z}{\partial x^2} = -\sin x - \cos(x - y), \]
\[ \frac{\partial^2 z}{\partial y^2} = -\cos y - \cos(x - y), \]
\[ \frac{\partial^2 z}{\partial x \partial y} = \cos(x - y), \]

故在点 \( P_0 \)，有
\[ A = -\frac{1 + \sqrt{3}}{2}, \quad B = \frac{\sqrt{3}}{2}, \quad C = -\frac{1 + \sqrt{3}}{2}, \]
\[ AC - B^2 = \frac{1 + 2\sqrt{3}}{4} \geq 0. \]

于是，函数 \( z \) 在点 \( P_0 \) 取得极大值 \( z(P_0) = \frac{3}{2} \sqrt{3} \).

3637. \( z = \sin x \sin y \sin(x + y) \) (\( 0 \leq x \leq \pi; \ 0 \leq y \leq \pi \)).

解：解方程组
\[
\begin{align*}
\frac{\partial z}{\partial x} &= \sin y \sin(2x + y) = 0, \quad (1) \\
\frac{\partial z}{\partial y} &= \sin x \sin(x + 2y) = 0, \quad (2)
\end{align*}
\]
由 (1) 及 (2) 可得下列四个方程组:

I: \begin{align*}
\sin x &= 0, \\
\sin y &= 0.
\end{align*}

II: \begin{align*}
\sin x &= 0, \\
\sin(2x + y) &= 0.
\end{align*}

III: \begin{align*}
\sin y &= 0, \\
\sin(x + 2y) &= 0.
\end{align*}

IV: \begin{align*}
\sin(2x + y) &= 0, \\
\sin(x + 2y) &= 0.
\end{align*}
考虑到 \(0 \leq x \leq \pi, \ 0 \leq y \leq \pi\)，于是得原方程组 (1) 与 (2) 的六个解

\(P_1(0, 0), P_2(0, \pi), P_3(\pi, 0), P_4(\pi, \pi), P_5(\frac{\pi}{3}, \frac{\pi}{3}), P_6(\frac{2\pi}{3}, \frac{2\pi}{3})\).

由于所考虑的区域是闭正方形 \(0 \leq x \leq \pi, \ 0 \leq y \leq \pi\)，故点 \(P_1, P_2, P_3, P_4\) 都是此区域的边界点。因此 \(P_1, P_2, P_3, P_4\) 不是函数 \(z\) 达极值的点 (根据极值的定义，首先要求函数在所考虑的点的某邻域中有定义)。由于

\[z_{xx}^2 = 2 \sin y \cos(2x + y), \quad z_{yy}^2 = 2 \sin x \cos(x + 2y),\]

在点 \(P_5\) 有 \(AC-B^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 > 0\) 且 \(A = -\sqrt{3} < 0\)。故函数 \(z\) 在点 \(P_5\) 取得极大值

\[z(P_5) = \frac{3 \sqrt{3}}{8}; \quad \text{在点} \ P_6 \ \text{有} \ AC-B^2 = (\sqrt{3})(\sqrt{3})\]
\(- \left(\frac{\sqrt{3}}{2}\right)^2 \geq 0\) 且 \(A = \sqrt{3} > 0\)，故函数 \(z\) 在点 \(P_0\) 取得极小值 \(z(P_0) = -\frac{3}{8} \sqrt{3}\).

3638. \(z = x - 2y + \ln \sqrt{x^2 + y^2} + 3 \arctan \frac{y}{x}\).

解

解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 1 + \frac{x}{x^2 + y^2} - \frac{3y}{x^2 + y^2} = 0,
\frac{\partial z}{\partial y} &= -2 + \frac{y}{x^2 + y^2} + \frac{3x}{x^2 + y^2} = 0
\end{align*}
\]

得静止点 \(P_0(1, 1)\).

\[
\frac{\partial^2 z}{\partial x^2} = -\frac{x^2 + 6xy + y^2}{(x^2 + y^2)^2}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 6xy - y^2}{(x^2 + y^2)^2},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = -\frac{3x^2 - 2xy + 3y^2}{(x^2 + y^2)^2}.
\]

在点 \(P_0\) 有 \(A = \frac{3}{2}, \ B = -\frac{1}{2}, \ C = -\frac{3}{2}\) 及 \(AC - B^2 = -\frac{5}{2} < 0\)，故无极值。

3639. \(z = xy \ln(x^2 + y^2)\).

解

解方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= y \ln(x^2 + y^2) + \frac{2x^2 y}{x^2 + y^2} = 0, \quad (1) \\
\frac{\partial z}{\partial y} &= x \ln(x^2 + y^2) + \frac{2xy^2}{x^2 + y^2} = 0. \quad (2)
\end{align*}
\]

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将（1）式乘以x减去（2）式乘以y，得

\[
\frac{2xy}{x^2+y^2}(x^2-y^2) = 0.
\]

于是，x = 0，y = 0，x = y，x = -y为四组解，对应地得静止点P₁(0, 1)，P₂(0, -1)，P₃(1, 0)，P₄(-1, 0)，P₅\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)，P₆\left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)，P₇\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)及P₈\left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)。

代入原式，不难看出，函数z在点P₁、P₂、P₃及P₄均无极值（领域内函数值可正可负）。由于

\[
\frac{\partial^2 z}{\partial x^2} = \frac{2xy(x^2+3y^2)}{(x^2+y^2)^2}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{2xy(3x^2+y^2)}{(x^2+y^2)^2},
\]

\[
\frac{\partial^2 z}{\partial x \partial y} = \ln(x^2+y^2) + \frac{2(x^4+y^4)}{(x^2+y^2)^2}.
\]

在点P₅及P₆，A = 2，B = 0，C = 2及AC - B² = 4 > 0，故函数z在点P₅及P₆取得极小值z(P₅) = z(P₆) = -\frac{1}{2e} ≈ -0.184。

在点P₇及P₈，A = -2，B = 0，C = -2及AC - B² = 4 > 0，故函数z在点P₇及P₈取得极大值z(P₇) = z(P₈) = \frac{1}{2e} ≈ 0.184。

3640. z = x + y + 4\sin x \sin y.

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解 方程组

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 1 + 4 \cos x \sin y = 0, \\
\frac{\partial z}{\partial y} &= 1 + 4 \sin x \cos y = 0.
\end{align*}
\] (1)

(2) - (1) 得 \( \sin(x - y) = 0 \)，故 \( x - y = n\pi; \)

(2) + (1) 得 \( \sin(x + y) = \frac{1}{2} \)，故 \( x + y = m\pi - \)

\((-1)^n \frac{\pi}{6}. \)

于是，得静止点 \( P_0(x_0, y_0) \)，其中

\[
\begin{align*}
x_0 &= (-1)^{m+1} \frac{\pi}{12} + (m+n) \frac{\pi}{2}, \\
y_0 &= (-1)^{m+1} \frac{\pi}{12} + (m-n) \frac{\pi}{2}.
\end{align*}
\]

在点 \( P_0 \)，有

\[
AC - B^2 = (-4 \sin x_0 \sin y_0) (-4 \sin x_0 \sin y_0)
\]

\[
- (4 \cos x_0 \cos y_0)^2
\]

\[
= 16 (\sin x_0 \sin y_0 - \cos x_0 \cos y_0)
\]

\[
\cdot (\sin x_0 \sin y_0 + \cos x_0 \cos y_0)
\]

\[
= -16 \cos(x_0 + y_0) \cos(x_0 - y_0)
\]

\[
= -16 \cos \left( m\pi - (-1)^n \frac{\pi}{6} \right) \cos n\pi
\]

\[
= -16 (-1)^{n+n} \cos \frac{\pi}{6}.
\]

当 \( m \) 及 \( n \) 有相同的奇偶性时，\( m + n \) 为偶数，\( AC - B^2 = 0 \)，故无极值。当 \( m \) 及 \( n \) 有不同的奇偶性时，\( m + n \)
为奇数，$AC - B^2$ = 0，故有极值。看 $A$ 的符号决定取得极大值还是极小值。由于

$$A = -4 \sin x_0 \sin y_0 = 2[\cos(x_0 + y_0) - \cos(x_0 - y_0)]$$

$$= 2[(-1)^n \cos \frac{x}{6} - (-1)^n],$$

故当 $m$ 为奇数及 $n$ 为偶数时，$A < 0$，取得极大值；当 $m$ 为偶数及 $n$ 为奇数时，$A > 0$，取得极小值。极值为

$$z(x_0, y_0) = m\pi + \left(\frac{x}{6} + \sqrt{3}\right)(-1)^{n+1} + 2 \cdot (-1)^n.$$

### 3641. $z = (x^2 + y^2)e^{-(x^2+y^2)}.$

解 方程组

$$\begin{cases}
\frac{\partial z}{\partial x} = 2 x e^{-(x^2+y^2)} (1 - x^2 - y^2) = 0, \\
\frac{\partial z}{\partial y} = 2 y e^{-(x^2+y^2)} (1 - x^2 - y^2) = 0
\end{cases}$$

得静止点 $P_0(0,0)$ 及 $P(x_0, y_0)$，其中 $x_0^2 + y_0^2 = 1$。

在点 $P_0$ 有 $z = 0$，而当 $(x, y) \neq (0, 0)$ 时 $z > 0$，故函数 $z$ 在点 $P_0$ 取得极小值 $z = 0$。

由1437题知，在满足 $x_0^2 + y_0^2 = 1$ 的点 $(x_0, y_0)$ 的邻域内，不论是 $x^2 + y^2 = 1$ 还是 $x^2 + y^2 < 1$，均有

$$z(x, y) = (x^2 + y^2)e^{-(x^2+y^2)} \ll e^{-1}.$$ 但是点 $(x_0, y_0)$ 的邻域内总有 $x^2 + y^2 = 1$ 的点 $(x, y)$，因此，函数 $z$ 在点 $(x_0, y_0)$ 取得弱极大值 $z = e^{-1}$。

### 3642. $u = x^2 + y^2 + z^2 - 2x + 4y - 6z.$

解 

$$du = 2(x+1)dx + 2(y+2)dy + 2(z-3)dz.$$
令 \( \frac{\partial u}{\partial x} = 2(x + 1) = 0 \), \( \frac{\partial u}{\partial y} = 2(y + 2) = 0 \),

\( \frac{\partial u}{\partial z} = 2(z - 3) = 0 \), 得静止点 \( P_0 (-1, -2, 3) \).

在该点由于

\( d^2 u = 2 (dx^2 + dy^2 + dz^2) \geq 0 \)

（当 \( dx^2 + dy^2 + dz^2 \neq 0 \) 时），

故函数 \( u \) 在点 \( P_0 \) 取得极小值 \( u(P_0) = -14 \).

3643. \( u = x^3 + y^2 + z^2 + 12xy + 2z \).

解 \( du = (3x^2 + 12y)dx + (2y + 12x)dy + (2z + 2)dz \).

令 \( \frac{\partial u}{\partial x} = 3x^2 + 12y = 0 \), \( \frac{\partial u}{\partial y} = 2y + 12x = 0 \),

\( \frac{\partial u}{\partial z} = 2z + 2 = 0 \), 得静止点 \( P_0 (0, 0, -1) \) 及 \( P_1 (24, -144, -1) \).

\( d^2 u = 6xdx^2 + 2d^2y + 2d^2z + 24dxdy \).

在点 \( P_0 \), 有

\( d^2 u = 2d^2y^2 + 2d^2z^2 + 24dxdy = 2d^2y^2 + 2d^2y(dy + 12dx) \),

当 \( dz = 0 \), \( dy \geq 0 \) 及 \( dy + 12dx < 0 \) 时, \( d^2u \leq 0 \); 而当 \( dx, dy \) 及 \( dz \) 均大于零时, \( d^2u > 0 \). 因此 \( d^2u \) 的符号不定, 故无极值.

在点 \( P_1 \), 有

\[
\begin{align*}
&d^2 u = 144d^2x^2 + 2d^2y^2 + 2d^2z^2 + 24dxdy \\
&= (12dx + dy)^2 + dy^2 + 2d^2z^2 \\
&\geq 0 \quad (当 \; dx^2 + dy^2 + dz^2 \neq 0 \; 时),
\end{align*}
\]

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故函数 $u$ 在点 $P_1$ 取得极小值 $u(P_1) = -6913$.

3644. $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z} \quad (x \gg 0, \quad y \gg 0, \quad z \gg 0)$.

解  $du = \left(1 - \frac{y^2}{4x^2}\right)dx + \left(\frac{y}{2x} - \frac{z^2}{y^2}\right)dy$

$+ \left(\frac{2z}{y} - \frac{2}{z^2}\right)dz$

令 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$，得方程组

$$
\begin{align*}
1 - \frac{y^2}{4x^2} &= 0, \\
\frac{y}{2x} - \frac{z^2}{y^2} &= 0, \\
\frac{2z}{y} - \frac{2}{z^2} &= 0.
\end{align*}
$$

解之得静止点 $P_0 \left(\frac{1}{2}, 1, 1\right)$.

$$
d^2u = \frac{y^2}{2x^3} d^2x^2 + \frac{2z}{y} d^2y + \left(\frac{1}{2x} + \frac{2z^2}{y^2}\right)dy^2$

$- \frac{4z}{y^2} dy dz + \left(\frac{2}{y} + \frac{4}{z^3}\right)dz^2.$

在点 $P_0$，有

$$
d^2u = 4d^2x^2 - 4dxdy + 3dy^2 - 4dydz + 6dz^2$

$= (2d - dy)^2 + dy^2 + (dy - 2dz)^2 + 2dz^2 \geq 0$

(当 $d^2x^2 + dy^2 + dz^2 \neq 0$ 时)，
故函数 \( u \) 在点 \( P_0 \) 取得极小值 \( u(P_0) = 4 \)。

3345. \( u = xyz^2(a-x-2y-3z) \) （a=0）。

解
\[
du = yz^2(a-2x-2y-3z)dx + 2xyz^2(a-x-3y-3z)dy + 3xyz^2(a-x-2y-4z)dz.
\]

令 \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0 \)，得方程组
\[
\begin{align*}
yz^2(a-2x-2y-3z) &= 0 \\
2xyz^2(a-x-3y-3z) &= 0 \\
3xyz^2(a-x-2y-4z) &= 0.
\end{align*}
\]

解之得静止点 \( P_0(\frac{a}{7}, \frac{a}{7}, \frac{a}{7}) \); 直线 \( x=0 \)，2y+3z=a；平面 \( y=0 \)，平面 \( z=0 \)。

同 3625 题的方法，不难确定：直线 \( x=0 \)，2y+3z=a 及平面 \( z=0 \) 上的点不取得极值。当 \( xz^2(a-x-3z) > 0 \) 取得极小值 \( u=0 \)；当 \( xz^2(a-x-3z) < 0 \) 取得极大值 \( u=0 \)；当 \( xz^2(a-x-3z) = 0 \) 不取得极值。

在点 \( P_0 \)，有
\[
d^2u = -\frac{2a^5}{7^5} \left( dx^2 + 3dy^2 + 6dz^2 + 2dx \cdot dy + 6dy \cdot dz + 3dx \cdot dz \right) = -\frac{a^5}{7^5} \left( (dx+2dy+3dz)^2 + dx^2 + 2dy^2 + 3dz^2 \right) < 0 \quad (当 \ dx^2 + dy^2 + dz^2 \neq 0 \ 时)，
\]
故函数 \( u \) 在点 \( P_0 \) 取得极小值 \( u(P_0) = \frac{a^7}{7^7} \)。
解之得静止点 \( P_0 \left( \frac{1}{2} \sqrt[15]{\frac{1}{16} a^4 b}, \frac{1}{4} \sqrt[15]{16 a^4 b} \right) \).

\[
\begin{align*}
\frac{2x}{y} - \frac{a^2}{x^2} &= 0, \\
\frac{2y}{z} - \frac{x^2}{y^2} &= 0, \\
\frac{2z}{b} - \frac{y^2}{z^2} &= 0.
\end{align*}
\]

3646. \( u = \frac{a^2}{x^2} + \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{b} \) \((x > 0, y > 0, z \geq 0, a \geq 0, b \geq 0)\).

解 \( \frac{du}{dx} = \left( \frac{2x}{y} - \frac{a^2}{x^2} \right) dx + \left( \frac{2y}{z} - \frac{x^2}{y^2} \right) dy \\
+ \left( \frac{2z}{b} - \frac{y^2}{z^2} \right) dz. \)

令 \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0 \)，得方程组

\[
\begin{align*}
\frac{2x}{y} - \frac{a^2}{x^2} &= 0, \\
\frac{2y}{z} - \frac{x^2}{y^2} &= 0, \\
\frac{2z}{b} - \frac{y^2}{z^2} &= 0.
\end{align*}
\]

解得静止点 \( P_0 \left( \frac{1}{2} \sqrt[15]{\frac{1}{16} a^4 b}, \frac{1}{4} \sqrt[15]{16 a^4 b} \right) \).

\[
\begin{align*}
\frac{d^2 u}{dx^2} &= \frac{2a^2}{x^3} \frac{dx^2}{dx} + \frac{2}{y} \frac{dx^2}{dx} \frac{dy}{dx} - \frac{4}{y^2} \frac{dx}{dx} \frac{dy}{dx} + \frac{2}{z} \frac{dy^2}{dz} \\
+ \frac{2x^2}{y^3} \frac{dy^2}{dz} - \frac{4y}{z^2} \frac{dy}{dz} \frac{dz}{dz} + \frac{2}{b} \frac{dz^2}{dz^2} + \frac{2y^2}{z^3} \frac{dz^2}{dz^2}, \\
= \frac{2a^2}{x^3} \frac{dx^2}{dx} + \frac{2}{y} \left( \frac{dx}{dx} - \frac{x}{y} \frac{dy}{dx} \right) + \frac{2}{z} \left( \frac{dy}{dy} - \frac{y}{z} \frac{dz}{dz} \right) \\
+ \frac{2}{b} \frac{dz^2}{dz^2}.
\end{align*}
\]
在点 $P_0$, $x \gg 0$, $y \gg 0$, $z \gg 0$, $d^2u \gg 0$ (当 $dx^2 + dy^2 + dz^2 \neq 0$ 时)，故函数 $u$ 在点 $P_0$ 取得极小值

$$u(P_0) = \frac{15a b}{4} \sqrt{\frac{a}{16b}}.$$ 

3647. $u = \sin x + \sin y + \sin z - \sin(x + y + z)$

$(0 \leq x \leq \pi, \ 0 \leq y \leq \pi, \ 0 \leq z \leq \pi).$

解

$$du = [\cos x - \cos(x + y + z)]dx$$

$$+ [\cos y - \cos(x + y + z)]dy$$

$$+ [\cos z - \cos(x + y + z)]dz.$$ 

令 $\frac{du}{dx} = \frac{du}{dy} = \frac{du}{dz} = 0$，得方程组

$$\begin{cases}
\cos x - \cos(x + y + z) = 0, \\
\cos y - \cos(x + y + z) = 0, \\
\cos z - \cos(x + y + z) = 0.
\end{cases}$$

注意到 $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $0 \leq z \leq \pi$，解之得静止点 $P_0(0, 0, 0), P_1(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ 及 $P_2(\pi, \pi, \pi)$。

在点 $P_1$, 有

$$d^2u = -\sin x dx^2 - \sin y dy^2 - \sin z dz^2$$

$$+ \sin(x + y + z)[d(x + y + z)]^2$$

$$= -dx^2 - dy^2 - dz^2 - (dx + dy + dz)^2 \ll 0,$$

故函数 $u$ 在点 $P_1$ 取得极大值 $u(P_1) = 4$.

由于 $P_0$ 与 $P_2$ 是所考虑区域 $0 \leq x \leq \pi, 0 \leq y \leq \pi, 0 \leq z \leq \pi$ 的边界点，故函数在点 $P_0$ 与 $P_2$ 不达极值（根据极值定义，首先要求函数在所考虑的点的某邻域中有定义）。但如果放宽要求，对于边界点，仅将
其函数值与属于所考虑的区域而与此边界点很接近的点的函数值相比较，则在边界点也可引入达极值和达弱极值的概念。今对于点 $P_0$ 及 $P_2$ 的邻域中且属于上述区域的点 $(x, y, z)$，显然有 $\sin x \geq 0$, $\sin y \geq 0$, $\sin z \geq 0$。又

$$
\sin(x + y + z) = \sin x \cos y \cos z - \sin x \sin y \sin z + \cos x \sin y \cos z + \cos x \cos y \sin z
$$

$$
\leq \sin x + \sin y + \sin z - \sin x \sin y \sin z,
$$

故 $u \geq 0$。而当 $x = y = 0$ 时或 $x = y = \pi$ 时恒有 $u = 0$。因此，函数 $u$ 在点 $P_0$ 及 $P_2$ 都达到极小值 $u(P_0) = u(P_2) = 0$（按上述边界点达极值的意义）。

3648. $u = x_1 x_2^2 \cdots x_n^n (1 - x_1 - 2x_2 - \cdots - nx_n)$

$(x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0)$。

解 先考虑满足 $1 - x_1 - 2x_1 - \cdots - nx_n = 0, x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0$ 的点 $(x_1, x_2, \cdots, x_n)$. 显然函数 $u$ 在这种点不达到极值（因为，例如，若保持 $x_2, x_3, \cdots, x_n$ 不变，而将 $x_1$ 增大任意小的值，就有 $u \leq 0$，但将 $x_1$ 减小任意小的值，则有 $u \geq 0$），故下面只需考察满足 $1 - \sum_{k=1}^n kx_k \neq 0, x_1 \geq 0, \cdots, x_n \geq 0$ 的点 $(x_1, x_2, \cdots, x_n)$。

我们有

$$
du = u \sum_{k=1}^n \frac{k}{x_k^2} dx_k - \frac{u}{1 - \sum_{k=1}^n kx_k}
$$
\[
\begin{align*}
\frac{k}{x_i} - \frac{k}{u} & = 0 \quad (k = 1, 2, \ldots, n) \\
\frac{1}{1 - \sum_{i=1}^{n} k x_i} & = 0
\end{align*}
\]

考虑到 \( x_i > 0 \) 及 \( 1 - \sum_{i=1}^{n} k x_i \neq 0 \)，故有 \( u = 0 \)。

解方程组

得静止点 \( P_0(x_1, x_2, \ldots, x_n) \)，其中

\[
x_1 = x_2 = \cdots = x_n = \frac{2}{n^2 + n + 2} = x_3.
\]

\[
d^2 u = \left[ \sum_{i=1}^{n} \left( \frac{k}{x_i} - \frac{k}{u} \right) d x_i \right] d u
\]

\[
+ u \left[ \sum_{i=1}^{n} \left( -\frac{k}{x_i^2} \right) d x_i^3 + \frac{1}{(1 - \sum_{i=1}^{n} k x_i)^2} \right]
\]

\[
\cdot \left( \sum_{k=1}^{n} k d x_k \right) \left( -\sum_{k=1}^{n} k d x_k \right)
\]

在点 \( P_0 \)，有

\[
d^2 u = -\frac{n}{x_0^2} \left[ \sum_{k=1}^{n} k d x_k^2 + \left( \sum_{k=1}^{n} k d x_k \right)^2 \right]
\]
\[\begin{align*}
&= -x_0^{\frac{z(z+1)}{2} - 1} \left[ \sum_{k=1}^{n} k d \frac{x_k^2}{2} + \left( \sum_{k=1}^{n} k d x_k \right)^2 \right] \\
&
\leq 0 \quad (\text{当} \sum_{k=1}^{n} d x_k^2 \neq 0 \text{时}) ,
\end{align*}\]

故函数在点\( P_0 \)取得极大值
\[ u(P_0) = \left( \frac{\frac{2}{n^2 + n + 2}}{2} \right)^{n^2 + n + 2} . \]

3649. \( u = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{2}{x_n} \) \( (x_i \Rightarrow 0, i = 1, 2, \cdots, n) \).

解 设 \( y_1 = x_1, y_2 = \frac{x_2}{x_1}, \cdots, y_i = \frac{x_i}{x_{i-1}}, \cdots, y_n = \frac{x_n}{x_{n-1}} \).

则 \( x_n = y_1 y_2 \cdots y_n, \ y_i \Rightarrow 0 \ (k = 1, 2, \cdots, n) \).

且

\[ u = y_1 + y_2 + y_3 + \cdots + \frac{2}{y_1 y_2 \cdots y_n} . \]

记 \( A = y_1 y_2 \cdots y_n \), 则可得

\[ du = \sum_{i=1}^{n} \left( 1 - \frac{2}{Ay_k} \right) dy_k . \]

令 \( \frac{\partial u}{\partial y_k} = 0 \) 得方程组

\[ 1 - \frac{2}{Ay_k} = 0 \quad (k = 1, 2, \cdots, n) . \]

解之得静止点 \( P_0 (y_1, y_2, \cdots, y_n) \), 其中

\[ y_1 = y_2 = \cdots = y_n = 2^{x+1} = y_0 . \]

在点 \( P_0 \), 有
\[ d^2 u \bigg|_{p = p_0} = \frac{2}{A} \sum_{k=1}^{n} \frac{1}{y_k} \frac{d y_k}{d x} + \frac{2}{A y_k^2} \left( \sum_{k=1}^{n} \frac{d y_k}{d x} \right)^2 \bigg|_{p = p_0} \]
\[ = \frac{1}{y_0} \left[ \sum_{k=1}^{n} \frac{d y_k}{d x} + \left( \sum_{k=1}^{n} \frac{d y_k}{d x} \right)^2 \right] = 0 \]

（当 \( \sum_{k=1}^{n} \frac{d y_k}{d x} \neq 0 \) 时）。

故函数 \( u \) 在 \( p_0 \) 点取得极小值，也即为

\[ x_1 = y_1 = 2^{\frac{1}{n+1}}, \]
\[ x_2 = y_2 x_1 = 2^{\frac{2}{n+1}}, \]
\[ \ldots \ldots \ldots \]
\[ x_k = y_k x_{k-1} = 2^{\frac{k}{n+1}}, \]
\[ \ldots \ldots \ldots \]
\[ x_n = y_n x_{n-1} = 2^{\frac{n}{n+1}} \]

处，函数 \( u \) 取得极小值 \( u = (n + 1)2^{\frac{1}{n+1}} \)。

3650. 惠更斯问题。在 \( a \) 和 \( b \) 二正数间插入 \( n \) 个数 \( x_1, x_2, \ldots, x_n \)，使得分数

\[ u = \frac{x_1 x_2 \cdots x_n}{(a + x_1)(x_1 + x_2) \cdots (x_n + b)} \]

的值是最大。

解 记 \( w = \frac{1}{u} = (a + x_1)(1 + \frac{x_2}{x_1})(1 + \frac{x_3}{x_2}) \cdots (1 + \frac{b}{x_n}). \)

设 \( y_1 = \frac{x_2}{x_1}, y_2 = \frac{x_3}{x_2}, \ldots, y_n = \frac{b}{x_n} \)，并记

\[ A = y_1 y_2 \cdots y_n, \]
\[ x_1 = \frac{b}{y_1 y_2 \cdots y_n A}, \]

\[ w = \left( a + \frac{b}{A} \right) (1 + y_1)(1 + y_2) \cdots (1 + y_n). \]

又记 \( m = a + \frac{b}{A} \)，则有

\[
dw = \sum_{k=1}^{n} \frac{w}{1+y_k} dy_k - \frac{wb}{mA} \sum_{k=1}^{n} \frac{dy_k}{y_k},
\]

\[
= w \sum_{k=1}^{n} \left( \frac{y_k}{1+y_k} - \frac{b}{mA} \right) \frac{dy_k}{y_k}.
\]

令 \( \frac{dw}{dy_k} = 0 \) 得方程组

\[ \frac{y_k}{1+y_k} = \frac{b}{mA} \quad (k=1, 2, \cdots, n). \]

解之得静止点 \( P_0(y_1, y_2, \cdots, y_n) \)，其中

\[ y_1 = y_2 = \cdots = y_n = \left( \frac{b}{a} \right)^{1/(n+1)} = y_0. \]

在点 \( P_0 \)，有

\[
d^2u \bigg|_{P=P_0} = w \sum_{k=1}^{n} d \left( \frac{y_k}{1+y_k} - \frac{b}{mA} \right) \frac{dy_k}{y_k} \bigg|_{P=P_0}
\]

\[
= w \sum_{k=1}^{n} d \left( \frac{y_k}{1+y_k} \right) \left( \frac{dy_k}{y_0} \right) \bigg|_{P=P_0}
\]

\[
- w \sum_{k=1}^{n} \frac{dy_k}{y_0} \left[ d \left( \frac{1}{1 + \frac{a}{b} A} \right) \bigg|_{P=P_0} \right]
\]

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\[
= \frac{w(P_0)}{y_0(1+y_0)^2} \sum_{i=1}^{n} d^2 y_i^2 + \frac{w(P_0)}{y_0(1+\frac{a}{b} A)^2} \bigg|_{P=P_0} \cdot \sum_{i=1}^{n} \left[ d y_k \left( \sum_{k=1}^{n} \frac{a A}{b y_k} d y_k \right) \right] \\
= \frac{w(P_0)}{y_0(1+y_0)^2} \left[ \sum_{i=1}^{n} d y_i^2 + \left( \sum_{i=1}^{n} d y_i \right)^2 \right] \\
\geq 0 \quad (\text{当} \sum_{i=1}^{n} d y_i^2 \neq 0 \text{时})
\]

故函数 \( w \) 在点 \( P_0 \) 取得极小值，从而函数 \( u \) 在

\[
\begin{align*}
x_1 &= \frac{b}{A} = \frac{b}{y_0^2} = \frac{b}{a} \cdot a y_0^{-a} = a y_0^{a+1} \cdot y_0^{-a} = a y_0, \\
x_2 &= x_1 y_1 = a y_0^2, \\
x_3 &= x_2 y_2 = a y_0^3, \\
&\vdots \\
x_n &= \frac{b}{y_n} = \frac{b}{a} a y_0^{-a-1} = a y_0^{a+1} y_0^{-1} = a y_0^n.
\end{align*}
\]

即数 \( a, x_1, x_2, \ldots, x_n, b \) 构成有公比 \( y_0 = \left( \frac{b}{a} \right)^{\frac{1}{n+1}} \) 的几何级数时，其值最大，并且 \( u \) 的最大值为

\[
u = \frac{1}{a(1+y_0)^{n+1}} = \left( a \frac{1}{a+1} + b \frac{1}{n+1} \right)^{-(n+1)}.
\]

求变量 \( x \) 和 \( y \) 的隐函数 \( z \) 的极值。

3651. \( x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0 \)
解 微分得
\[(x - 1) dx + (y - 1) dy + (z - 2) dz = 0.\]

显见，当 \(x = 1, \ y = -1\) 时 \(dz = 0\). 代入原方程可解得 \(z = 6\) 及 \(z = -2\). 又 \(z = 2\) 时为不可微的，为判断极值，求二阶微分，得
\[d x^2 + d y^2 + (z - 2) d^2 z + d z^2 = 0.\]

以 \(x = 1, \ y = -1, \ z = 6\) 代入，并考虑 \(dz = 0\)，得
\[d^2 z = -\frac{1}{4} (d x^2 + d y^2) \leq 0 \quad (当 \ d x^2 + d y^2 \neq 0 时),\]
故当 \(x = 1, \ y = -1\) 时，隐函数 \(z\) 取极小值 \(z = 6\).
同法可判断得：当 \(x = 1, \ y = -1\) 时，隐函数 \(z\) 也取得极小值，且其值为 \(z = -2\).

不难看出，\(z = 2\) 是球的切面平行于 \(Oz\) 轴的地方，因此函数 \(z\) 不取得极值.

3652. \(x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0\).

解 微分一次，得
\[(2x - z + 2) dx + (2y - z + 2) dy + (2z - x - y + 2) dz = 0.\]

解方程组
\[
\begin{align*}
2x - z + 2 &= 0, \\
2y - z + 2 &= 0, \\
x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 &= 0
\end{align*}
\]
得 : \(x_1 = y_1 = -(3 + \sqrt{6}), \ z_1 = -(4 + 2\sqrt{6});\)
\(x_2 = y_2 = -(3 - \sqrt{6}), \ z_2 = 2\sqrt{6} - 4.\)

再微分一次，并注意到 \(dz = 0\)，即得
$$2dx^2 + 2dy^2 + (2z - x - y + 2)d^2z = 0.$$  

在点 \( (x_1, y_1, z_1) \), \( d^2z = \frac{1}{\sqrt{6}} (dx^2 + dz^2 + dy^2) \geq 0 \)，故当 \( x = y = -(3 + \sqrt{6}) \) 时，取得极小值 \( z = -(4 + 2\sqrt{6}) \)。同理可知，当 \( x = y = -(3 - \sqrt{6}) \) 时，取得极大值 \( z = 2\sqrt{6} - 4 \)。

对于 \( dx \) 的系数 \( 2z - x - y + 2 = 0 \) 时代表的情况，与上题类似也不取得极值。

3653. \((x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 - z^2) \)。

解 微分一次，得

\[
2(x^2 + y^2 + z^2)(xdx + ydy + zdz)
= a^2(xdx + ydy - zdz).
\]

令 \( dz = 0 \)，得方程

\[
(2(x^2 + y^2 + z^2) - a^2)(xdx + ydy) = 0.
\]

解之，得 \( x = y = 0 \) 及 \( x^2 + y^2 + z^2 = \frac{a^2}{2} \)。

以 \( x = y = 0 \) 代入原方程，解得 \( z = 0 \)。这是隐函数的一个奇点。把原式看作 \( z^2 \) 的一个方程，去去根，可解出

\[
z^2 = -(a^2 + x^2 + y^2) + \sqrt{a^4 + 3a^2(x^2 + y^2)}.
\]

显然 \( z \) 有正负两支在 \((0, 0, 0)\) 点相交。因此，不认为 \( z \) 在 \((0, 0, 0)\) 点取得极值。

以 \( x^2 + y^2 + z^2 = \frac{a^2}{2} \) 代入原方程，解得

\[
x^2 + y^2 = \frac{3}{8}a^2, \quad z^2 = \frac{a^2}{8}.
\]
为考虑极值，将一次微分式改写为
\[
\left[2(x^2 + y^2 + z^2) - a^2\right](xdx + ydy) + \\
\left[2(x^2 + y^2 + z^2) + a^2\right]zdz = 0.
\]
将上式再微分一次，注意到 \(dz = 0\) 及 \(x^2 + y^2 + z^2 = \frac{a^2}{2}\)，即得
\[
a^2zd^2z = -2(xdx + ydy)^2,
\]
故当 \(x^2 + y^2 = \frac{3}{8}a^2\)，\(z = \frac{a}{2\sqrt{2}}\)时，\(d^2z \leq 0\)，函数 \(z\) 取得弱极大值 \(z = \frac{a}{2\sqrt{2}}\)；当 \(x^2 + y^2 = \frac{3}{8}a^2\)，
\[
z = -\frac{a}{2\sqrt{2}}\)时，\(d^2z \geq 0\)，函数 \(z\) 取得弱极小值 \(z =
\[
-\frac{a}{2\sqrt{2}}.
\]
求下列函数的条件极值点：
3654. \(z = xy\)，若 \(x + y = 1\)。
解 设 \(F(x, y) = xy + \lambda(x + y - 1)\)，解方程组
\[
\begin{align*}
\frac{\partial F}{\partial x} &= y + \lambda = 0, \\
\frac{\partial F}{\partial y} &= x + \lambda = 0, \\
x + y &= 1
\end{align*}
\]
得 \(x = y = -\lambda = \frac{1}{2}\)，\(z = \frac{1}{4}\)。由于当 \(x \to \pm \infty\) 时，\(y \to \pm \infty\)，故 \(z = xy \to -\infty\)。从而得知，点 \(x = \frac{1}{2}\)，\(y = \frac{1}{2}\)。
为条件极值点，且 \( z = \frac{1}{4} \) 为极大值。

如将 \( z = xy \) 改写为 \( z = y(1-y) \)，则成为普通极值，易知极大值点为 \( y = \frac{1}{2} \)，从而 \( x = 1 - \frac{1}{2} = \frac{1}{2} \)，

\[ z = \frac{1}{4} . \]

3655. \( z = \frac{x}{a} + \frac{y}{b} \)，若 \( x^2 + y^2 = 1 \)。

解 设 \( F(x, y) = \frac{x}{a} + \frac{y}{b} + \lambda (x^2 + y^2 - 1) \)。解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= \frac{1}{a} + 2\lambda x = 0, \\
\frac{\partial F}{\partial y} &= \frac{1}{b} + 2\lambda y = 0, \\
x^2 + y^2 &= 1
\end{align*}
\]

可得

\[
\lambda = \pm \frac{\sqrt{a^2 + b^2}}{2|ab|}, \quad x = \pm \frac{bc}{\sqrt{a^2 + b^2}},
\]

\[
y = \mp \frac{ac}{\sqrt{a^2 + b^2}},
\]

其中 \( \varepsilon = \text{sgn} \ \cdot ab \neq 0 \)。相应地，\( z = \mp \frac{\sqrt{a^2 + b^2}}{|ab|} \)。

由于函数 \( z \) 在闭圆周 \( x^2 + y^2 = 1 \) 上连续且不为常数，故必取得最大值和最小值并且最大值与最小值
不相等。这里可疑点仅有两个。

因此，当 \( x = -\frac{be}{\sqrt{a^2 + b^2}}, y = -\frac{ae}{\sqrt{a^2 + b^2}} \) 时，函数

值 \( z = -\frac{\sqrt{a^2 + b^2}}{|ab|} \) 必为最小值，从而是极小值；当

\( x = \frac{be}{\sqrt{a^2 + b^2}}, y = \frac{ae}{\sqrt{a^2 + b^2}} \) 时，\( z = \frac{\sqrt{a^2 + b^2}}{|ab|} \) 为最大值，从而是极大值。

3856. \( z = x^2 + y^2 \)，若 \( x + \frac{y}{b} = 1 \)。

解 令 \( F(x, y) = x^2 + y^2 + \lambda \left( \frac{x}{a} + \frac{y}{b} - 1 \right) \)。解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2x + \frac{1}{a}\lambda = 0, \\
\frac{\partial F}{\partial y} &= 2y + \frac{1}{b}\lambda = 0, \\
\frac{x}{a} + \frac{y}{b} &= 1
\end{align*}
\]

可得

\( \lambda = -\frac{2a^2b^2}{a^2 + b^2}, \quad x = \frac{ab^2}{a^2 + b^2}, \quad y = \frac{a^2b}{a^2 + b^2} \).

由于当 \( x \to \infty, y \to \infty \) 时，\( z \to \infty \)，函数 \( z \) 必在有限处取得最小值。这里可疑点仅一个。因此，当

\( x = \frac{ab^2}{a^2 + b^2}, \quad y = \frac{a^2b}{a^2 + b^2} \) 时，函数 \( z \) 取得极小值。
$$z = \frac{a^2b^2}{a^2+b^2}.$$

注  如果用二阶微分判别，则易从
$$d^2z = 2(dx^2 + dy^2) \geq 0$$
（不论 $dx$, $dy$ 之间有何约束条件，此式恒成立）可
知 $z = \frac{a^2b^2}{a^2+b^2}$ 为极小值。

3657. $z = Ax^2 + 2Bxy + Cy^2$, 若 $x^2 + y^2 = 1$.
解 设 $F(x, y) = Ax^2 + 2Bxy + Cy^2 - \lambda(x^2 + y^2 - 1)$. 解方程组

$$\begin{align*}
\frac{\partial F}{\partial x} &= 2[(A-\lambda)x + By] = 0, \quad (1) \\
\frac{\partial F}{\partial y} &= 2[Bx + (C-\lambda)y] = 0, \quad (2) \\
x^2 + y^2 &= 1. \quad (3)
\end{align*}$$

由 $x^2 + y^2 = 1$ 知 $x, y$ 不全为零，故 $\lambda$ 必须满足方程

$$\begin{vmatrix}
A-\lambda & B \\
B & C-\lambda
\end{vmatrix} = \lambda^2 - (A+C)\lambda + (AC - B^2) = 0. \quad (4)$$

当 $(A-C)^2 + 4B^2 = 0$ 时，所研究的函数为常数；当 $(A-C)^2 + 4B^2 \neq 0$ 时，方程 (4) 有两个不等的实根，记为 $\lambda_1$ 和 $\lambda_2$ ($\lambda_1 > \lambda_2$). 由方程组 (1), (2), (3) 可解出
\[ x_{1,2} = \frac{\pm (\lambda_1 - C)}{\sqrt{B^2 + (\lambda_1 - C)^2}}, y_{1,2} = \frac{\pm (\lambda_1 - A)}{\sqrt{B^2 + (\lambda_1 - A)^2}}. \]

\[ x_{3,4} = \frac{\pm (\lambda_2 - C)}{\sqrt{B^2 + (\lambda_2 - C)^2}}, y_{3,4} = \frac{\pm (\lambda_2 - A)}{\sqrt{B^2 + (\lambda_2 - A)^2}}. \]

相应地，有

\[ z(x_1, y_1) = Ax_1^2 + 2Bx_1y_1 + Cy_1^2 \]
\[ = (Ax_1 + By_1)x_1 + (Bx_1 + Cy_1)y_1. \]

由 (1)、(2) 可解得

\[ Ax_1 + By_1 = \lambda_1 x_1, Bx_1 + Cy_1 = \lambda_1 y_1, \]

故得

\[ z(x_1, y_1) = \lambda_1 x_1^2 + \lambda_1 y_1^2 = \lambda_1(x_1^2 + y_1^2) = \lambda_1. \]

同理可得

\[ z(x_2, y_2) = \lambda_1, z(x_3, y_3) = z(x_4, y_4) = \lambda_2. \]

由于函数 \( z \) 在单位球面上连续且不为常数，故必取得最大值和最小值并且最大值和最小值不相等。这里可疑点仅四个 \( (x_i, y_i) \) \( (i = 1, 2, 3, 4) \)，而 \( z(x_1, y_1) = z(x_2, y_2) = \lambda_1, z(x_3, y_3) = z(x_4, y_4) = \lambda_2. \) 于是，当 \( x = x_{1,2}, y = y_{1,2} \)，函数 \( z \) 取得最大值 \( z = \lambda_1 \)，而也是极大值；当 \( x = x_{3,4}, y = y_{3,4} \)，函数 \( z \) 取得最小值 \( z = \lambda_2 \)，而也是极小值。

3658. \[ z = \cos^2 x + \cos^2 y, \] \( \text{若} x - y = \frac{\pi}{4}. \)

解 设 \( F(x, y) = \cos^2 x + \cos^2 y + \lambda(x - y - \frac{\pi}{4}). \)

解方程组
\[
\begin{aligned}
\frac{\partial F}{\partial x} &= -\sin 2x + \lambda = 0, \\
\frac{\partial F}{\partial y} &= -\sin 2y - \lambda = 0, \\
x - y &= \frac{\pi}{4},
\end{aligned}
\]

可得
\[
x_k = \frac{\pi}{8} + \frac{k\pi}{2}, \quad y_k = -\frac{\pi}{8} + \frac{k\pi}{2} (k = 0, \pm 1, \pm 2, \cdots).
\]

相应地，当 $k$ 为偶数时，$z = 1 + \frac{1}{\sqrt{2}}$；当 $k$ 为奇数时，

\[z = 1 - \frac{1}{\sqrt{2}}.
\]

由于所给连续函数 $z$ 必在任意有限区域内取得最大值和最小值，而且 $z$ 又是关于 $x, y$ 的周期（周期为 $\pi$）函数，故当 $k$ 为偶数时，函数 $z$ 在点 $(x_k, y_k)$ 取得最大值 $z = 1 + \frac{1}{\sqrt{2}}$，从而是极大值；当 $k$ 为奇数时，函数 $z$ 在点 $(x_k, y_k)$ 取得最小值 $z = 1 - \frac{1}{\sqrt{2}}$，从而是极小值。

3659. $u = x - 2y + 2z$, 若 $x^2 + y^2 + z^2 = 1$。

解 设 $F(x, y, z) = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$。

解方程组
\[
\begin{aligned}
\frac{\partial F}{\partial x} &= 1 + 2\lambda x = 0, \\
\frac{\partial F}{\partial y} &= -2 + 2\lambda y = 0, \\
\frac{\partial F}{\partial z} &= 2 + 2\lambda z = 0, \\
x^2 + y^2 + z^2 &= 1,
\end{aligned}
\]
可得
\[ x = \pm \frac{1}{3}, \quad y = \pm \frac{2}{3}, \quad z = \pm \frac{2}{3}. \]
相应地，\[ u = \pm 3.\]
由于所给函数在闭球面上连续且不为常数，故必取得最大值及最小值并且最大值与最小值不相等。这里可疑点仅两个，于是，当 \[ x = \frac{1}{3}, \quad y = -\frac{2}{3}, \quad z = \frac{2}{3} \] 时，函数 \( u \) 取得最大值 \( u = 3 \)，因而也是极大值；当 \[ x = -\frac{1}{3}, \quad y = \frac{2}{3}, \quad z = -\frac{2}{3} \] 时，函数 \( u \) 取得最小值 \( u = -3 \)，因而也是极小值。

3660. \( u = x^m y^n z^p \)，若 \( x + y + z = a \) （\( m > 0, \quad n \gg 0, \quad p \gg 0, \quad a \gg 0 \)）。
解 设 \( w = \ln u = m \ln x + n \ln y + p \ln z \)。
\[ F(x, y, z) = w - \frac{1}{\lambda} (x + y + z - a). \]
解方程组
\[
\begin{align*}
\frac{\partial F}{\partial x} &= \frac{m}{x} - \frac{1}{\lambda} = 0, \\
\frac{\partial F}{\partial y} &= \frac{n}{y} - \frac{1}{\lambda} = 0, \\
\frac{\partial F}{\partial z} &= \frac{p}{z} - \frac{1}{\lambda} = 0,
\end{align*}
\]
\[ x + y + z = a \]

*）编者注：应加上条件 \( x > 0, \quad y > 0, \quad z > 0 \)。
可得
\[x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p} .\]

相应地，\[u = -\frac{a^m \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}} .\]

连续函数 \(w\) 定义在平面 \(x+y+z=a\) 于第一卦限内的部分，边界由三条直线
\[
\begin{cases}
x + y = a, \\
z = 0, \\
x + z = a, \\
y = 0
\end{cases}
\]

组成。当点 \(P\) 趋于边界上的点时，显然有 \(w \rightarrow -\infty\)。
因此，函数 \(w\) 在区域内取得最大值。由于可疑点仅一个，故当 \(x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}\)
\[z = \frac{ap}{m+n+p}\]时，函数 \(u\) 取得最大值
\[u = -\frac{a^m \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}, \quad \text{因而也是极大值。}\]

3661. \(u = x^2 + y^2 + z^2, \quad \text{若} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (a>b>c>0).\)

解 设 \(F(x, y, z) = x^2 + y^2 + z^2 + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) - 1\)。解方程组

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\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2x \left(1 + \frac{\lambda}{a^2}\right) = 0, \\
\frac{\partial F}{\partial y} &= 2y \left(1 + \frac{\lambda}{b^2}\right) = 0, \\
\frac{\partial F}{\partial z} &= 2z \left(1 + \frac{\lambda}{c^2}\right) = 0, \\
x^2 \frac{1}{a^2} + y^2 \frac{1}{b^2} + z^2 \frac{1}{c^2} &= 1
\end{align*}
\]

可得
\[
x = \pm a, \quad y = z = 0; \quad x = 0, \quad y = \pm b; \\
x = y = 0, \quad z = \pm c.
\]
相应地，有
\[
u(\pm a, 0, 0) = a^2, \quad u(0, \pm b, 0) = b^2, \quad u(0, 0, \pm c) = c^2.
\]
由于 \(a \gg b \gg c \gg 0\)，故连续函数 \(u\) 在点 \((\pm a, 0, 0)\) 取得最大值 \(a^2\)，因而也是极大值；在点 \((0, 0, \pm c)\) 取得最小值 \(c^2\)，因而也是极小值。

在点 \((0, \pm b, 0)\) 处，对应的 \(\lambda = -b^2\)，且
\[
d^2 F = 2 \left(1 + \frac{\lambda}{a^2}\right) dx^2 + 2 \left(1 + \frac{\lambda}{b^2}\right) dy^2 \\
+ 2 \left(1 + \frac{\lambda}{c^2}\right) dz^2
\]
\[
= 2 \left(1 - \frac{b^2}{a^2}\right) dx^2 + 2 \left(1 - \frac{b^2}{c^2}\right) dz^2.
\]
把 \(x, z\) 当自变量，\(y\) 看成由条件 \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\) 所确定的 \(x\) 和 \(z\) 的函数。在点 \((0, \pm b, 0)\)，有 \(d^2 u = d^2 F\).
而 $1 - \frac{b^2}{a^2} \geq 0$, $1 - \frac{b^2}{c^2} \leq 0$. 因此, $d^2u$ 的符号不定，从而函数 $u$ 在点 $(0, \pm b, 0)$ 不取得极值。

3662. 令 $u = x y^2 z^3$, 若 $x + 2y + 3z = a$ ($x > 0, y > 0, z > 0$, $a > 0$).

解 设 $w = \ln u = \ln x + 2\ln y + 3\ln z$,

$$F(x, y, z) = w - \frac{1}{\lambda}(x + 2y + 3z - a).$$

解方程组

$$
\begin{align*}
\frac{\partial F}{\partial x} &= \frac{1}{x} - \frac{1}{\lambda} = 0, \\
\frac{\partial F}{\partial y} &= \frac{2}{y} - \frac{2}{\lambda} = 0, \\
\frac{\partial F}{\partial z} &= \frac{3}{z} - \frac{3}{\lambda} = 0, \\
x + 2y + 3z &= a
\end{align*}
$$

可得

$$x = y = z = a.$$

类似3660题的讨论可知，函数 $u$ 当 $x = y = z = \frac{a}{6}$ 时取得极大值 $u = \left(\frac{a}{6}\right)^6$.

3663. 令 $u = xyz$, 若 $x^2 + y^2 + z^2 = 1$, $x + y + z = 0$.

解 设 $F(x, y, z) = xyz + \lambda(x^2 + y^2 + z^2 - 1) + \mu(x + y + z)$. 解方程组

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\[
\begin{align*}
\frac{\partial F}{\partial x} &= yz + 2\lambda x + \mu = 0, \\
\frac{\partial F}{\partial y} &= xz + 2\lambda y + \mu = 0, \\
\frac{\partial F}{\partial z} &= xy + 2\lambda z + \mu = 0,
\end{align*}
\]

(1) \quad (2) \quad (3)

\[x^2 + y^2 + z^2 = 1, \quad (4)\]

\[x + y + z = 0. \quad (5)\]

(1) - (2), (2) - (3), 得

\[
\begin{align*}
(x - y)(2\lambda - z) &= 0, \\
y - z)(2\lambda - x) &= 0.
\end{align*}
\]

(6) \quad (7)

由(6)，若 \(x - y = 0\)，代入(5)得 \(z = -2x\)。再代入(4)，解得静止点

\[P_1\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\]和

\[P_2\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right).
\]

如果 \(x - y \neq 0\)，则 \(z = 2\lambda\)。由(7)，若 \(y - z = 0\)，

类似上面解法可得静止点

\[P_3\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\]

和

\[P_4\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right); \text{若} \ y - z \neq 0, \text{则}
\]

\[x = 2\lambda, \text{故} \ x = z, \text{类似上面解法又可得静止点}
\]

\[P_5\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\]和

\[P_6\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right).
\]

相应地，有
\[ u(P_1) = u(P_3) = u(P_5) = -\frac{1}{3\sqrt{6}}. \]

\[ u(P_2) = u(P_4) = u(P_6) = \frac{1}{3\sqrt{6}}. \]

类似前面各题的讨论可知，函数 \( u \) 在点 \( P_1, P_3 \) 及 \( P_5 \) 取得极小值 \( u = -\frac{1}{3\sqrt{6}} \); 在点 \( P_2, P_4 \) 及 \( P_6 \) 取得极大值 \( u = \frac{1}{3\sqrt{6}} \).

\[ 3664. \quad u = \sin x \sin y \sin z, \quad \text{若} \quad x + y + z = \frac{\pi}{2} \]

\[ (x \geq 0, \quad y \geq 0, \quad z \geq 0). \]

解 由 \( x + y + z = \frac{\pi}{2} \) 及 \( x \geq 0, \quad y \geq 0, \quad z \geq 0 \) 不难得出

\[ 0 \leq x \leq \frac{\pi}{2}, \quad 0 \leq y \leq \frac{\pi}{2}, \quad 0 \leq z \leq \frac{\pi}{2}. \]

设 \( w = \ln u = \ln \sin x + \ln \sin y + \ln \sin z \),

\[ F(x, y, z) = w + \lambda (x + y + z - \frac{\pi}{2}). \]

解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= \cot x + \lambda = 0, \\
\frac{\partial F}{\partial y} &= \cot y + \lambda = 0, \\
\frac{\partial F}{\partial z} &= \cot z + \lambda = 0,
\end{align*}
\]

\[ x + y + z = \frac{\pi}{2} \]
并注意到点 \( P(x, y, z) \) 在第一三角形，即得静止点 \( P_0 \)。
\( \left( \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6} \right) \)。

类似3660题的讨论，当点 \((x, y, z)\) 趋于平面 \( x + y + z = \frac{\pi}{2} \) 在第一三角形部分的边界时，\( u \to 0 \)；而在边界内部 \( u \geq 0 \)。因此，函数 \( u \) 在边界内部取得最大值，故在点 \( P_0 \) 取得极大值 \( u(P_0) = \frac{1}{8} \)。

3665. \( u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \)，若 \( x^2 + y^2 + z^2 = 1 \)，\( x \cos \alpha + y \cos \beta + z \cos \gamma = 0 \) \((a \gg b \gg c > 0, \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1)\)。

解 设 \( F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \lambda (x^2 + y^2 + z^2 - 1) + \mu(x \cos \alpha + y \cos \beta + z \cos \gamma) \)。

解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2 \left( \frac{1}{a^2} - \lambda \right) x + \mu \cos \alpha = 0, \\
\frac{\partial F}{\partial y} &= 2 \left( \frac{1}{b^2} - \lambda \right) y + \mu \cos \beta = 0, \\
\frac{\partial F}{\partial z} &= 2 \left( \frac{1}{c^2} - \lambda \right) z + \mu \cos \gamma = 0, \\
x^2 + y^2 + z^2 &= 1, \\
x \cos \alpha + y \cos \beta + z \cos \gamma &= 0, \\
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1.
\end{align*}
\]
将(1)、(2)、(3)三式分别乘以x, y, z，然后相加，并注意到(4)、(5)两式，即得
\[ \lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = u(x, y, z). \tag{7} \]
再将(1)、(2)、(3)三式分别乘以cosα, cosβ, cosγ，然后相加，并注意到(5)、(6)两式，即得
\[ \mu = -2\left( \frac{x\cos\alpha}{a^2} + \frac{y\cos\beta}{b^2} + \frac{z\cos\gamma}{c^2} \right). \tag{8} \]
将(8)式代入(1)、(2)、(3)，得
\[
\begin{align*}
\left( \frac{\sin^2\alpha}{a^2} - \lambda \right) x - \frac{\cos\alpha\cos\beta}{b^2} y - \frac{\cos\alpha\cos\gamma}{c^2} z &= 0, \\
-\frac{\cos\alpha\cos\beta}{a^2} x + \left( \frac{\sin^2\beta}{b^2} - \lambda \right) y - \frac{\cos\beta\cos\gamma}{c^2} z &= 0, \tag{9} \\
-\frac{\cos\alpha\cos\gamma}{a^2} x - \frac{\cos\beta\cos\gamma}{b^2} y + \left( \frac{\sin^2\gamma}{c^2} - \lambda \right) z &= 0.
\end{align*}
\]
要\[ \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \]为方程组(9)的非零解，必须有
\[
\begin{vmatrix}
\sin^2\alpha - a^2\lambda & -\cos\alpha\cos\beta & -\cos\alpha\cos\gamma \\
-\cos\alpha\cos\beta & \sin^2\beta - b^2\lambda & -\cos\beta\cos\gamma \\
-\cos\alpha\cos\gamma & -\cos\beta\cos\gamma & \sin^2\gamma - c^2\lambda
\end{vmatrix} = 0.
\]
展开计算可得
\[
\lambda \left[ \lambda^2 - \left( \frac{\sin^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2} + \frac{\sin^2\gamma}{c^2} \right) \lambda + \left( \frac{\cos^2\alpha}{b^2c^2} + \frac{\cos^2\beta}{c^2a^2} \right) \right] + \left( \frac{\cos^2\gamma}{a^2b^2} \right) = 0. \tag{10}
\]
由(7)知\( \lambda \neq 0 \)，且不难验证(10)式在消去\( \lambda \)后得到

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的二次方程有两个不等的实根 $\lambda_1 < \lambda_2$.

固定 $\lambda = \lambda_1$，代入方程组(9)，可得到关于$\lambda$，$y, z$)的一个自由度的一个解系，再代入方程(4)，可得对应于$\lambda = \lambda_1$的两个静止点$P_1(x_1, y_1, z_1)$ 和 $P_2(x_2, y_2, z_2)$. 由(7)知，对应的$u(P_1) = u(P_2) = \lambda_1$.

同理可求得对应于$\lambda = \lambda_2$的两个静止点$P_3(x_3, y_3, z_3)$ 和 $P_4(x_4, y_4, z_4)$，且有$u(P_3) = u(P_4) = \lambda_2$.

$P_1, P_2, P_3, P_4$为满足方程组(1)～(5)的一切解所对应的点。类似前面各题的讨论可知，函数$u$在点$P_1$及$P_2$取得极小值$\lambda_1$，而在点$P_3$及$P_4$取得极大值$\lambda_2$。

$$u = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$$

若 $Ax + By + Cz = 0, \quad x^2 + y^2 + z^2 = R^2, \quad \frac{\xi}{\cos \alpha} = \frac{\eta}{\cos \beta} = \frac{\zeta}{\cos \gamma}$，

其中 $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

解 设 $F(x, y, z) = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 + \lambda(Ax + By + Cz) + \mu(x^2 + y^2 + z^2 - R^2)$，

记 $\xi = \rho \cos \alpha, \eta = \rho \cos \beta, \zeta = \rho \cos \gamma, \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2}$。

解方程组

$$\begin{align*}
\frac{\partial F}{\partial x} &= 2(x - \rho \cos \alpha) + \lambda A + 2 \mu x = 0, \quad (1) \\
\frac{\partial F}{\partial y} &= 2(y - \rho \cos \beta) + \lambda B + 2 \mu y = 0, \quad (2) \\
\frac{\partial F}{\partial z} &= 2(z - \rho \cos \gamma) + \lambda C + 2 \mu z = 0, \quad (3) \\
x^2 + y^2 + z^2 &= R^2, \quad (4) \\
Ax + By + Cz &= 0, \quad (5) \\
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1. \quad (6)
\end{align*}$$
将 (1)、(2)、(3) 三式分别乘以 $A, B, C$, 然后相加，并注意到 (5) 式，即得

$$-2\rho (A\cos\alpha + B\cos\beta + C\cos\gamma) + \lambda (A^2 + B^2 + C^2) = 0,$$

$$\lambda = \frac{2\rho (A\cos\alpha + B\cos\beta + C\cos\gamma)}{A^2 + B^2 + C^2}.$$  (7)

再将 (1)、(2)、(3) 三式分别乘以 $x, y, z$, 然后相加，并注意到 (4) 式和 (5) 式，即得

$$2(1 + \mu) R^2 = 2\rho (x\cos\alpha + y\cos\beta + z\cos\gamma).$$  (8)

又将 (1)、(2)、(3) 三式分别乘以 $\cos\alpha, \cos\beta, \cos\gamma$, 然后相加，并注意到 (6) 式，即得

$$2(1 + \mu) (x\cos\alpha + y\cos\beta + z\cos\gamma)$$

$$= 2\rho - \lambda (A\cos\alpha + B\cos\beta + C\cos\gamma)$$

$$= 2\rho \left[ 1 - \frac{(A\cos\alpha + B\cos\beta + C\cos\gamma)^2}{A^2 + B^2 + C^2} \right].$$  (9)

由 (8)，(9) 可得

$$(1 + \mu)^2 R^2 = (1 + \mu) \rho (x\cos\alpha + y\cos\beta + z\cos\gamma)$$

$$= \rho^2 \left[ 1 - \frac{(A\cos\alpha + B\cos\beta + C\cos\gamma)^2}{A^2 + B^2 + C^2} \right].$$

即

$$1 + \mu = \pm \frac{\rho}{R} \sqrt{1 - \frac{(A\cos\alpha + B\cos\beta + C\cos\gamma)^2}{A^2 + B^2 + C^2}}.$$  (10)

由 (1)，(2)，(3) 可得

$$x = \frac{2\rho\cos\alpha - \lambda A}{2(1 + \mu)}, \quad y = \frac{2\rho\cos\beta - \lambda B}{2(1 + \mu)},$$

$$z = \frac{2\rho\cos\gamma - \lambda C}{2(1 + \mu)}.$$
把（7）式和（10）式代入上式，即可得 $P_1(x_1, y_1, z_1)$ 和 $P_2(x_2, y_2, z_2)$，其中 $P_1$ 对应于（10）式取正号，而 $P_2$ 对应于（10）式取负号。下面求 $u(P_1)$ 和 $u(P_2)$。由（9）、（10）可得

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \pm R \sqrt{1 - \frac{(A \cos \alpha + B \cos \beta + C \cos \gamma)^2}{A^2 + B^2 + C^2}}.$$ 

于是，

$$u(P_1) = (x_1 - \rho \cos \alpha)^2 + (y_1 - \rho \cos \beta)^2 + (z_1 - \rho \cos \gamma)^2$$

$$= (x_1^2 + y_1^2 + z_1^2) - 2 \rho (x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma) + \rho^2$$

$$= R^2 + \rho^2 - 2 \rho R \sqrt{1 - \frac{(A \cos \alpha + B \cos \beta + C \cos \gamma)^2}{A^2 + B^2 + C^2}}.$$ 

同理可得

$$u(P_2) = R^2 + \rho^2 + 2 \rho R \sqrt{1 - \frac{(A \cos \alpha + B \cos \beta + C \cos \gamma)^2}{A^2 + B^2 + C^2}}.$$ 

类似以前各题的讨论可知：$u(P_2)$ 为极大值，$u(P_1)$ 为极小值。

3667. $u = x_1^2 + x_2^2 + \cdots + x_n^2$，若 $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} = 1$ 令 $a_i \geq 0$； $i = 1, 2, \cdots, n$。

解 设 $F(x_1, x_2, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 + \lambda \left( \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} - 1 \right)$。解方程组

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\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= 2x_i + \frac{\lambda}{a_i} = 0 \quad (i = 1, 2, \ldots, n), \\
\sum_{i=1}^{n} \frac{x_i}{a_i} &= 1
\end{align*}
\]

可得静止点 \( P_0(x_1, x_2, \ldots, x_n) \)，其中
\[
x_i = \frac{1}{a_i} \left( \sum_{j=1}^{n} \frac{1}{a_j^2} \right)^{-1} \quad (i = 1, 2, \ldots, n).
\]

由于 \( d^2 u = d^2 F = 2 \sum_{i=1}^{n} d^2 x_i \gg 0 \) （它不受约束条件的限制），故当 \( x_i = \frac{1}{a_i} \left( \sum_{j=1}^{n} \frac{1}{a_j^2} \right)^{-1} \) 时，函数 \( u \) 取极小值
\[
u = \sum_{i=1}^{n} \left[ \frac{1}{a_i} \left( \sum_{j=1}^{n} \frac{1}{a_j^2} \right)^{-1} \right] = \left( \sum_{i=1}^{n} \frac{1}{a_i^2} \right)^{-1}
\]

3668. \( u = x_1^p + x_2^p + \cdots + x_n^p \quad (p \gg 1) \)，若 \( x_1 + x_2 + \cdots + x_n = a \quad (a \gg 0) \)。

设 \( F(x_1, x_2, \ldots, x_n) = x_1^p + x_2^p + \cdots + x_n^p + \lambda (x_1 + x_2 + \cdots + x_n - a) \)。解方程组
\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= px_i^{p-1} + \lambda = 0 \quad (i = 1, 2, \ldots, n), \\
\sum_{i=1}^{n} x_i &= a
\end{align*}
\]

得 \( x_i = \frac{a}{n} \quad (i = 1, 2, \ldots, n) \)。由于
\[
\frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} 
p(p-1)x_i^{p-2}, & i = j, \\ 0, & i \neq j, \end{cases}
\]

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故当 \( x_i = \frac{a_i}{n} \quad (i = 1, 2, \cdots, n) \) 时，

\[
d^2 F = p(p-1) \sum_{i=1}^{n} \left(\frac{a_i}{n}\right)^{p-2} d x_i^2 \geq 0 \quad (\text{当} \sum_{i=1}^{n} d x_i^2 \neq 0 \text{ 时})，\]

它不受约束条件的限制，故函数 \( u \) 取得极小值 \( u = \frac{a_i^p}{n^{p-1}} \).

这里应该指出的是，对于一般的实数 \( p \)，应限定 \( x_i \gg 0 \)。

3669. \( u = \frac{\alpha_1}{x_1} + \frac{\alpha_2}{x_2} + \cdots + \frac{\alpha_n}{x_n} \)，若 \( \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 1 \quad (\alpha_i \gg 0, \beta_i \gg 0; \ i = 1, 2, \cdots, n) \).\]

解设 \( F'(x_1, x_2, \cdots, x_n) = \frac{\alpha_1}{x_1} + \frac{\alpha_2}{x_2} + \cdots + \frac{\alpha_n}{x_n} + \lambda(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n - 1) \).

解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= -\frac{\alpha_i}{x_i^2} + \lambda \beta_i = 0 \quad (i = 1, 2, \cdots, n), \\
\sum_{i=1}^{n} \beta_i x_i &= 1
\end{align*}
\]

得 \( x_i = \sqrt{\frac{\alpha_i}{\beta_i}} \left( \sum_{j=1}^{i} \sqrt{\frac{\alpha_i \beta_j}{\alpha_j \beta_i}} \right)^{-1} \quad (i = 1, 2, \cdots, n) \)。由于

\[
d^2 F = 2 \sum_{i=1}^{n} \frac{\alpha_i}{x_i^3} d x_i^2 \geq 0 ,
\]

*) 编者注：本题应加条件 \( x_i \gg 0 \quad (i = 1, 2, \cdots, n) \).
故当 \( x_i = \sqrt{\frac{\alpha_i}{\beta_i}} \left( \sum_{i=1}^{n} \sqrt{\alpha_i \beta_i} \right)^{-1} \) 时，函数 \( u \) 取得极小值

\[
u = \left( \sum_{i=1}^{n} \sqrt{\alpha_i \beta_i} \right)^2.
\]

3670. \( u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \)，若 \( x_1 + x_2 + \cdots + x_n = a \) (\( a \gg 0 \),\( \alpha_i \gg 1 \), \( i = 1, 2, \cdots, n \))

解 设 \( w = \ln u = \sum_{i=1}^{n} \alpha_i \ln x_i \),

\[
F(x_1, x_2, \cdots, x_n) = w - \frac{1}{\lambda} \left( \sum_{i=1}^{n} x_i - a \right)
\]

\[
= \sum_{i=1}^{n} \left( \alpha_i \ln x_i - \frac{x_i}{\lambda} \right) + \frac{a}{\lambda}.
\]

解方程组

\[
\begin{cases}
\frac{\partial F}{\partial x_i} = \frac{\alpha_i}{x_i} - \frac{1}{\lambda} = 0 \\
\sum_{i=1}^{n} x_i = a
\end{cases} \quad (i = 1, 2, \cdots, n),
\]

得 \( x_i = \frac{a \alpha_i}{a_1 + a_2 + \cdots + a_n} \) (\( i = 1, 2, \cdots, n \)). 由于

\[
d^2 w = - \sum_{i=1}^{n} \frac{\alpha_i}{x_i^2} \, dx_i^2 < 0 \quad (\text{当} \sum_{i=1}^{n} dx_i^2 \neq 0 \text{ 时})
\]

不论 \( dx_i \) 之间有什么约束条件恒成立，故函数 \( w \) 当

\[
x_i = \frac{a \alpha_i}{a_1 + a_2 + \cdots + a_n} \quad (i = 1, 2, \cdots, n)
\]

时取得极大值。

*) 编者注：本题应加条件 \( x_i > 0 \) (\( i = 1, 2, \cdots, n \)).
即函数当 \( x_i = \frac{a i}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \) 时取得极大值

\[
u = \left( \frac{a}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \alpha_1^{a_1} \alpha_2^{a_2} \cdots \alpha_n^{a_n} \right).
\]

3671 若 \( \sum_{i=1}^{n} x_i^2 = 1 \)，求二次型 \( u = \sum_{i, j=1}^{n} a_{ij} x_i x_j \) (\( a_{ii} = a_{ii} \)) 的
极值。

解 设 \( F(x_1, x_2, \cdots, x_n) = u - \lambda (x_1^2 + x_2^2 + \cdots + x_n^2 - 1) \)。解方程组

\[
\begin{align*}
\frac{1}{2} \frac{\partial F}{\partial x_1} &= (a_{11} - \lambda) x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0, \quad (1) \\
\frac{1}{2} \frac{\partial F}{\partial x_2} &= a_{21} x_1 + (a_{22} - \lambda) x_2 + \cdots + a_{2n} x_n = 0, \quad (2) \\
& \quad \cdots \cdots \cdots \\
\frac{1}{2} \frac{\partial F}{\partial x_n} &= a_{n1} x_1 + a_{n2} x_2 + \cdots + (a_{nn} - \lambda) x_n = 0, \quad (n) \\
x_1^2 + x_2^2 + \cdots + x_n^2 &= 1. \quad (n + 1)
\end{align*}
\]

前 \( n \) 个方程要有非零解，必须矩阵 \((a_{ii})\) 的特征方程 \(|A - \lambda E| = 0\) 有解，其中 \(A\) 为以 \(a_{ii}\) 为元素的对称矩阵，\(E\) 为单位矩阵。由线性代数中关于欧氏空间的理论知，此特征方程必有 \( n \) 个实根，即有 \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) 满足 \(|A - \lambda E| = 0\)。对于任一个 \( \lambda_k \)，代入方程 (1) ~ (n)，可求得 \((x_1, x_2, \cdots, x_n)\) 的一个解空间，解空间的维数，等于 \( \lambda_k \) 的重数。解空间中的单位元即方程组 (1) ~ (n + 1) 的根。当 \( \lambda_k \) 是单重根时，解空
间是一维的，单位元向有两个。当λₖ是多重根时，
对应λₖ的单位元向就有无穷多个了。

对于λₖ的解（x₁, x₂, ..., xₙ），显然满足方程组
(1)～(n+1)。因此，有∑_{i=1}^{n} a_{ij} x_j = \lambda_i x_i (i = 1, 2, ..., n)，从而得

\[ u(x_1, x_2, ..., x_n) = \sum_{i, j=1}^{n} a_{ij} x_j = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} a_{ij} x_j \]

\[ = \sum_{i=1}^{n} \lambda_i x_i^2 = \lambda_i \sum_{i=1}^{n} x_i^2 = \lambda_i. \]

由于函数u在n维球面x₁^2 + x₂^2 + ... + xₙ^2 = 1上连续，故必取得最大值和最小值。于是，对应于λ₁和λₙ的解，分别使函数u取得最大值λ₁和最小值λₙ，因而也是u的极大值和极小值，或是u的弱极大值和弱极小值，视λ₁和λₙ的重数而定（多重时为弱极值）。由线性代数中把d²F化标型的方法，可证：对于不等于λ₁和λₙ的λ₂，二次型不取得极值。

3672. 若n ≥ 1及x ≥ 0，y ≥ 0，证明不等式

\[ \frac{x^n + y^n}{2} \geq \left( \frac{x + y}{2} \right)^n. \]

证 考虑函数z = \frac{x^n + y^n}{2}在条件x + y = a (a ≥ 0, x ≥ 0, y ≥ 0)下的极值问题。设

\[ F(x, y) = \frac{1}{2} (x^n + y^n) + \lambda (x + y - a). \]

解方程组
\[
\begin{align*}
\frac{\partial F}{\partial x} &= \frac{n}{2} x^{n-1} + \lambda = 0, \\
\frac{\partial F}{\partial y} &= \frac{n}{2} y^{n-1} + \lambda = 0, \\
x + y &= a
\end{align*}
\]
可得 \(x = y = \frac{a}{2}\).

将点 \((\frac{a}{2}, \frac{a}{2})\) 与边界点 \((0, a), (a, 0)\) 的函数值进行比较（注意到 \(n \geqslant 1\)):

\(z(0, a) = z(a, 0) = \frac{a^n}{2} \geqslant (\frac{a}{2})^n = z(\frac{a}{2}, \frac{a}{2}) (n \geqslant 1)\),

即知函数 \(z\) 当 \(x + y = a\) 时的最小值为 \((\frac{a}{2})^n\). 从而有

\[
\frac{x^n + y^n}{2} \geqslant (\frac{a}{2})^n
\]

(当 \(x + y = a, x \geqslant 0, y \geqslant 0\) 时). \hspace{1cm} (1)

下面我们证明

\[
\frac{x^n + y^n}{2} \geqslant \left(\frac{x + y}{2}\right)^n \hspace{1cm} (\text{当 } x \geqslant 0, y \geqslant 0 \text{ 时}). \hspace{1cm} (2)
\]
当 \(x = y = 0\) 时，不等式 (2) 显然成立；当 \(x \geqslant 0, y \geqslant 0\) 且 \(x, y\) 不同时为零时，令 \(x + y = a\)，则 \(a > 0\). 于是，由不等式 (1) 即得

\[
\frac{x^n + y^n}{2} \geqslant \left(\frac{a}{2}\right)^n = \left(\frac{x + y}{2}\right)^n.
\]
由此可知，不等式 (2) 成立，证毕。

3673. 证明和尔宾不等式

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\[ \sum_{i=1}^{n} a_i x_i \leq \left( \sum_{i=1}^{n} a_i \right)^{\frac{1}{k}} \left( \sum_{i=1}^{n} x_i^k \right)^{\frac{1}{k'}} \]

\((a_i \geq 0, x_i \geq 0, i = 1, 2, \ldots, n, k > 1, \frac{1}{k} + \frac{1}{k'} = 1)\).

证　我们首先证明函数

\[ u = \left( \sum_{i=1}^{n} a_i \right)^{\frac{1}{k}} \left( \sum_{i=1}^{n} x_i^k \right)^{\frac{1}{k'}} \]

在条件 \( \sum_{i=1}^{n} a_i x_i = A \quad (A > 0) \) 下的最小值是 \( A \)。为此，对 \( n \) 用数学归纳法。

当 \( n = 1 \) 时，显然有

\[ (a_1)^{\frac{1}{k}} (x_1^k)^{\frac{1}{k'}} = a_1 x_1 = A. \]

设当 \( n = m \) 时，命题为真，故对任意 \( m \) 个数 \( a_1, a_2, \ldots, a_m (a_i \geq 0) \)，当 \( \sum_{i=1}^{m} a_i x_i = A \quad (x_1 \geq 0, \ldots, x_m \geq 0) \) 时，必有

\[ A \leq \left( \sum_{i=1}^{m} a_i \right)^{\frac{1}{k}} \left( \sum_{i=1}^{m} x_i^k \right)^{\frac{1}{k'}}. \]

我们证明当 \( n = m + 1 \) 时命题也真。设 \( \sum_{i=1}^{m+1} a_i x_i = A \)，

\[ u = a^{\frac{1}{k}} \left( \sum_{i=1}^{m+1} x_i^k \right)^{\frac{1}{k'}} \]

其中 \( a = \sum_{i=1}^{m+1} a_i \)。求 \( u \) 的最小值。令

\[ F(x_1, x_2, \ldots, x_{m+1}) = u(x_1, x_2, \ldots, x_{m+1}) - \lambda \left( \sum_{i=1}^{m+1} a_i x_i - A \right). \]

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解方程组

$$
\begin{align*}
\frac{\partial F}{\partial x_i} &= \frac{1}{k^i} \left( \sum_{i=1}^{m+1} x_i^{k^i} \right) \frac{1}{k^i - 1} (k^i - 1) - \lambda a_i = 0 \\
\sum_{i=1}^{m+1} a_i x_i &= A
\end{align*}
$$

得

$$
\frac{x_i^{k_i - 1}}{a_i} = \frac{\lambda}{\alpha} \left( \sum_{i=1}^{m+1} x_i^{k_i} \right)^{1/k_i} = \mu^{k_i - 1} \quad (i = 1, 2, \ldots, m+1).
$$

（这里引入了记号 \( \mu \)），即

$$
x_i = (a_i \mu^{k_i - 1})^{1/(k_i - 1)} = a_i^{1/(k_i - 1)} \mu = \mu a_i^{1/(k_i - 1)},
$$

从而有

$$
\mu \sum_{i=1}^{m+1} a_i x_i^{k_i - 1} = \mu \sum_{i=1}^{m+1} a_i = \mu \alpha = A,
$$

$$
\mu = \frac{A}{\alpha}.
$$

于是，解得满足极值必要条件的唯一解

$$
x_i^0 = \frac{A}{\alpha} a_i^{1/(k_i - 1)} \quad (i = 1, 2, \ldots, m+1).
$$

对应的函数值为

$$
u_0 = u(x_1^0, x_2^0, \ldots, x_{m+1}^0) = \alpha^\frac{1}{k_i} \left[ \sum_{i=1}^{m+1} \left( \frac{A}{\alpha} a_i^{1/(k_i - 1)} \right)^{k_i} \right]^{1/k_i}
= \alpha^\frac{1}{k_i} \frac{A}{\alpha} \left[ \sum_{i=1}^{m+1} a_i^{(k_i - 1)/k_i} \right]^{1/k_i}
= \alpha^{1/(k_i - 1)} A \left( \sum_{i=1}^{m+1} a_i \right)^{1/\alpha}
= A a_i^{1/(k_i - 1)} \frac{1}{\alpha} = A.
$$
所研究的区域 \( \sum_{i=1}^{m+1} a_i x_i = A \), \( x_i \geq 0 \) \((i = 1, 2, \cdots, m+1)\) 是在维空间中一个 \( m \) 维平面在第一卦限的部份. 其边界各 \( m+1 \) 个 \( m-1 \) 维平面 (之一部分) 所组成, \( x_i = 0 \), \( \sum_{i=1}^{m+1} a_i x_i = A \) \((a_i > 0, x_i \geq 0, i = 1, 2, \cdots, m+1)\). 在这些界面上, 求

\[
\begin{align*}
    u(x_1, x_2, \cdots, x_{m+1}) &= u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{m+1}) \\
    &= a^\frac{1}{i-1} \left( \sum_{i=1}^{i-1} x_i^{i-1} + \sum_{i=1}^{m+1} x_i^{i-1} \right)^{\frac{1}{i-1}}
\end{align*}
\]

的最小值变为求 \( m \) 个变量的最小值. 以估计 \( x_{m+1} = 0 \), \( \sum_{i=1}^{m} a_i x_i = A \) 的最小值为例. 根据归纳法假设, 注意到

\[
    a = \sum_{i=1}^{m+1} a_i \geq \sum_{i=1}^{m} a_i,
\]

即有

\[
    u(x_1, x_2, \cdots, x_m, 0) = a^\frac{1}{i-1} \left( \sum_{i=1}^{m} x_i^{i-1} \right)^{\frac{1}{i-1}} \geq \left( \sum_{i=1}^{m} a_i \right)^{\frac{1}{m+1}} \left( \sum_{i=1}^{m} x_i^{i-1} \right)^{\frac{1}{i-1}} \geq \sum_{i=1}^{m} a_i x_i = A.
\]

因此, \( u \) 在边界面的最小值不小于 \( A \). 由此可知, \( u \) 在区域上的最小值为 \( u(x_1^0, x_2^0, \cdots, x_{m+1}^0) = A \), 故命题当 \( n = m+1 \) 时为真. 于是, 由归纳法可知

\[
    \left( \sum_{i=1}^{m} a_i \right)^{\frac{1}{m+1}} \left( \sum_{i=1}^{m} x_i^{i-1} \right)^{\frac{1}{i-1}} \geq A,
\]

当 \( \sum_{i=1}^{m} a_i x_i = A \), \( x_i \geq 0 \) \((i = 1, 2, \cdots, n)\) 时. (1)
下面我们证明和尔策不等式

\[
\sum_{i=1}^{n} a_i x_i \leq \left( \sum_{i=1}^{n} a_i^k \right)^{\frac{1}{k}} \left( \sum_{i=1}^{n} x_i^k \right)^{\frac{1}{k}} \quad (a_i \geq 0, \ x_i \geq 0) \quad (2)
\]

成立。事实上，若 \( \sum_{i=1}^{n} a_i x_i = 0 \)，则 (2) 式显然成立；
若 \( \sum_{i=1}^{n} a_i x_i \geq 0 \)，令 \( \sum_{i=1}^{n} a_i x_i = A \)，则 \( A \geq 0 \)。于是，根据不等式 (1) 知

\[
\left( \sum_{i=1}^{n} a_i^k \right)^{\frac{1}{k}} \left( \sum_{i=1}^{n} x_i^k \right)^{\frac{1}{k}} \geq A = \sum_{i=1}^{n} a_i x_i,
\]

故不等式 (2) 成立。证毕。

注。和尔策 (Hölder) 不等式是一个重要而常用的不等式，而且还可推广到一般的形式，证明方法也很多。

3674. 对于 \( n \) 阶行列式 \( A = |a_{ij}| \) 证明哈达马不等式

\[
A^2 \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right).
\]

证 证法一
为区别起见，以下用 \( A \) 表矩阵 \((a_{ij})\)，\( |A| \) 表行列式 \( |a_{ii}| \)。考虑函数 \( u = |A| = |a_{ii}| \) 在条件 \( \sum_{j=1}^{n} a_{ij}^2 = S_i \) (\( i = 1, 2, \cdots, n \)) 下的极值问题，其中 \( S_i > 0 (i = 1, 2, \cdots, n) \)。

由于上述 \( n \) 个条件限制下的 \( n^2 \) 元点集是有界闭集，故连续函数 \( u \) 必在其上取得最大值和最小值。下面我们求函数 \( u \) 满足条件极值的必要条件。设
\[ F = u - \sum_{i=1}^{n} \lambda_i \left( \sum_{j=1}^{n} a_{ij}^2 - S_i \right). \]

由于函数 \( u \) 是多项式，当按第 \( i \) 行展开时，有

\[ u = |A| = \sum_{i=1}^{n} a_{ij} A_{ii}, \]

其中 \( A_{ii} \) 是 \( a_{ij} \) 的代数余子式。解方程组

\[ \frac{\partial F}{\partial a_{ij}} = A_{ij} - 2\lambda_i a_{ij} = 0 \quad (i, j = 1, 2, \ldots, n) \]

得 \( a_{ij} = \frac{A_{ij}}{2\lambda_i} \)。当 \( i \neq k \) 时，有

\[ \sum_{j=1}^{n} a_{ij} a_{kj} = \sum_{j=1}^{n} \frac{A_{ij} a_{kj}}{2\lambda_i} = \frac{1}{2\lambda_i} \sum_{j=1}^{n} A_{ij} a_{jj} = 0, \]

故当函数 \( u \) 满足极值的必要条件时，行列式不同的两行所对应的向量必专交。若以 \( A' \) 表示 \( A \) 的转置矩阵，则由行列式的乘法得

\[ u^2 = |A'| \cdot |A| = \begin{vmatrix} S_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & S_n & 0 \end{vmatrix} = \prod_{i=1}^{n} S_i. \]

因此，函数 \( u \) 满足极值的必要条件时，必有

\[ u = \pm \sqrt{\prod_{i=1}^{n} S_i}. \]

由于显然函数 \( u \) 在条件 \( \sum_{i=1}^{n} a_{ii}^2 = S_i \quad (i = 1, 2, \ldots, n) \) 下不恒为常数，故
\[ u_{\text{max}} = \sqrt{\prod_{i=1}^{n} S_i}, \quad u_{\text{min}} = -\sqrt{\prod_{i=1}^{n} S_i}. \]

从而

\[ |A|^2 \leq \prod_{i=1}^{n} S_i, \]

当 \[ \sum_{j=1}^{n} a_{ij}^2 = S_i \quad (i = 1, 2, \ldots, n) \] 时，(1)

下面我们证明

\[ |A|^2 \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right). \] (2)

若至少有一个 \( i \)，使 \( \sum_{j=1}^{n} a_{ij}^2 = 0 \)，则 \( a_{ij} = 0 \) \((j = 1, 2, \ldots, n)\)。从而 \(|A| = 0\)。于是不等式 (2) 显然成立。

若对一切 \( i (i = 1, 2, \ldots, n) \)，都有 \( \sum_{j=1}^{n} a_{ij}^2 \neq 0 \)，令

\[ S_i = \sum_{j=1}^{n} a_{ij}^2, \quad S_i \geq 0 \quad (i = 1, 2, \ldots, n). \] 于是，根据不等式 (1) 即得

\[ |A|^2 \leq \prod_{i=1}^{n} S_i = \prod_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right). \]

故不等式 (2) 成立。证毕。

证法二

如将原题归一化，则也可获证。设

\[ \overline{a_{ij}} = \frac{a_{ij}}{\left( \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}} \quad (i, j = 1, 2, \ldots, n), \]

则有
设 \( \bar{A} = \sum_{i=1}^{n} \bar{a}^2_i = 1 \quad (i=1,2,\cdots,n) \)。
从而原命题就可转化为证明不等式
\[ |A| \leq 1, \]
其中 \( \sum_{i=1}^{n} a^2_{ij} = 1 \quad (i=1,2,\cdots,n) \)。

设 \( F = |A| + \sum_{i=1}^{n} \lambda_i \left( \sum_{i=1}^{n} a^2_{ii} - 1 \right) \)。解方程组
\[ \frac{\partial F}{\partial a_{ij}} = A_{ij} + 2\lambda_i a_{ii} = 0, \]
其中 \( A_{ij} \) 为 \( a_{ij} \) 的代数余子式 \((i,j=1,2,\cdots,n)\)。于上式两端乘以 \( a_{ij} \) 并对 \( j=1,2,\cdots,n \) 求和，即得
\[ |A| + 2\lambda_i = 0 \quad (i=1,2,\cdots,n). \]
从而有
\[ \lambda_i = -\frac{|A|}{2} \quad (i=1,2,\cdots,n), \]
也即
\[ A_{ii} = a_{ii} |A| \quad (i,j=1,2,\cdots,n). \]
故得
\[
\begin{vmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{vmatrix} =
\begin{vmatrix}
a_{11} & \cdots & a_{1n} & |A| \\
\vdots & \ddots & \vdots & \vdots \\
a_{n1} & \cdots & a_{nn} & |A|
\end{vmatrix},
\]
上式左端的行列式叫做 \( A \) 的附属行列式，记为 \( A^* \)。
由线性代数知识可知，当 \( |A| = 0 \) 时，\( A^* = 0 \)。当 \( |A| \neq 0 \) 时，\( |A| |A^*| = \begin{vmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n \)。故有
\[ 475 \]
\[ |A^*| = |A|^{\ast-1}. \text{ 于是,} \]
\[ |A|^{\ast-1} = |A|^{\ast+1}. \]
由于\( |A| \)的极值必须满足上式，故不难推知\( |A|_{\text{max}} = 1, |A|_{\text{min}} = -1 \)。从而得知：当\( \sum_{i-1}^{1} a_i^2 = 1 (i=1, 2, \cdots, n) \)时，恒有
\[ |A|^2 \leq 1 \text{ 或 } |A^T| \leq 1. \]
求下列函数在指定域内的上确界\( (\text{sup}) \)和下确界\( (\text{inf}) \)。

3675. \( z = x - 2y - 3 \)，若 \( 0 \leq x \leq 1 \)，\( 0 \leq y \leq 1 \)，\( 0 \leq x + y \leq 1 \)。

解 以

D表区域 \( 0 \leq x \leq 1 \)，\( 0 \leq y \leq 1 \)，\( 0 \leq x + y \leq 1 \)，它是一个有界闭区域（为一闭三角形），故连续函数 \( z \) 在其上必有最大值和最小值。由于 \( z \) 是 \( x, y \) 的线性函数，故不存在静止点，因此，最大值与最小值都在 \( D \) 的边界上达到。\( D \) 的边界为三条直线段：\( y = 0 \) (\( 0 \leq x \leq 1 \))，\( x = 0 \) (\( 0 \leq y \leq 1 \))，\( x + y = 1 \) (\( 0 \leq x \leq 1 \)); 在其上 \( z \) 分别变成一元函数：\( z = x - 3 \) (\( 0 \leq x \leq 1 \))，\( z = -2y - 3 \) (\( 0 \leq y \leq 1 \))，\( z = 3x - 5 \) (\( 0 \leq x \leq 1 \))。由于这些函数都是一元线性函数，故也无静止点，其最大值与最小值必在此三线段的端点（即点 \( (0, 0), (1, 0), (0, 1) \)）达到。由此可知，\( z \) 在 \( D \) 上的最大值与最小值必在此三点 \( (0, 0), (1, 0), (0, 1) \) 中达到。

由于

\[ z(0, 0) = -3, \ z(1, 0) = -2, \ z(0, 1) = -5, \]
故
\[ \sup z = -2, \; \inf z = -5. \]

3676. \( z = x^2 + y^2 - 12x + 16, \) 若 \( x^2 + y^2 \leq 25. \)

解：考虑函数 \( z \) 在区域 \( x^2 + y^2 \leq 25 \) 内的静止点：

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x - 12 = 0, \\
\frac{\partial z}{\partial y} &= 2y + 16 = 0.
\end{align*}
\]

在区域内无解，故连续函数 \( z \) 的最大值与最小值必在边界 \( x^2 + y^2 = 25 \) 上达到。

考虑函数 \( z \) 在边界 \( x^2 + y^2 = 25 \) 上的条件极值。设 \( F(x, y) = z - \lambda(x^2 + y^2 - 25) \)。解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2x - 12 - 2\lambda x = 0, \\
\frac{\partial F}{\partial y} &= 2y + 16 - 2\lambda y = 0, \\
x^2 + y^2 &= 25
\end{align*}
\]

可得静止点 \( P_1(3, -4) \) 及 \( P_2(-3, 4) \)。由于

\[ z(3, -4) = -75, \; z(-3, 4) = 125, \]

故得

\[ \sup z = 125, \; \inf z = -75. \]

3677. \( z = x^2 - xy + y^2, \) 若 \( |x| + |y| \leq 1. \)

解：求函数 \( z \) 在区域 \( |x| + |y| \leq 1 \) 内的静止点：

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2x - y = 0, \\
\frac{\partial z}{\partial y} &= 2y - x = 0.
\end{align*}
\]

解得静止点 \( P_0(0, 0) \)。相应地，\( z(P_0) = 0. \)
再在边界：$x \geq 0, \ y \geq 0, \ x+y=1$ 上求静止点。设 $F_1 = x^2 - xy + y^2 - \lambda (x+y-1)$。
解方程组
\[
\begin{align*}
\frac{\partial F_1}{\partial x} &= 2x - y - \lambda = 0, \\
\frac{\partial F_1}{\partial y} &= 2y - x - \lambda = 0, \\
&\quad x + y = 1
\end{align*}
\]
得静止点 $P_1 \left( \frac{1}{2}, \frac{1}{2} \right)$。相应地，$z(P_1) = \frac{1}{4}$。

同理可在另外三条边界线：$x \geq 0, \ y \leq 0, \ x-y=1$ 上；$x \leq 0, \ y \geq 0, \ x-y=-1$ 上；$x \leq 0, \ y \leq 0, \ x+y=-1$ 上分别求得静止点 $P_2 \left( \frac{1}{2}, -\frac{1}{2} \right)$，
$P_3 \left( -\frac{1}{2}, \frac{1}{2} \right)$ 及 $P_4 \left( -\frac{1}{2}, -\frac{1}{2} \right)$. 相应地，$z(P_2) = z(P_3) = \frac{1}{4}$，
$z(P_4) = \frac{3}{4}$。

最后，在上述四条边界线的端点 $P_5(1,0), P_6(0,1), P_7(-1,0)$ 及 $P_8(0,-1)$ 上求得函数值：
$z(P_5) = z(P_6) = z(P_7) = z(P_8) = 1$。
比较 $z(P_i)$ ($i=0,1,2,\ldots,8$)，即得
\[
\sup z = 1, \ \inf z = 0.
\]

3678. $u = x^2 + 2y^2 + 3z^2$, 若 $x^2 + y^2 + z^2 \leq 100$。
解  该函数 $u$ 在区域 $x^2 + y^2 + z^2 \leq 100$ 内的静止点为 $P_0(0,0,0)$，而在边界 $x^2 + y^2 + z^2 = 100$ 上的静止点为 $P_1(10,0,0), P_2(-10,0,0), P_3(0,10,0)$,
$P_4(0, -10, 0), P_5(0, 0, 10)$ 及 $P_6(0, 0, -10)$. 相应地, $u(P_0) = 0, u(P_1) = u(P_2) = 100, u(P_3) = u(P_4) = 200, u(P_5) = u(P_6) = 300$. 于是,

$$\sup u = 300, \inf u = 0.$$ 3679. $u = x + y + z$, 若 $x^2 + y^2 \leq z \leq 1$.

解 所讨论的立体区域由曲面 $x^2 + y^2 = z$ $(0 \leq z \leq 1)$ 和平面 $z = 1, x^2 + y^2 \leq 1$ 所围成，两个曲面的交线为 $x^2 + y^2 = z = 1$.

显见在立体区域内部无静止点. 在边界面 $z = 1, x^2 + y^2 \leq 1$ 的内部，$u(x, y, 1) = x + y + 1$ 也无静止点. 在边界面 $x^2 + y^2 = z$ $(0 \leq z \leq 1)$ 上，有

$$u = x + y + x^2 + y^2 \quad (x^2 + y^2 \leq 1).$$

解方程组

$$\begin{cases} \frac{\partial u}{\partial x} = 1 + 2x = 0, \\ \frac{\partial u}{\partial y} = 1 + 2y = 0 \end{cases}$$

得静止点 $P_1\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$. 相应地, $u(P_1) = -\frac{1}{2}$.

在边界线 $x^2 + y^2 = z = 1$ 上, 设

$$F(x, y) = x + y + 1 + \lambda(x^2 + y^2 - 1).$$

解方程组

$$\begin{cases} \frac{\partial F}{\partial x} = 1 + 2\lambda x = 0, \\ \frac{\partial F}{\partial y} = 1 + 2\lambda y = 0, \\ x^2 + y^2 = 1 \end{cases}$$

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得静止点 \( P_2 \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right) \) 及 \( P_3 \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1 \right) \) 相应地，\( u(P_2) = 1 + \sqrt{2} \)，\( u(P_3) = 1 - \sqrt{2} \)。于是，

\[
\sup u = 1 + \sqrt{2}, \inf u = -\frac{1}{2}.
\]

3680. 求函数

\[
u = (x + y + z)e^{-\left(x + 2y + 3z\right)}
\]

在域 \( x \geq 0, y \geq 0, z \geq 0 \) 内的下确界 (\( \inf \)) 与上确界 (\( \sup \))。

解 函数 \( u \) 在区域 \( x \geq 0, y \geq 0, z \geq 0 \) 上是连续函数。因此，把区域扩大包括边界时，上、下确界不变，下面再扩大后的区域加以讨论。

显然当 \( x \geq 0, y \geq 0, z \geq 0 \) 时 \( u \geq 0 \)，且 \( u(0, 0, 0) = 0 \)。故 \( \inf u = 0 \)。

在区域内，由于

\[
\frac{\partial u}{\partial x} = e^{-\left(x + 2y + 3z\right)} \left[ 1 - (x + y + z) \right],
\]

\[
\frac{\partial u}{\partial y} = e^{-\left(x + 2y + 3z\right)} \left[ 1 - 2(x + y + z) \right],
\]

\[
\frac{\partial u}{\partial z} = e^{-\left(x + 2y + 3z\right)} \left[ 1 - 3(x + y + z) \right],
\]

而 \( e^{-\left(x + 2y + 3z\right)} \neq 0 \)。故函数 \( u \) 在域内无静止点。

又因

\[
u = (x + y + z)e^{-\left(x + 2y + 3z\right)} = (x + y + z) e^{-\left(x + y + z\right)}
\]

\[
\cdot e^{-\left(y + 2z\right)} \leq (x + y + z) e^{-\left(x + y + z\right)} \to 0 \quad ((x + y + z) \to +\infty)
\]
故函数 $u$ 的最大值必在有限的边界上达到。考虑界面:

$$ x = 0 ; \quad u(0, y, z) = (y + z) e^{-(x + z^2)} , \quad y \geq 0 , \quad z \geq 0 . $$

$$ y = 0 ; \quad u(x, 0, z) = (x + z) e^{-(y + z^2)} , \quad x \geq 0 , \quad z \geq 0 . $$

$$ z = 0 ; \quad u(x, y, 0) = (x + y) e^{-(x + y^2)} , \quad x \geq 0 , \quad y \geq 0 . $$

同样可证明，这些界面上无静止点。

最后考虑边界线: $x = 0 , \quad y = 0 , \quad z \geq 0 ,

$$ u(0, 0, z) = z e^{-3z} $$

可解得静止点 $P_1(0, 0, \frac{1}{3})$，相应地，$u(P_1) = \frac{1}{3} e^{-1}$。

同法在边界线: $x = 0 , \quad z = 0 , \quad y \geq 0$ 上可解得静止点 $P_2(0, \frac{1}{2}, 0)$；在边界线: $y = 0 , \quad z = 0 , \quad x \geq 0$

上可解得静止点 $P_3(1, 0, 0)$，相应地，$u(P_2) = \frac{1}{2} e^{-1}$,

$u(P_3) = e^{-1}$。至于边界线的一端为原点，另一端伸向无穷远，均已讨论过。于是，

$$ \sup u = e^{-1} . $$

3581. 证明：函数 $z = (1 + e^y) \cos x - ye^x$ 有无穷多个极大值而无一极小值。

证。解方程组

$$ \frac{\partial z}{\partial x} = -(1 + e^y) \sin x = 0 , $$

$$ \frac{\partial z}{\partial y} = e^x (\cos x - 1 - y) = 0 $$

得 $x = k \pi , \quad y = (-1)^k - 1 (k = 0 , \pm 1 , \pm 2 , \cdots )$。

由于
\[
\frac{\partial^2 z}{\partial x^2} = -(1 + e^x) \cos x, \quad \frac{\partial^2 z}{\partial x \partial y} = -e^x \sin x, \\
\frac{\partial^2 z}{\partial y^2} = e^y (\cos x - 2 - y),
\]

故在点 \((2m\pi, 0) (m = 0, \pm 1, \cdots), A = -2, B = 0, C = -1\) 及 \(AC - B^2 = 2 > 0\) ，此时函数 \(z\) 取得极小值；
而在点 \((2m + 1)\pi, -2) (m = 0, \pm 1, \cdots), A = 1 + e^{-2}, B = 0, C = -e^{-2}\) 及 \(AC - B^2 = -e^{-2} - e^{-4} < 0\) ，此时函数 \(z\) 无极值。

3682．函数 \(f(x, y)\) 在点 \(M_0(x_0, y_0)\) 有极小值的充分条件是否为此函数在沿着过 \(M_0\) 点的每一条直线上有极小值呢？
解 研究函数
\[
f(x, y) = (x - y^2)(2x - y^2).
\]
对于每一条通过原点的直线 \(y = kx (-\infty < x < +\infty)\) 均有
\[
f(x, kx) = (x - k^2 x^2)(2x - k^2 x^2)
= x^2 (1 - k^2 x)(2 - k^2 x),
\]
当 \(0 < |x| < \frac{1}{k^2}\) 时，\(f(x, kx) > 0\)。但是\(f(0, 0) = 0\)，
因此，函数 \(f(x, y)\) 在直线 \(y = kx\) 上在原点取得极小值零。

对于通过原点的另一条直线 \(x = 0\) 有 \(f(0, y) = y^4\)，
故在原点也取得极小值零。

因此，函数 \(f(x, y)\) 在一切通过原点的直线上均有极小值。但是，
\[
f(a, \sqrt{1.5}a) = -0.25a^2 < 0 \ (a > 0),
\]
因此，函数 \( f(x, y) \) 在 \((0, 0)\) 点不取得极小值。

此例说明：尽管 \( f(x, y) \) 在沿着过点 \( M_0 \) 的每一条直线上在 \( M_0 \) 均有极小值，但却不能保证 \( f(x, y) \)
作为二元函数在点 \( M_0 \) 一定有极小值。

3683. 分解已知正数 \( a \) 为 \( n \) 个正的因数，使得它们的倒数的和为最小。

解　按题设，我们应求函数 \( u = \sum_{i=1}^{n} \frac{1}{x_i} \) 在条件 \( a = \prod_{i=1}^{n} x_i \)
或 \( \ln a = \sum_{i=1}^{n} \ln x_i \ (a > 0, \ x_i > 0) \) 下的极值。设 \( F(x_1, x_2, \ldots, x_n) = u + \lambda \left( \sum_{i=1}^{n} \ln x_i - \ln a \right) \)。解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= -\frac{1}{x_i^2} + \frac{\lambda}{x_i} = 0 \ (i = 1, 2, \ldots, n), \\
a &= \prod_{i=1}^{n} x_i
\end{align*}
\]

可得 \( x_i = \frac{1}{\lambda} \ (i = 1, 2, \ldots, n) \)。从而解得

\( x_1^0 = x_2^0 = \cdots = x_n^0 = a^n, u(x_1^0, x_2^0, \ldots, x_n^0) = na^{-\frac{1}{n}} \).

当点 \( P(x_1, x_2, \ldots, x_n) \) 趋向于边界时，至少有一个 \( x_i \to 0 \)，即 \( \frac{1}{x_i} \to +\infty \)，而 \( u \geq \frac{1}{x_i} \)，故 \( u \to +\infty \).

因此，函数 \( u \) 必在区域内部取得最小值。于是，将正数 \( a \) 分为 \( n \) 个相等的正的因数 \( a^n \) 时，其倒数和 \( na^{-\frac{1}{n}} \)
最小。

3684. 分解已知正数 \( a \) 为 \( n \) 个相加数，使得它们的平方和为最小。
解 考虑函数 \( u = \sum_{i=1}^{n} x_i^2 \) 在条件 \( a = \sum_{i=1}^{n} x_i (a > 0) \) 下的极值．设 \( F(x_1, x_2, \ldots, x_n) = u + \lambda \left( \sum_{i=1}^{n} x_i - a \right) \)．解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= 2x_i + \lambda = 0 \quad (i = 1, 2, \ldots, n), \\
\sum_{i=1}^{n} x_i &= a
\end{align*}
\]

得 \( x_1^0 = x_2^0 = \cdots = x_n^0 = \frac{a}{n} \)，\( u(x_1^0, x_2^0, \ldots, x_n^0) = \frac{a^2}{n} \)．

当\( n \)个相加数中有若干个相加数\( \to \pm \infty \)时，平方和\( \to +\infty \)．因此，函数\( u \)必在有限区域内取得最小值．于是，将正数\( a \)分解为\( n \)个相等的相加数\( \frac{a}{n} \)时，其平方和\( \frac{a^2}{n} \)最小。

3685. 分解已知正数\( a \)为\( n \)个正的因数，使得它们的已知正乘幂的和为最小。

解 考虑函数 \( u = \sum_{i=1}^{n} x_i^{\alpha_i} \) （\( \alpha_i > 0 \)）在条件 \( \ln a = \sum_{i=1}^{n} \ln x_i \) （\( a > 0 \)，\( x_i > 0 \)）下的极值．设 \( F = u - \lambda \left( \sum_{i=1}^{n} \ln x_i - \ln a \right) \)．解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x_i} &= \alpha_i x_i^{\alpha_i - 1} - \frac{\lambda}{x_i} = 0 \quad (i = 1, 2, \ldots, n), \quad (1) \\
\sum_{i=1}^{n} \ln x_i &= \ln a, \quad (2)
\end{align*}
\]
由（1）得 $x_i = \left( \frac{\lambda}{\alpha_i} \right)^{1/\alpha_i}$. 代入（2），得

$$\ln \alpha + \sum_{i=1}^{n} \frac{\ln \alpha_i}{\alpha_i} = \ln \lambda \sum_{i=1}^{n} \frac{1}{\alpha_i}.$$ 

令 $\beta = \sum_{i=1}^{n} \frac{1}{\alpha_i}$，则有

$$\lambda = \alpha \prod_{i=1}^{n} \alpha_i^{1/\alpha_i} = \left( a^{\frac{1}{\alpha_i}} \prod_{i=1}^{n} \alpha_i^{1/\alpha_i} \right)^{\frac{1}{\beta}}.$$

$$x_i = \frac{1}{a_i} \left( \alpha \prod_{i=1}^{n} \alpha_i^{1/\alpha_i} \right)^{\frac{1}{n}} \left( \sum_{i=1}^{n} \frac{1}{\alpha_i} \right)^{1/\beta} \left( \alpha \prod_{i=1}^{n} \alpha_i^{1/\alpha_i} \right)^{1/\alpha_i}.$$

$$u = \sum_{i=1}^{n} \frac{\lambda}{\alpha_i} = \beta \lambda = \left( \sum_{i=1}^{n} \frac{1}{\alpha_i} \right)^{\frac{1}{\beta}} \left( \sum_{i=1}^{n} \frac{1}{\alpha_i} \right)^{1/\alpha_i}.$$

显然，函数 $u$ 在区域内部达到最小值，于是，所求得的 $u$ 即为最小值。

3686. 已知在平面上的 $n$ 个质点 $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_n(x_n, y_n)$，其质量分别为 $m_1, m_2, \ldots, m_n$.

$P(x, y)$ 点在怎样的位置，这一体系对于此点的转动惯量为最小？

设 $f(x, y) = \sum_{i=1}^{n} m_i [(x - x_i)^2 + (y - y_i)^2]$. 解方
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2 \sum_{i=1}^{n} m_i (x - x_i) = 0, \\
\frac{\partial f}{\partial y} &= 2 \sum_{i=1}^{n} m_i (y - y_i) = 0
\end{align*}
\]

得

\[x_0 = \frac{1}{M} \sum_{i=1}^{n} m_i x_i, \quad y_0 = \frac{1}{M} \sum_{i=1}^{n} m_i y_i,\]

其中 \(M = \sum_{i=1}^{n} m_i.\)

当 \(x \to \infty\) 或 \(y \to \infty\) 时，显然 \(f \to +\infty\). 因此，点

\(P(x_0, y_0)\) 即为所求。

3687. 已知容积为 \(V\) 的开顶长方浴盆，当其尺寸怎样时，有

最小的表面积？

解 设浴盆长、宽、高分别为 \(x, y, h\)，则考虑函数

\(S = 2 (x + y)h + xy\) 在条件 \(V = xyh (x \gg 0, y \gg 0,\)

\(h \gg 0\) 下的极值。

设 \(F(x, y, h) = S - \lambda(xy - V)\). 解方程组

\[
\begin{align*}
\frac{\partial F}{\partial x} &= y + 2h - \lambda xyh = 0, & (1) \\
\frac{\partial F}{\partial y} &= x + 2h - \lambda xyh = 0, & (2) \\
\frac{\partial F}{\partial h} &= 2 (x + y) - \lambda xy = 0, & (3) \\
xyh &= V.
\end{align*}
\]

(1)，(2)，(3) 可改写为
\[
\frac{1}{h} + \frac{2}{y} = \lambda = \frac{1}{h} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y},
\]

故有
\[
x_0 = y_0 = 2h_0 = \sqrt{2V}, \quad h_0 = \frac{1}{2} \sqrt{2V} = \frac{3}{\sqrt{4}}.
\]

从实际问题的常识可以断定，一定在某一处达到最小。因此，当长宽均为 \(\sqrt{2V}\)，高为 \(\frac{3}{\sqrt{4}}\) 时，浴盆的表面积最小，且最小表面积为 \(S = 3\sqrt{4V^2}\)。

从数学上来考虑，应讨论 \(x, y, h\) 趋于边界的情况。当 \(x, y, h\) 中有任一个趋于零，例如，\(h \to +0\)，则由
\[
V = xyh
\]
即可断定 \(xy \to +\infty\)。但是，\(S \to xy\)，故 \(S \to +\infty\)。当 \(x, y, h\) 中有任一个趋于 \(+\infty\) 时，一定引起至少数有一个趋于零。重复上面的讨论可知 \(S \to +\infty\)。因此，连续函数 \(S\) 必在区域内部取得最小值。

3688. 横断面为半圆形的圆柱形的张口浴盆，其表面积等于 \(S\)，当其尺寸怎样时，此盆有最大的容积？

解：设圆柱半径为 \(r\)，高为 \(h\)，则考虑函数
\[
V = \frac{1}{2} \pi r^2 h
\]
在条件 \(S = \pi(r^2 + rh)(r \geq 0, h \geq 0\) 下的极值。为简单起见，忽略系数 \(\frac{1}{2} \pi\)。设
\[
F = r^2 h - \lambda \left( r^2 + rh - \frac{S}{\pi} \right).
\]
解方程组
\[
\begin{aligned}
\frac{\partial F}{\partial r} &= 2rh - \lambda(2r + h) = 0, \\
\frac{\partial F}{\partial h} &= r^2 - \lambda r = 0, \\
r^2 + rh &= \frac{S}{\pi}
\end{aligned}
\]

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得
\[ r_0 = \sqrt{\frac{S}{3\pi}}, \quad h_0 = 2\sqrt{\frac{S}{3\pi}}. \]
从而有
\[ V_0 = \frac{1}{2} \pi r_0^2 h_0 = \sqrt{\frac{S^3}{27\pi^3}}. \]
由实际情况知，\( V_0 \) 一定达到最大体积。因此，当
\[ h_0 = 2r_0 = 2\sqrt{\frac{S}{3\pi}} \]
时，体积 \( V_0 = \sqrt{\frac{S^3}{27\pi^3}} \) 最大。
从数学角度看，由 \( r^2 + rh = \frac{S}{\pi} \) 知 \( r^2 \) 和 \( rh \) 恒有界。当 \( r \to +0 \) 或 \( h \to +0 \) 时必有 \( V \to 0 \)。当 \( h \to +\infty \) 时，
由 \( rh \) 有界可推出 \( r \to +0 \)。因而 \( V \to 0 \)（显然不可能 \( r \to +\infty \)）。于是，体积 \( V \) 必在区域内部达到最大值。

3589. 在球面 \( x^2 + y^2 + z^2 = 1 \) 上求出一点，这点到 \( n \) 个已知点 \( M_i(x_i, y_i, z_i) \)（\( i = 1, 2, \ldots, n \)）距离的平方和为最小。<br>
解 考虑函数
\[ u = \sum_{i=1}^{n} [(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2] \]
在条件 \( x^2 + y^2 + z^2 = 1 \) 下的极值。设 \( F(x, y, z) = u - \lambda (x^2 + y^2 + z^2 - 1) \)。<br>
解方程组<br>
\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2 \left[ \sum_{i=1}^{n} (x-x_i) - \lambda x \right] = 2 \left[ (n-\lambda) x - \sum_{i=1}^{n} x_i \right] = 0, \\
\frac{\partial F}{\partial y} &= 2 \left[ (n-\lambda) y - \sum_{i=1}^{n} y_i \right] = 0, \\
\frac{\partial F}{\partial z} &= 2 \left[ (n-\lambda) z - \sum_{i=1}^{n} z_i \right] = 0, \\
&\quad x^2 + y^2 + z^2 = 1.
\end{align*}
\]
由 (1), (2), (3) 得

\[ x = \frac{1}{n-\lambda} \sum_{i=1}^{n} x_i, \quad y = \frac{1}{n-\lambda} \sum_{i=1}^{n} y_i, \quad z = \frac{1}{n-\lambda} \sum_{i=1}^{n} z_i, \]

代入 (4), 得

\[(n-\lambda)^2 = \left( \sum_{i=1}^{n} x_i \right)^2 + \left( \sum_{i=1}^{n} y_i \right)^2 + \left( \sum_{i=1}^{n} z_i \right)^2 = N^2\]

\((N \geq 0)\). 于是，得

\[ x' = \frac{1}{N} \sum_{i=1}^{n} x_i, \quad y' = \frac{1}{N} \sum_{i=1}^{n} y_i, \quad z' = \frac{1}{N} \sum_{i=1}^{n} z_i \]

及

\[ x'' = -\frac{1}{N} \sum_{i=1}^{n} x_i, \quad y'' = -\frac{1}{N} \sum_{i=1}^{n} y_i, \quad z'' = -\frac{1}{N} \sum_{i=1}^{n} z_i. \]

从而，

\[ u(x', y', z') = \sum_{i=1}^{n} (x' - x_i)^2 + (y' - y_i)^2 + (z' - z_i)^2 \]

\[ = n(x'^2 + y'^2 + z'^2) - 2x' \sum_{i=1}^{n} x_i - 2y' \sum_{i=1}^{n} y_i \]

\[ - 2z' \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) \]

\[ = n - \frac{2}{N} \left[ \sum_{i=1}^{n} x_i \right]^2 + \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) \]

\[ + \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) \]

\[ = n - 2N + \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2). \]

同理可求得

\[ u(x'', y'', z'') = n + 2N + \sum_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2) \]

\[ \geq u(x', y', z'). \]
由于函数 $u$ 在闭球面 $x^2 + y^2 + z^2 = 1$ 上连续，于其取得最大值及最小值。于是，当 $x = x'$，$y = y'$，$z = z'$ 时，$u$ 最小（同时也证明了当 $x = x''$，$y = y''$，$z = z''$ 时，$u$ 最大）。

3690. 由直圆柱及以直圆锥作顶构成一个体，当已知体的全表面积等于 $Q$ 时，求它的尺寸大小，使得体的体积为最大。

解  设圆柱部分的底半径为 $R$，高为 $h$；圆锥部分的母线与底面的夹角为 $\alpha$，则有 $\pi R^2 + 2\pi R h + \frac{\pi R^2}{\cos \alpha} = Q$（常数）（$R > 0$，$h > 0$，$0 < \alpha < \frac{\pi}{2}$）。考虑函数 $V(a, h, R) = \pi R^2 h + \frac{1}{3} \pi R^8 \tan \alpha$ 在上述条件下的极值。设

$$F(a, h, R) = 3R^2 h + R^8 \tan \alpha - \lambda \left(R^2 + 2Rh + \frac{R^2}{\cos \alpha} - \frac{Q}{\pi}\right).$$

解方程组

$$\begin{align*}
\frac{\partial F}{\partial a} &= \frac{R^8}{\cos \alpha} - \frac{\lambda R^2 \sin \alpha}{\cos^2 \alpha} = 0, \\
\frac{\partial F}{\partial h} &= 3R^2 - 2R \lambda = 0, \\
\frac{\partial F}{\partial R} &= 6Rh + 3R^2 \tan \alpha - \left(2R + 2h + \frac{2R}{\cos \alpha}\right) \lambda = 0, (3) \\
R^2 + 2Rh + \frac{R^2}{\cos \alpha} &= \frac{Q}{\pi}. (4)
\end{align*}$$

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由 (2) 得 \( \lambda = \frac{3}{2} R \). 代入 (1)，得 \( \sin \alpha = \frac{2}{3} \). 由于 \( 0 \leq \alpha \leq \frac{\pi}{2} \)，故由 \( \sin \alpha = \frac{2}{3} \)，得 \( \cos \alpha = \frac{\sqrt{5}}{3} \)，\( \tan \alpha = \frac{2}{\sqrt{5}} \). 代入 (3)，得

\[
6Rh + \frac{6}{\sqrt{5}} R^2 = 3R^2 + 3Rh + \frac{9}{\sqrt{5}} R^2,
\]

即

\[
Rh = R^2 + \frac{R^2}{\sqrt{5}} \text{或 } h = \left( 1 + \frac{1}{\sqrt{5}} \right) R.
\]

代入 (4)，得

\[
R^2 + \left( 2 + \frac{2}{\sqrt{5}} \right) R^2 + \frac{3}{\sqrt{5}} R^2 = \frac{Q}{\pi}.
\]

于是，

\[
R = \frac{\sqrt{2} (\sqrt{5} - 1)}{4} \sqrt{\frac{Q}{\pi}}.
\]

相应地，有

\[
V_0 = \pi R^2 h + \frac{1}{3} \pi R^3 \tan \alpha = \left( 1 + \frac{1}{\sqrt{5}} + \frac{2}{3 \sqrt{5}} \right) \pi R^3
\]

\[
= \left( 1 + \frac{5}{3 \sqrt{5}} \right) \pi R^3. \quad R = \frac{3 + \sqrt{5}}{5} \pi, \quad \frac{3 - \sqrt{5}}{4} \frac{Q}{\pi}
\]

\[
= \frac{\sqrt{2} (\sqrt{5} - 1)}{12 \pi} \sqrt{\frac{Q^3}{\pi}}.
\]

现在讨论边界情况. 由 (4) 知 \( R^2 \)，\( Rh \) 及 \( \frac{R^2}{\cos \alpha} \)

均为正的有界量．

(i) 当 \( R \to +0 \) 时，由 \( Rh \) 及 \( \frac{R^2}{\cos \alpha} \) 有界可知
\[ V = \pi(Rh)R + \frac{\pi}{3}\left(\frac{R^2}{\cos\alpha}\right)\sin\alpha \cdot R \to 0. \]

(ii) 当 \( h \to +0 \) （所研究的体退化为圆锥）时，需要求圆锥全表面积 \( \pi R^2 + \frac{\pi R^2}{\cos\alpha} = Q \) （常数）时圆锥体积 \( V = \frac{1}{3}\pi R^3\tan\alpha \) 的最大值，用 \( l \) 表圆锥的斜高，即 \( l = \frac{R}{\cos\alpha} \)，

\[ R \tan\alpha = \sqrt{\frac{R^2}{\cos^2\alpha} - R^2} = \sqrt{l^2 - R^2}. \]

于是，\( l = \frac{Q - \pi R^2}{\pi R} \)， \( V = \frac{1}{3}\pi R^2\sqrt{l^2 - R^2} \)，故

\[ V^2 = \frac{1}{9}QR^2(Q - 2\pi R^2)(0 < R < \sqrt{\frac{Q}{\pi}}). \]

由此易知 \( V^2 \) （从而 \( V \)）当 \( R^2 = \frac{Q}{4\pi} \) （即 \( R = \frac{1}{2} \sqrt{\frac{Q}{\pi}} \）时达最大值，并且最大体积 \( V_1 = \frac{1}{6}\sqrt{6} \sqrt{\frac{Q^3}{\pi}} \).

不难验证 \( V_1 < V_0 \).

(iii) 当 \( h \to +\infty \) 时，由 \( Rh \) 有界知 \( R \to +0 \).

由(i)知 \( V \to 0 \).

(iv) 当 \( \alpha \to \frac{\pi}{2} - 0 \) 时，由 \( \frac{R^2}{\cos\alpha} \) 有界可知 \( R \to +0 \), 由(i)知 \( V \to 0 \).

(v) 当 \( \alpha \to +0 \)（所研究的体退化为圆柱）时，可以求得达到最大体积的尺寸为 \( h = 2R \) 及 \( Q = \sqrt{54\pi V^2} \) （参看1563题），即

\[ V_2 = \sqrt{\frac{Q^3}{54\pi}} = \frac{\sqrt{6}}{18} \sqrt{\frac{Q^3}{\pi}}. \]

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不难证明 \( V_2 < V_0 \).

综上所述，我们得到当
\[
R = \frac{\sqrt{2}}{4} \left( \frac{\sqrt{3}}{4} - 1 \right) \sqrt{\frac{Q}{\pi}},
\]
\( \alpha = \arcsin \frac{2}{3} \) 时，所研究的体积 \( V \) 达到最大值
\[
V_0 = \frac{\sqrt{2} \left( \frac{\sqrt{3}}{4} - 1 \right)}{12} \sqrt{\frac{Q^3}{\pi}}.
\]

3691. 一个体，其体积等于 \( V \), 形为直角平行六面体，上底及下底为正的四角锥，当角锥的侧面对它们的底成怎样的倾角，体的全表面积为最小？

解 设长方体两底（正方形）边长为 \( a \)，高为 \( h \)，棱锥侧面与底面的夹角为 \( \alpha \)，则
\[
V = a^2 h + \frac{1}{3} a^3 \tan \alpha.
\]
考虑函数
\[
S = 4ah + \frac{2a^2}{\cos \alpha}
\]
在上述条件下的极值。设
\[
F = S - \lambda \left( a^2 h + \frac{1}{3} a^3 \tan \alpha - V \right).
\]
解方程组

\[
\begin{align*}
\frac{\partial F}{\partial a} &= 4h + \frac{4a}{\cos \alpha} - 2\lambda ah - \lambda a^2 \tan \alpha = 0, \quad (1) \\
\frac{\partial F}{\partial h} &= 4a - \lambda a^2 = 0, \quad (2) \\
\frac{\partial F}{\partial \alpha} &= \frac{2a^2 \sin \alpha}{\cos^2 \alpha} - \frac{\lambda a^3}{3 \cos^2 \alpha} = 0, \quad (3) \\
\lambda a^2 h + \frac{1}{3} \lambda a^3 \tan \alpha &= V. \quad (4)
\end{align*}
\]

由 (2), (3) 可得 \( \alpha = \arcsin \frac{2}{3} \)，同3690题进一步可求出 \( a \) 和 \( h \)。

类似3687题的讨论，当 \( a \to + \infty \), \( a \to + \infty \), \( h \to + \infty \), \( \alpha \to \frac{\pi}{2} - 0 \) 等情况均能证明 \( S \to + \infty \)。对于边
界为 $a=0$ 及 $h=0$ 这两种退化情况，类似 3690 题，
可证明此时的全表面积比 $a=\arcsin\frac{2}{3}$ 时的全表面积
为大。于是，当 $a=\arcsin\frac{2}{3}$ 时，体的全表面积最小。

3692. 已知矩形的周长为 $2p$，将它绕其一边旋转而构成
一体积，求所得体积为最大的那个矩形。
解  设 矩 形 的 边 长 为 $x$ 及 $y$，则考虑函数 $V=\pi y^2 x$
在条件 $x+y=p$ 下的极值。设 $F=V-\lambda(x+y-p)$。
解方程组

$$
\begin{align*}
\frac{\partial F}{\partial x} &= \pi y^2 - \lambda = 0, \\
\frac{\partial F}{\partial y} &= 2\pi xy - \lambda = 0, \\
\lambda &= x + y = p
\end{align*}
$$

得 $x = \frac{p}{3}$, $y = \frac{2p}{3}$。

由于在边界上，一边为零，一边为 $p$，推出 $V=0$。
于是，当矩形的两边分别为 $\frac{p}{3}$ 及 $\frac{2p}{3}$ 时，旋转体的体积
最大。

3693. 已知三角形的周长为 $2p$，
求出这样的三角形，当它
绕着自己的一边旋转所构
成的体积最大。
解  如图 6.43 所示，以 $AC$
为轴旋转，取参数：高 $h$ 及
二角 $\alpha, \beta$。考虑函数

![图 6.43]
\[ V = \frac{1}{3} \pi h^3 (\tan \alpha + \tan \beta) \]

在条件 \[ \frac{h}{\cos \alpha} + \frac{h}{\cos \beta} + h (\tan \alpha + \tan \beta) = 2p \]

下的极值。为计算简单起见，略去常数 \( \frac{1}{3} \pi \)。设 \( F = h^3 (\tan \alpha + \tan \beta) - \lambda \left( \frac{h}{\cos \alpha} + \frac{h}{\cos \beta} + h \tan \alpha + h \tan \beta - 2p \right) \)。

解方程组

\[
\begin{align*}
\frac{\partial F}{\partial h} &= 3h^2 (\tan \alpha + \tan \beta) - \lambda \left( \frac{1}{\cos \alpha} + \frac{1}{\cos \beta} + \tan \alpha + \tan \beta \right) = 0, \quad (1) \\
\frac{\partial F}{\partial \alpha} &= \frac{h^3}{\cos^2 \alpha} - \lambda h \left( \frac{\sin \alpha}{\cos^2 \alpha} + \frac{1}{\cos^2 \alpha} \right) = 0, \quad (2) \\
\frac{\partial F}{\partial \beta} &= \frac{h^3}{\cos^2 \beta} - \lambda h \left( \frac{\sin \beta}{\cos^2 \beta} + \frac{1}{\cos^2 \beta} \right) = 0, \quad (3) \\
h \left( \frac{1}{\cos \alpha} + \frac{1}{\cos \beta} + \tan \alpha + \tan \beta \right) &= 2p. \quad (4)
\end{align*}
\]

由 (2) 及 (3) 得 \( \alpha = \beta \) 及 \( \lambda = \frac{h^2}{1 + \sin \alpha} = \frac{h^2}{1 + \sin \beta} \)。

代入 (1) 式，得 \( \sin \alpha = \sin \beta = \frac{1}{3} \)。于是，\( h \tan \alpha = \frac{h}{3 \cos \alpha} \)，代入 (4) 式，即得 \( \frac{h}{\cos \alpha} = \frac{3}{4} p \)。从而，得三边分别为

\[ AB = BC = \frac{3}{4} p, \quad AC = 2 h \tan \alpha = \frac{p}{2}. \]

讨论边界情况。当 \( h \to 0 \) 或 \( h \to p \) 时，显然有
$V \to 0$. 对于三角 $\alpha$ 及 $\beta$ 必有大小限制：$0 \leq \alpha \leq \frac{\pi}{2}$，
$-\alpha \leq \beta \leq \alpha$ (注意 $\alpha, \beta$ 的方向规定不同)，当 $\alpha \to +0$
或 $\alpha \to \frac{\pi}{2} - 0$ 或 $\beta \to -\alpha$ 时，同样均有 $V \to 0$。于是，
当三角形的三边长分别为 $\frac{b}{2}, \frac{3b}{4}$ 及 $\frac{3b}{4}$，并绕长为 $\frac{b}{2}$
的边旋转时，所得的体积最大。

3694. 在半径为 $R$ 的半球内嵌入有最大体积的直角平行六面体。

解 不妨设此长方体的一个底面与半球所在的底面重合，另外四个顶点在半球球面上，且半球面在直角坐标系下的方程为

$$x^2 + y^2 + z^2 = R^2, \quad z \geq 0.$$ 

又设长方体的长、宽、高分别为 $2x$, 2$y$ 及 $z$ ($x \geq 0$, $y \geq 0$, $z \geq 0$)。考虑函数 $V = 4xyz$ 在上述条件下的
极值。设 $F = xyz - \lambda(x^2 + y^2 + z^2 - R^2)$。

解方程组

$$\begin{cases}
\frac{\partial F}{\partial x} = yz - 2\lambda x = 0, \\
\frac{\partial F}{\partial y} = xz - 2\lambda y = 0, \\
\frac{\partial F}{\partial z} = xy - 2\lambda z = 0,
\end{cases}$$

$$x^2 + y^2 + z^2 = R^2$$

可得 $x = y = z = \frac{R}{\sqrt{3}}$. 
由于在边界上（即 $x \to +0$ 或 $y \to +0$ 或 $z \to +0$ 时）显然 $V \to 0$，故当直角平行六面体的长、宽、高为 $\frac{2R}{\sqrt{3}}$，$\frac{2R}{\sqrt{3}}$ 及 $\frac{R}{\sqrt{3}}$ 时，其体积最大。

3895. 在已知的直圆锥内嵌入有最大体积的直角平行六面体。

解 不妨设直圆锥的底面半径为 $R$，高为 $H$，且长方体的一个面与直圆锥的底面重合，两个边长为 $2x$ 和 $2y$，四个顶点在直圆锥面上，高为 $z$。过直圆锥的高和长方体底面的对角线作一截面，如图6.44所示，则 $CD = H$，$EK = FG = z$，$AD = R$，$DE = \sqrt{x^2 + y^2}$，$(H - z) \cdot R - H \cdot \sqrt{x^2 + y^2} (R, H$ 为常数)。考虑函数 $V = 4xyz$ 在上述条件下的极值 ($x \to 0$, $y \to 0$, $z \to 0$)。为计算简单计，略去常数4。设

$$F = xyz - \lambda \left[ H \sqrt{x^2 + y^2} - (H - z)R \right].$$

解方程组

$$\begin{align*}
\frac{\partial F}{\partial x} &= yz - \frac{\lambda H x}{\sqrt{x^2 + y^2}} = 0, \\
\frac{\partial F}{\partial y} &= xz - \frac{\lambda H y}{\sqrt{x^2 + y^2}} = 0, \\
\frac{\partial F}{\partial z} &= xy - \lambda R = 0, \\
(H - z) R - H \sqrt{x^2 + y^2}. & \quad (4)
\end{align*}$$

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由 (1), (2) 得 \(x = y\), 代入 (3), 得 \(x = y = \sqrt{\lambda R}\)。
又由 (1) 可得 \(z = \frac{\lambda H}{\sqrt{2\lambda R}}\). 将 \(x, y, z\) 代入 (4) 得

\[H = \frac{\lambda H}{\sqrt{2\lambda R}} = \frac{H}{R} \sqrt{2\lambda R},\]

解之得 \(\lambda = \frac{2}{9} R\), 从而有

\[x = y = \sqrt{\frac{2}{3}} R, z = \sqrt{\frac{1}{3}} H, \quad V = \frac{\sqrt{2}}{36} R^2 H.\]

显然，在所论区域的边界上（即 \(x \to +0\) 或 \(y \to +0\) 或 \(z \to +0\) 时），有 \(V \to 0\), 故当直角平行六面体的高等于 \(\frac{1}{3}\) 圆锥的高时, 其体积最大。

3696. 在椭球

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\]

内嵌入有最大体积的直角平行六面体。
解 此直角平行六面体的对称中心为原点, 设其中一个顶点为 \((x, y, z)\), 则按题意, 我们应考虑函数 \(V = 8xyz\) 在条件 \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\) (\(x > 0, \ y > 0, \ z > 0\)) 下的极值. 为计算简便计, 略去常数 \(a\), 设 \(F = xyz - \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\). 解方程组

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\[
\begin{align*}
\frac{\partial F}{\partial x} &= yz - 2\lambda \cdot \frac{x}{a^2} = 0, \\
\frac{\partial F}{\partial y} &= xz - 2\lambda \cdot \frac{y}{b^2} = 0, \\
\frac{\partial F}{\partial z} &= xy - 2\lambda \cdot \frac{z}{c^2} = 0, \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1
\end{align*}
\]

得 \( x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}} \)，这时 \( V = \frac{8}{3 \sqrt{3}} \cdot abc \neq 0 \)。

现在讨论边界情况。当 \( x \to a - 0, y \to b - 0, z \to c - 0 \) 中有任一个成立时，则另两个变量必皆趋于零；又若 \( x, y, z \) 中有一个趋于零时，则体积 \( V \) 趋于零。总之，在边界上，恒有 \( V \to 0 \)。于是，具有最大体积的直角平行六面体的长、宽、高分别为 \( \frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}, \frac{2c}{\sqrt{3}} \)。

3697. 直圆锥的母线 \( l \) 与底平面成为角 \( \alpha \)。试求此直圆锥中

嵌入其最大全表面积的直角平行六面体。

解 设圆锥的底半径为 \( R \)，高为 \( H \)，则有 \( R = l \cos \alpha \)，

\( H = l \sin \alpha \)， \( \frac{H}{R} = \tan \alpha \)。内接长方体的放置方法与

3695题相同。设底面的两边分别为 \( 2d \cos \theta \) 和 \( 2d \sin \theta \)，

高为 \( h \)，则 \( 0 < d < R, \ 0 < h < H, \ 0 < \theta < \frac{\pi}{2} \)，且 \( h, \)}
由条件 $\frac{H-h}{H} = \frac{d}{R}$ 约束，此条件可改写为

$$d \cdot \tan \alpha + h = H = l \sin \alpha.$$ 

所求的表面积为

$$S = 4(d^2 \sin 2\theta + dh \sin \theta + dh \cos \theta).$$

(i) 固定 $d$ 和 $h$，考虑 $S = S(\theta)$ 的变化情况。由一元函数极值法，不难断定，仅有 $S'(\frac{\pi}{4}) = 0$。

$S(\theta)$ 在 $\frac{\pi}{4}$ 处达到最大值 $S = 4(d^2 + \sqrt{2} dh)$，即底面为正方形时，$S$ 才取得最大值。因此，原问题可化为在条件 $d \cdot \tan \alpha + h = l \sin \alpha$ ($d \geq 0$，$h \geq 0$) 下，求数 $S = 4(d^2 + \sqrt{2} dh)$ 的极值。

(ii) 此问题的边界值：当 $d \to +0$ （此时 $h \to H - 0$）时，显然 $S \to 0$；而当 $h \to +0$ （这时 $d \to R - 0$）时，$S \to 4R^2$。在后一种情况下，表面积退化为上、下两个正方形面积之和。

(iii) 在区域内部，设

$$F = 4(d^2 + \sqrt{2} dh) - \lambda(d \cdot \tan \alpha + h - l \sin \alpha).$$

解方程组

$$ \begin{cases} \frac{\partial F}{\partial d} = 3d + 4\sqrt{2} h - \lambda \tan \alpha = 0, \\ \frac{\partial F}{\partial h} = 4\sqrt{2} d - \lambda = 0, \\ d \cdot \tan \alpha + h = l \sin \alpha. \end{cases} \tag{1}$$

由 (2) 得 $\lambda = 4\sqrt{2} d$，代入 (1)，得

$$h = (\tan \alpha - \sqrt{2})d. \tag{4}$$
由 $h \Rightarrow 0$ 及 $d \Rightarrow 0$ 知，当 $\tan \alpha \leq \sqrt{2}$ 时，方程组在所研究的区域内无解。此时，$S$ 的最大值必在边界上达到，即在 $h \rightarrow 0$ 时达到 $4R^2$。当 $\tan \alpha \geq \sqrt{2}$ 时，将 (4) 式代入 (3) 式，可得

$$
d = \frac{l \sin \alpha}{2 \tan \alpha - \sqrt{2}}, \quad h = l \sin \alpha \cdot \frac{\tan \alpha - \sqrt{2}}{2 \tan \alpha - \sqrt{2}}.
$$

此时

$$
S = 4 \left( d^2 + \sqrt{2} \frac{dh}{dx} \right) = \frac{2l^2 \sin^2 \alpha}{\sqrt{2} \tan \alpha - 1} = \frac{2R^2 \tan^2 \alpha}{\sqrt{2} \tan \alpha - 1}.
$$

由于 $(\tan \alpha - \sqrt{2})^2 = \tan^2 \alpha - 2(\sqrt{2} \tan \alpha - 1) \geq 0$，故

$$
\frac{\tan^2 \alpha}{\sqrt{2} \tan \alpha - 1} \geq 2.
$$

从而，$S \geq 4R^2$，即在该点的值大于边界上的值。因此，它为最大值。于是，当 $\tan \alpha \geq \sqrt{2}$，

长方体底面为正方形，边长为 $2d \sin \frac{\pi}{4} = \frac{l \sin \alpha}{\sqrt{2} \tan \alpha - 1}$，

高 $h = l \sin \alpha \cdot \frac{\tan \alpha - \sqrt{2}}{2 \tan \alpha - \sqrt{2}}$ 时，全表面积为最大。

3638. 在椭圆抛物面 $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$，$z = c$ 的一段中嵌入有最大体积的直角平行六面体。

解：设长方体的长、宽、高为 $2x$，$2y$ 及 $h = c - z$，则

按题设考虑函数 $V = 4xyz = 4xy(c-z)$ 在条件 $x^2 + \frac{y^2}{b^2} = \frac{z}{c}$ ($x > 0$，$y > 0$，$0 \leq z < c$) 下的极值。为计算简单起见，作 $F$ 时略去常数 $4$，令 $F = xy(c-z) - \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} \right)$。

解方程组

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\[
\begin{align*}
\frac{\partial F}{\partial x} &= y(c-z) - 2\lambda \cdot \frac{x}{a^2} = 0, \quad (1) \\
\frac{\partial F}{\partial y} &= x(c-z) - 2\lambda \cdot \frac{y}{b^2} = 0, \quad (2) \\
\frac{\partial F}{\partial z} &= -xy + \frac{\lambda}{c} = 0, \quad (3) \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{z}{c}. \quad (4)
\end{align*}
\]

将(1)、(2)、(3)三式分别乘以x、y、(c-z)，
比较即得，\(\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{c-z}{2c}\)。代入(4)式，可得

\[
x = \frac{a}{2}, \quad y = \frac{b}{2}, \quad z = \frac{c}{2}, \quad h = c - z = \frac{c}{2}.
\]

由于边界上V趋于零，故长方体的最大值必在区域中达到。于是，当平行六面体的尺寸为a、b及\(\frac{c}{2}\)时，其体积最大。

**3699.** 求点\(M_0(x_0, y_0, z_0)\)至平面\(Ax+By+Cz+D=0\)的最近距离。
解  按题设，我们应求函数

\[
r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2
\]

在条件\(Ax+By+Cz+D=0\)下的极值。设

\[
F(x, y, z) = r^2 + \lambda(Ax+By+Cz+D).
\]

解方程组
\[ \begin{align*}
\frac{\partial F}{\partial x} &= 2(x-x_0) + \lambda A = 0, \\
\frac{\partial F}{\partial y} &= 2(y-y_0) + \lambda B = 0, \\
\frac{\partial F}{\partial z} &= 2(z-z_0) + \lambda C = 0, \\
Ax + By + Cz + D &= 0.
\end{align*} \]

由 (1), (2), (3) 可得

\[x = x_0 - \frac{1}{2} \lambda A, \quad y = y_0 - \frac{1}{2} \lambda B, \quad z = z_0 - \frac{1}{2} \lambda C. \quad (5)\]

代入 (4)，得

\[\lambda = \frac{2(Ax_0 + By_0 + Cz_0 + D)}{A^2 + B^2 + C^2}. \quad (6)\]

将 (5), (6) 代入 \( r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \) 中，得

\[r = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \]

当 \( x, y, z \) 中有任一个趋于无穷时，\( r \) 趋于无穷。因此，在区域内 \( r \) 必取最小值。

于是，点 \( M_0(x_0, y_0, z_0) \) 至平面 \( Ax + By + Cz + D = 0 \) 的最短距离为

\[r = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \]

3700. 求空间二直线

\[
\frac{x-x_1}{m_1} = \frac{y-y_1}{n_1} = \frac{z-z_1}{p_1}
\]

和

\[
\frac{x-x_2}{m_2} = \frac{y-y_2}{n_2} = \frac{z-z_2}{p_2}
\]

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之间的最短距离。
解 显然，当两直线不平行时，直线上一点趋于无穷远处时，与另一直线上各点的距离，都趋于无穷。因此，不平行两直线的最短距离必在有限处达到。

为了书写简洁，我们采用向量的表达形式。用
\[ \vec{r}_1(t) = \overrightarrow{l}_1 t + \overrightarrow{r}_{10} \]
表示直线
\[ \frac{x-x_1}{m_1} = \frac{y-y_1}{n_1} = \frac{z-z_1}{p_1}, \quad (1) \]
\[ \vec{r}_2(s) = \overrightarrow{l}_2 s + \overrightarrow{r}_{20} \]
表示直线
\[ \frac{x-x_2}{m_2} = \frac{y-y_2}{n_2} = \frac{z-z_2}{p_2}, \quad (2) \]
其中 \( t, s \) 为参数，\( \overrightarrow{l}_1 = \{ m_1, n_1, p_1 \}, \overrightarrow{l}_2 = \{ m_2, n_2, p_2 \}, \)
\( \overrightarrow{r}_{10} = \{ x_1, y_1, z_1 \}, \overrightarrow{r}_{20} = \{ x_2, y_2, z_2 \}. \)
又记
\[ \overrightarrow{r}_0 = \overrightarrow{r}_{10} - \overrightarrow{r}_{20} = \{ x_1 - x_2, y_1 - y_2, z_1 - z_2 \}. \]
始端在直线 (2) 上，终端在直线(1)上的向量为：
\[ \vec{u}(t,s) = (\overrightarrow{l}_1 t + \overrightarrow{r}_{10}) - (\overrightarrow{l}_2 s + \overrightarrow{r}_{20}) \]
\[ = \overrightarrow{l}_1 t - \overrightarrow{l}_2 s + \overrightarrow{r}_0. \quad (3) \]

本题即求 \( |\vec{u}(t,s)| \) 的最小值，它必在有限的 \( t, s \) 上取得。令
\[ w = |\vec{u}(t,s)|^2 = |\overrightarrow{l}_1 t - \overrightarrow{l}_2 s + \overrightarrow{r}_0|^2 \]
\[ = l_1^2 t^2 + l_2^2 s^2 + r_0^2 - 2(\overrightarrow{l}_1 \cdot \overrightarrow{l}_2) st + 2(\overrightarrow{l}_1 \cdot \overrightarrow{r}_0) t \]
\[ - 2(\overrightarrow{l}_2 \cdot \overrightarrow{r}_0) s, \]
其中 \( l_1^2 = \overrightarrow{l}_1 \cdot \overrightarrow{l}_1, l_2^2 = \overrightarrow{l}_2 \cdot \overrightarrow{l}_2, r_0 = \overrightarrow{r}_0 \cdot \overrightarrow{r}_0. \)
\( w \) 取得极值的必要条件为
\[
\frac{\partial w}{\partial t} = 2(l_1^2 t - (\vec{l}_1 \cdot \vec{l}_2) s + (\vec{l}_1 \cdot \vec{r}_0)) = 0,
\]
\[
\frac{\partial w}{\partial s} = 2(l_2^2 s - (\vec{l}_1 \cdot \vec{l}_2) t - (\vec{l}_2 \cdot \vec{r}_0)) = 0.
\]
由此可解得唯一的静止点 \((t_0, s_0)\):
\[
t_0 = \frac{l_2^2 (\vec{l}_1 \cdot \vec{r}_0) - (\vec{l}_1 \cdot \vec{l}_2)(\vec{l}_2 \cdot \vec{r}_0)}{l_1^2 l_2^2 - (\vec{l}_1 \cdot \vec{l}_2)^2},
\]
\[
s_0 = \frac{l_1^2 (\vec{l}_2 \cdot \vec{r}_0) - (\vec{l}_1 \cdot \vec{l}_2)(\vec{l}_1 \cdot \vec{r}_0)}{l_1^2 l_2^2 - (\vec{l}_1 \cdot \vec{l}_2)^2}.
\]
于是 \( \vec{u}(t_0, s_0) \) 即为所求的最短距离，下面计算 \( |\vec{u}(t_0, s_0)| \)。
令 \( A = \sqrt{l_1^2 l_2^2 - (\vec{l}_1 \cdot \vec{l}_2)^2} \)，显然有
\[
A^2 = |\vec{l}_1|^2 |\vec{l}_2|^2 \left[ |\vec{l}_1| \cdot |\vec{l}_2| \cos(\vec{l}_1, \vec{l}_2) \right]^2
\]
\[
= |\vec{l}_1|^2 |\vec{l}_2|^2 \sin^2(\vec{l}_1, \vec{l}_2) = |\vec{l}_1 \times \vec{l}_2|^2,
\]
即
\[
A = |\vec{l}_1 \times \vec{l}_2|.
\]
将 \( t_0 \) 及 \( s_0 \) 代入 (3) 式，得
\[
\vec{u}(t_0, s_0) = -\frac{1}{A^2} (\vec{l}_1 \cdot \vec{r}_0) [l_2^2 \vec{l}_1 - (\vec{l}_1 \cdot \vec{l}_2) \vec{l}_2]
\]
\[
- \frac{1}{A^2} (\vec{l}_2 \cdot \vec{r}_0) [l_1^2 \vec{l}_2 - (\vec{l}_1 \cdot \vec{l}_2) \vec{l}_1] + \vec{r}_0.
\]
通过计算，不难得出
\[
\vec{u}(t_0, s_0) \cdot \vec{l}_1 = -\frac{1}{A^2} (\vec{l}_1 \cdot \vec{r}_0) [l_2^2 \vec{l}_1^2 - (\vec{l}_1 \cdot \vec{l}_2)^2] - \frac{1}{A^2}
\]
\[
(\vec{l}_2 \cdot \vec{r}_0) [l_1^2 (\vec{l}_1 \cdot \vec{l}_2) - (\vec{l}_1 \cdot \vec{l}_2) l_1^2] + (\vec{r}_0 \cdot \vec{l}_1) = 0,
\]
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\[ \vec{u}(t_0, s_0) \cdot \vec{l}_2 = 0. \]

因此，得知

\[ \vec{u}(t_0, s_0) \parallel \vec{l}_1 \times \vec{l}_2. \]

令 \( n_0 = \frac{\vec{l}_1 \times \vec{l}_2}{\mathcal{A}} \) ，则 \( |n_0| = 1, \)

\[ |\vec{u}(t_0, s_0)| = |\vec{u}(t_0, s_0) \cdot n_0| = \frac{|r_2 \cdot (\vec{l}_1 \times \vec{l}_2)|}{\mathcal{A}} \]

\[ = \pm \frac{1}{\mathcal{A}} \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \end{vmatrix}, \]

其中

\[ \mathcal{A} = \sqrt{\left( \frac{m_1}{n_2} \right)^2 + \left( \frac{n_1}{m_2} \right)^2 + \left( \frac{p_1}{n_2} \right)^2 + \left( \frac{p_1}{m_2} \right)^2}. \]

且正负号的选取保证所得结果为正值。

2701. 求抛物线 \( y = x^2 \) 和直线 \( x - y - 2 = 0 \) 之间的最短距离。

解：设 \((x_1, y_1)\) 为抛物线 \( y = x^2 \) 上任一点，\((x_2, y_2)\) 为直线 \( x - y - 2 = 0 \) 上的任一点。按题意，我们应求函数

\[ r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \]

在条件 \( y_1 - x_1^2 = 0 \) 及 \( x_2 - y_2 - 2 = 0 \) 下的极值。显然，由几何知，当两点 \((x_1, y_1)\) 和 \((x_2, y_2)\) 至少有一向无穷时，\( r \) 也必趋于无穷大。故 \( r \) 的最小值必在有限处达到。

设 \( F(x_1, x_2, y_1, y_2) = r^2 + \lambda_1(y_1 - x_1^2) + \lambda_2(x_2 - y_2 \ldots \)
解方程组
\[
\begin{align*}
\frac{\partial F}{\partial x_1} &= -2(x_2 - x_1) - 2\lambda_1 x_1 = 0 , \\
\frac{\partial F}{\partial x_2} &= 2(x_2 - x_1) + \lambda_2 = 0 , \\
\frac{\partial F}{\partial y_1} &= -2(y_2 - y_1) + \lambda_1 = 0 , \\
\frac{\partial F}{\partial y_2} &= 2(y_2 - y_1) - \lambda_2 = 0 , \\
y_1 &= x_1^2 , \\
x_2 - y_2 - 2 &= 0 .
\end{align*}
\]

得唯一的一组解 $x_1 = \frac{1}{2}, y_1 = \frac{1}{4}, x_2 = \frac{11}{8}, y_2 = \frac{5}{8}$.

于是，所求的最短距离为
\[
r_0 = \sqrt{\left(\frac{11}{8} - \frac{1}{2}\right)^2 + \left(\frac{5}{8} - \frac{1}{4}\right)^2} = \frac{7}{8}\sqrt{2} .
\]

3702. 求有心二次曲线
\[Ax^2 + 2Bxy + Cy^2 = 1\]
的半轴。

解 设 $(x_0, y_0)$ 为二次曲线 $Ax^2 + 2Bxy + Cy^2 = 1$ 上的点，则 $(-x_0, -y_0)$ 也为该曲线上的点。因此，原点 $(0, 0)$ 即为曲线的中心。按题意，应求函数 $u = x^2 + y^2$ 在条件 $Ax^2 + 2Bxy + Cy^2 = 1$ 下的极值。设
\[F = x^2 + y^2 - \lambda(Ax^2 + 2Bxy + Cy^2 - 1) .\]
解方程组
\[
\begin{align*}
-\frac{1}{2} \frac{\partial F}{\partial x} &= (\lambda A - 1)x + \lambda B y = 0,
-\frac{1}{2} \frac{\partial F}{\partial y} &= \lambda B x + (\lambda C - 1)y = 0,
A x^2 + 2B x y + C y^2 &= 1.
\end{align*}
\]

要方程组有非零解，\(\lambda\) 必须满足二次方程
\[
\begin{vmatrix}
\lambda A - 1 & \lambda B \\
\lambda B & \lambda C - 1
\end{vmatrix} = 0. \quad (1)
\]

由题设知二次曲线为有心的，因此 \(AC^2 - B^2 \neq 0\)。

由方程 (1) 可求得两根 \(\lambda_1\) 和 \(\lambda_2\) (\(\lambda_1 \geqslant \lambda_2\))。将 \(\lambda\) 的值代入方程组，求得对应于 \(\lambda_1\) 的解 \((x_1, y_1)\) 及对应于 \(\lambda_2\) 的解 \((x_2, y_2)\)。相应地，有

\[
u(x_1, y_1) = x_1^2 + y_1^2 = x_1[\lambda_1(Ax_1 + By_1)]
+ y_1[\lambda_1(Bx_1 + Cy_1)]
= \lambda_1(Ax_1^2 + 2Bx_1 y_1 + Cy_1^2) = \lambda_1,
\]

同理 \(u(x_2, y_2) = x_2^2 + y_2^2 = \lambda_2\)。

(1) 当 \(AC - B^2 > 0\) 且 \(A + C > 0\) (或 \(A > 0\)) 时，
由 (1) 解得
\[
\lambda_i = \frac{(A + C) \pm \sqrt{(A + C)^2 - 4(AC - B^2)}}{2(AC - B^2)} > 0,
\]

即有 \(\lambda_1 \geqslant \lambda_2 > 0\)。显然 \(u\) 的最大值及最小值必在区域内达到。因此，\(\lambda_1\) 及 \(\lambda_2\) 分别为 \(u\) 的最大值及最小值。此时，所对应的曲线为椭圆，长、短半轴的平方分别为 \(\lambda_1\) 及 \(\lambda_2\)。当 \(\lambda_1 = \lambda_2\) \((A = C, B = 0)\) 时为圆。
当 $AC < 0$ （或 $A < 0$）时，两根 $\lambda_i$ 均为负，
相应曲线无轨迹。

(ii) 当 $AC - B^2 < 0$ 时，$\lambda_1 > 0, \lambda_2 < 0$，此时只有一个极值 $\lambda_1$。对应的曲面为双曲面，$\lambda_1$ 为实半轴的
平方（$\lambda_2$ 表面上无意义，但实质上为虚半轴的平方），
其中特别是 $B = 0$ 时，曲线退化为一对相交直线。

3703．求有心二次曲面

$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Ey_2 + 2Fxz = 1$

的半轴。

解 同上题可知，曲面的中心为 $(0, 0, 0)$，按题意，
达到曲面半轴的点 $(x, y, z)$ 一定是函数 $u(x, y, z)
= x^2 + y^2 + z^2$ 在条件

$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Ey_2 + 2Fxz = 1$

下的静止点（但不一定是极值点。例如，椭球面的中
间轴所在的点）。设

$F = \frac{u}{2} - \lambda(Ax^2 + By^2 + Cz^2 + 2Dxy + 2Ey_2 + 2Fxz)
- 1$。

解方程组

\[
\begin{align*}
-\frac{1}{2} \frac{\partial F}{\partial x} &= (\lambda A - 1) x + \lambda Dy + \lambda Fz = 0, \\
-\frac{1}{2} \frac{\partial F}{\partial y} &= \lambda Dx + (\lambda B - 1) y + \lambda Ez = 0, \\
-\frac{1}{2} \frac{\partial F}{\partial z} &= \lambdaFx + \lambda Ey + (\lambda C - 1)z = 0, \\
Ax^2 + By^2 + Cz^2 + 2Dxy + 2Ey_2 + 2Fxz &= 1.
\end{align*}
\]
上述方程组要有非零解，λ必须满足三次方程
\[
\begin{vmatrix}
\lambda A - 1 & \lambda D & \lambda F \\
\lambda D & \lambda B - 1 & \lambda E \\
\lambda F & \lambda E & \lambda C - 1
\end{vmatrix} = 0.
\]
设三根为λ₁ ≥ λ₂ ≥ λ₃。对应于此三根可求出满足方程组的静止点。与3702题相同，可证明在这些静止点处
\(u(x, y, z)\)的值恰为 \(λ_i (i = 1, 2, 3)\)，即 \(λ_i\) 为曲面半轴的平方（严格地说，当 \(λ_i < 0\) 时不能认为它是半轴的平方）。

与二次曲面的情况类似，根据 \(λ_i\) 的正负可讨论曲面半轴的虚、实等问题，这对熟悉二次曲面分类的读者无实质性的困难，因此省略掉这些烦琐的讨论。

3704. 求用平面
\[Ax + By + Cz = 0\]
与圆柱
\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\]
相交所成椭圆的面积。
解 我们只要确定所得椭圆的长短半轴 \(a\) 及 \(b\)，即可按公式 \(S = \pi ab\) 求得椭圆的面积。

注意到原点 (0, 0, 0) 在原椭圆柱面的中心轴上，且截平面 \(Ax + By + Cz = 0\) 又通过它。因此，原点是截线椭圆的中心，从而长短半轴 \(a\) 及 \(b\) 的平方 \(a^2\) 及 \(b^2\)，分别为函数
\[u = x^2 + y^2 + z^2\]
在条件 \(Ax + By + Cz = 0\) 及
\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\]
下的最大值和最小值。设
\[F = u + 2λ(Ax + By + Cz) - μ(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1).\]

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于是，达到最大值、最小值的点的坐标必须满足方程组
\[
\begin{align*}
\frac{1}{2} \frac{\partial F}{\partial x} &= (1 - \frac{\mu}{a^2})x + \lambda A = 0, \\
\frac{1}{2} \frac{\partial F}{\partial y} &= (1 - \frac{\mu}{b^2})y + \lambda B = 0, \\
\frac{1}{2} \frac{\partial F}{\partial z} &= z + \lambda C = 0, \\
Ax + By + Cz &= 0, \\
x^2 + \frac{y^2}{b^2} &= 1. 
\end{align*}
\]  (1)  (2)  (3)  (4)  (5)

将 (1)、(2)、(3) 三式分别乘以 x, y, z 后，然后相加，得 \(x^2 + y^2 + z^2 = \mu\)，即从方程组可解得
\[u(x, y, z) = \mu.\]

由 (1)、(2)、(3)、(4) 知，若取 \(x, y, z\) 及 \(\lambda\) 不全为零，\(\mu\) 必须满足下列方程 (同时 \(\mu\) 只要满足下列方程，静止点 \((x, y, z)\) 也一定有解):

\[
\begin{vmatrix}
1 - \frac{\mu}{a^2} & 0 & 0 & A \\
0 & 1 - \frac{\mu}{b^2} & 0 & B \\
0 & 0 & 1 & C \\
A & B & C & 0
\end{vmatrix} = 0.
\]

展开后，得
\[
\frac{C^2}{a^2 b^2} \mu^2 = \left( \frac{B^2}{a^2} + \frac{A^2}{b^2} + \frac{C^2}{a^2} + \frac{C^2}{b^2} \right) \mu \\
+ (A^2 + B^2 + C^2) = 0.
\]
此方程有两正根，显然即为最大值及最小值 $a^2, b^2$，由韦达定理知

$$\frac{a^2}{b} = \frac{a^2b^2(A^2 + B^2 + C^2)}{C^2}.$$  

故椭圆面积 $\pi ab = \frac{\pi ab \sqrt{A^2 + B^2 + C^2}}{|C|}$ （$C \neq 0$）。

当 $C = 0$ 时，平面 $Ax + By = 0$ 过 $Oz$ 轴，显然得不到椭圆截面。

3705. 求用平面

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0$$

（其中 $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$）与椭球面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

相截所得截面的面积。

解：截面为一椭圆。与3704题一样，我们只要先考虑函数 $u = x^2 + y^2 + z^2$ 在条件

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0$$

及 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

下的极值 ($a > 0$, $b > 0$, $c > 0$)。设

$$F = u + 2\lambda_1(x \cos \alpha + y \cos \beta + z \cos \gamma) - \lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

解方程组

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\[
\begin{aligned}
\frac{1}{2} \frac{\partial F}{\partial x} & = (1 - \frac{\lambda_2}{a^2}) x + \lambda_1 \cos \alpha = 0, \\
\frac{1}{2} \frac{\partial F}{\partial y} & = (1 - \frac{\lambda_2}{b^2}) y + \lambda_1 \cos \beta = 0, \\
\frac{1}{2} \frac{\partial F}{\partial z} & = (1 - \frac{\lambda_2}{c^2}) z + \lambda_1 \cos \gamma = 0,
\end{aligned}
\]

\[x \cos \alpha + y \cos \beta + z \cos \gamma = 0,
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]

将（1），（2），（3）三式分别乘以x，y，z，然后相加，即得

\[u = x^2 + y^2 + z^2 = \lambda_2.
\]

由（1），（2），（3），（4）知，若要x，y，z及\lambda_1不全为零，\lambda_2必须满足下列方程

\[
\begin{vmatrix}
1 - \frac{\lambda_2}{a^2} & 0 & 0 & \cos \alpha \\
0 & 1 - \frac{\lambda_2}{b^2} & 0 & \cos \beta \\
0 & 0 & 1 - \frac{\lambda_2}{c^2} & \cos \gamma \\
\cos \alpha & \cos \beta & \cos \gamma & 0
\end{vmatrix} = 0.
\]

展开整理得

\[
\left( \frac{\cos^2 \alpha}{b^2c^2} + \frac{\cos^2 \beta}{a^2b^2} + \frac{\cos^2 \gamma}{c^2a^2} \right) \lambda_2^2 - \left( \frac{\cos^2 \alpha}{b^2} + \frac{\cos^2 \beta}{c^2} \right) \lambda_2 + 1 = 0.
\]
此方程有两正根, 显然即为椭圆的长短半轴的平方 $\bar{a}^2, \bar{b}^2$. 由韦达定理知

$$\frac{-\bar{a}^2 - \bar{b}^2}{\bar{a} \bar{b}} = \frac{a^2 b^2 c^2}{a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma}. $$

于是，所求椭圆的面积为

$$S = \pi ab = \frac{\pi abc}{\sqrt{a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma}}. $$

3706. 根据飞耳马原则，从 $A$ 点射出而达于 $B$ 点的光线，沿着需要最短时间的曲线传播.

假定点 $A$ 和点 $B$ 位于以平面所分开的不同的光介质中，并且光散播的速度在第一介质中等于 $v_1$, 而在第二介质中等于 $v_2$, 推出光的折射定律.

解　如图6·45所示，

光线从 $A$ 点射出，沿折线 $AMB$ 到达 $B$ 点，由 $A, B$ 作垂直于 $l$ 的直线 $AC$ 及 $BD$，

并与直线 $l$ 交于 $C$ 点及 $D$ 点。设 $AC = a, BD = b, CD = d$. 选择角度 $\alpha, \beta$ 为变量, 则

$$AM = \frac{a}{\cos \alpha}, \quad BM = \frac{b}{\cos \beta}, $$

$$CM = atg \alpha, \quad MD = btg \beta. $$

于是，我们的问题就是求函数

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\[ f(\alpha, \beta) = \frac{a}{v_1 \cos \alpha} + \frac{b}{v_2 \cos \beta} \]

在条件 \( \tan \alpha + \tan \beta = d \) 下的最小值，其中 \(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}, -\frac{\pi}{2} < \beta < \frac{\pi}{2} \)（当 \( M \) 在 \( C \) 与 \( D \) 之间时，\( \alpha > 0, \beta > 0 \)；当 \( M \) 在点 \( D \) 的左边时，\( \alpha < 0, \beta > 0 \)；当 \( M \) 在点 \( D \) 的右边时，\( \alpha > 0, \beta < 0 \)）显然 \( f(\alpha, \beta) \) 是连续函数，又当 \( \alpha \to \frac{\pi}{2}^-0 \) 时（这时点 \( M \) 从右边伸向无穷远，\( \beta \to -\frac{\pi}{2} + 0 \)），显然 \( f(\alpha, \beta) \to +\infty \)；当 \( \alpha \to -\frac{\pi}{2} + 0 \) 时（这时点 \( M \) 从左边伸向无穷远，\( \beta \to \frac{\pi}{2}^-0 \)），显然也有 \( f(\alpha, \beta) \to +\infty \)，由此可知 \( f(\alpha, \beta) \) 在有限处达到最小值，此处必为静止点。设

\[ F = \frac{a}{v_1 \cos \alpha} + \frac{b}{v_2 \cos \beta} - \lambda (\tan \alpha + \tan \beta - d). \]

注意到由

\[
\begin{align*}
\frac{\partial F}{\partial \alpha} &= \frac{a \sin \alpha}{v_1 \cos^2 \alpha} - \frac{\lambda a}{\cos^2 \alpha} = 0, \\
\frac{\partial F}{\partial \beta} &= \frac{b \sin \beta}{v_2 \cos^2 \beta} - \frac{\lambda b}{\cos^2 \beta} = 0,
\end{align*}
\]

即得

\[
\frac{\sin \alpha}{v_1} = \lambda, \quad \frac{\sin \beta}{v_2} = \lambda.
\]

于是，在静止点必满足

\[
\frac{\sin \alpha}{\sin \beta} = \frac{v_1}{v_2}.
\]

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由此可知，光的传播路径必须满足上述的关系。这就是著名的光线折射定律。此时，由点A到点B的光线传播所需要的时间最短。

3707. 当入射角怎样时，光线的折射（即入射线与出射线之间的角）为最小？
（此光线经过棱镜的折射角为α，折射系数为n）。
求出此最小的折射。

解 如图6·46所示，ABC为棱镜。∠BAC=α
为棱镜顶角（即棱镜的折射角），DE为入射光线，折射后从F点折射出棱镜，射出线为FG。IH和JH分别为入射点
和射出点的法线，它们相交于H（IH⊥AC，JH⊥AB）。入射线DE的延长线DM与射出线的反向延长线FL交于K。令∠DEI=β，∠GFJ=γ，∠GKM=δ，∠HEF=λ，∠EFH=μ。

按题意即问：当β在$(0, \frac{\pi}{2})$之间的一定范围内变化时，δ何时达到极小值。这是一元函数的极值问题，然因牵涉的变量关系太多，因此把它看作多元函数的条件极值问题。

由折射定律（3706题）可知。
\[ \sin \beta = \sin \lambda, \quad (1) \]
\[ \sin \gamma = n \sin \mu. \quad (2) \]

由几何关系不难求出 \( \alpha, \beta, \gamma, \delta, \lambda \) 及 \( \mu \) 之间的关系:
\[ \lambda + \mu = \alpha, \quad (3) \]
\[ \delta = \beta + \gamma - \alpha. \quad (4) \]

由于 \( \alpha \) 为常数，故从 (1)、(2)、(3)、(4) 四式中消去 \( \lambda, \mu \) 及 \( \gamma \) 就得到 \( \delta \) 作为 \( \beta \) 的函数。令
\[ F(\beta, \gamma, \lambda, \mu) = \beta + \gamma - \alpha + k_1 (\sin \beta - \sin \lambda) \]
\[ + k_2 (n \sin \mu - \sin \gamma) + k_3 (\lambda + \mu - \alpha). \]

静止点适合下列方程组
\[
\begin{align*}
\frac{\partial F}{\partial \beta} &= 1 + k_1 \cos \beta = 0, \quad (5) \\
\frac{\partial F}{\partial \gamma} &= 1 - k_2 \cos \gamma = 0, \quad (6) \\
\frac{\partial F}{\partial \lambda} &= -k_1 n \cos \lambda + k_3 = 0, \quad (7) \\
\frac{\partial F}{\partial \mu} &= k_2 n \cos \mu + k_3 = 0. \quad (8)
\end{align*}
\]

由 (7)、(8) 消去 \( k_3 \)，得
\[ k_1 \cos \lambda = -k_2 \cos \mu. \quad (9) \]

由 (5)、(6) 得
\[ k_1 = -\frac{1}{\cos \beta}, k_2 = \frac{1}{\cos \gamma}. \quad \text{代入 (9)，} \]

两边平方，即得
\[ \frac{\cos^2 \lambda}{\cos^2 \beta} = \frac{\cos^2 \mu}{\cos^2 \gamma}, \quad \text{或} \]
\[ \frac{1 - \sin^2 \lambda}{1 - \sin^2 \beta} = \frac{1 - \sin^2 \mu}{1 - \sin^2 \gamma}. \quad (10) \]

将 (1)、(2) 代入 (10)，得
\[ \frac{1 - \sin^2 \lambda}{1 - n^2 \sin^2 \lambda} = \frac{1 - \sin^2 \mu}{1 - n^2 \sin^2 \mu}. \]

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整理后得
\[(n^2-1)(\sin^2 \lambda - \sin^2 \mu) = 0.\]
由于 \(0 < \lambda < \frac{\pi}{2}\), \(0 < \mu < \frac{\pi}{2}\) 故 \(\sin \lambda = \sin \mu\) 或 \(\lambda = \mu\).
代入 (3)，得 \(\lambda = \mu = \frac{\alpha}{2}\). 从而 \(\beta = \gamma = \arcsin (n \sin \frac{\alpha}{2})\).
于是，
\[\delta = \beta + \gamma - \alpha = 2 \arcsin (n \sin \frac{\alpha}{2}) - \alpha.\]
所求得的 \(\beta\) 即为唯一的静止点．
根据物理知识，作为本题所讨论的对象，顶角较小的分光棱镜，在区域内确实存在着最小的折射．于是，
当入射角
\[\beta = \arcsin (n \sin \frac{\alpha}{2})\]
时，则
\[\delta = 2 \arcsin (n \sin \frac{\alpha}{2}) - \alpha\]
应为最小折射，至于作其它用途的各种棱镜，光线的折射路径不仅与顶角有关，而且大都与整个棱镜的构造有关，这已不属于本题所考虑的对象，因而也不再对它们进行讨论．

3708. 变量 \(x\) 和 \(y\) 满足线性方程式
\[y = ax + b.\]
它的系数需要确定．由于一系列的等精确测定的结果，对于量 \(x\) 和 \(y\) 得到值 \(x_i, y_i (i = 1, 2, \ldots, n)\).
利用最小二乘方的方法，求系数 \(a\) 和 \(b\) 的最可靠数值．
解 根据最小二乘方的方法，系数 \(a\) 和 \(b\) 的最可靠数值

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值是这样的：对于它们，误差的平方和

\[ M = \sum_{i=1}^{n} (a x_i - b - y_i)^2 \]

为最小。因此，上述问题可以通过求方程组

\[
\begin{align*}
\frac{\partial M}{\partial a} &= 2 \sum_{i=1}^{n} (a x_i + b - y_i) x_i = 0, \\
\frac{\partial M}{\partial b} &= 2 \sum_{i=1}^{n} (a x_i + b - y_i) = 0
\end{align*}
\]

的解来解决。记

\[ [x, y] = \sum_{i=1}^{n} x_i y_i, \quad [x, x] = \sum_{i=1}^{n} x_i^2, \]

\[ [x, 1] = \sum_{i=1}^{n} x_i, \quad [y, 1] = \sum_{i=1}^{n} y_i, \]

则上述方程组化为

\[
\begin{align*}
[a(x, x) + b(x, 1) &= [x, y], \\
[a(x, 1) + bn &= [y, 1].
\end{align*}
\]

系数行列式

\[
\Delta = \begin{vmatrix}
[x, x] & [x, 1] \\
[x, 1] & n
\end{vmatrix} = n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2
\]

\[= (n-1) \sum_{i=1}^{n} x_i^2 - 2 \sum_{i \neq j} x_i x_j = \sum_{i \neq j} (x_i - x_j)^2.\]

当\( \Delta \neq 0 \)时，方程组有唯一的一组解，且

\[
a = \frac{\begin{vmatrix}
[x, y] & [x, 1] \\
y, 1 & n
\end{vmatrix}}{\begin{vmatrix}
[x, x] & [x, 1] \\
[x, 1] & n
\end{vmatrix}} = \frac{n \sum_{i=1}^{n} x_i y_i - \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} y_i \right)}{\sum_{i \neq j} (x_i - x_j)^2}
\]
\[
\begin{align*}
   b &= \frac{\begin{vmatrix} x_i \quad x_i y_i \\ y_i \quad 1 \end{vmatrix} - \begin{vmatrix} x_i \quad y_i \\ 1 \quad 1 \end{vmatrix}}{egin{vmatrix} x_i \quad y_i \\ 1 \quad 1 \end{vmatrix} n} \\
   &= \frac{\left( \sum_{i=1}^{n} x_i y_i \right) - \left( \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i \right)}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.
\end{align*}
\]

显然，此时 \( M \) 为最小，因此，上述 \( a \) 和 \( b \) 即为所求。

3709. 在平面上已知 \( n \) 个点 \( M_i (x_i, y_i) \) (\( i = 1, 2, \ldots, n \))。
直线 \( x \cos \alpha + y \sin \alpha = p \) 在怎样的位置时，已知点与此直线的偏差的平方和为最小？

解 已知点与直线的偏差平方和
\[
M(a, p) = \sum_{i=1}^{n} (x_i \cos \alpha + y_i \sin \alpha - p)^2.
\]

记
\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i,
\]
\[
\bar{x} \bar{y} = \frac{1}{n} \sum_{i=1}^{n} x_i y_i, \quad \bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2, \quad \bar{y}^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2,
\]
则所求直线的参数 \( a \) 和 \( p \) 应满足方程
\[
\frac{\partial M}{\partial \alpha} = 2 \sum_{i=1}^{n} (x_i \cos \alpha + y_i \sin \alpha - p)(y_i \cos \alpha - x_i \sin \alpha)
\]
\[
= 2 \sum_{i=1}^{n} \left[ x_i y_i \cos 2\alpha + (y_i^2 - x_i^2) \frac{\sin 2\alpha}{2}
- y_i p \cos \alpha + x_i p \sin \alpha \right]
\]
\[
= n \left[ 2 \bar{x} \bar{y} \cos 2\alpha + (\bar{y}^2 - \bar{x}^2) \sin 2\alpha - 2p (\bar{y} \cos \alpha
- \bar{x} \sin \alpha) \right] = 0 , 
\]

(1)
\[\frac{\partial M}{\partial \rho} = -2 \sum_{i=1}^{n} (x_i \cos \alpha + y_i \sin \alpha - \rho) = -2n(x \cos \alpha + y \sin \alpha - \rho) = 0. \quad (2)\]

由（2）式，解得
\[\rho = x \cos \alpha + y \sin \alpha. \quad (3)\]

将（3）式代入（1）式，即可解出
\[\tan 2\alpha = \frac{2(x \bar{y} - \bar{x} y)}{(x - \bar{x})^2 + (y - \bar{y})^2}. \quad (4)\]

在（0, 2\pi）范围内，（4）式的解α共有四个:
\[\alpha_0, \alpha_0 + \frac{\pi}{2}, \alpha_0 + \pi, \alpha_0 + \frac{3\pi}{2}, \]

其中 0 ≤ α_0 ≤ \frac{\pi}{2}，将这四个解代入（3）式可以求出ρ。

根据习惯，取 ρ ≥ 0，故上述四个 α 只有两个满足 ρ ≥ 0 的要求**。记为 α_1, ρ_1; α_2, ρ_2。这样就得到两条互相垂直的直线:

\[\begin{align*}
\{ & x \cos \alpha_1 + y \sin \alpha_1 - \rho_1 = 0, \\
& x \cos \alpha_2 + y \sin \alpha_2 - \rho_2 = 0. \quad (5) \quad (6)
\end{align*}\]

显然，M(α, ρ) 一定在 ρ 为有限值的点上取得最小值，因此，只要比较 M(α_1, ρ_1) 和 M(α_2, ρ_2) 的值，M 较小的那条直线即为所求**。

*）当（4）式分母为零而分子不为零时，解为 2α = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}。当分子分母同时为零时，有无穷多个解，即任意一条过 n 个点的重心的直线均使 M(α,
\( p \) 为最小，具体的讨论不进行.

\( ** \) 也可能同时有一对或两对 \( p \) 使 \( p = 0 \)，但此时代表的直线仍具有互相垂直的两条，只是直线方程(5)或(6)有两种不同的表示法而已。

\( * * * \) 特殊情况下也可能有 \( M(\alpha_1, p_1) = M(\alpha_2, p_2) \)，此时使 \( M \) 取得最小值的直线有两条。

3710. 在区间 \((1, 3)\) 内用线性函数 \( ax + b \)，来近似地代替函数 \( x^2 \)，使得绝对偏差

\[ \Delta = \sup_{1 \leq x \leq 3} |x^2 - (ax + b)| \]

为最小。

解 考虑函数

\[ u(a, b) = \Delta^2 = \sup_{1 \leq x \leq 3} (x^2 - (ax + b))^2, \]

\[ f(x, a, b) = x^2 - (ax + b). \]

由于 \( \frac{\partial f}{\partial x} = 2x - a \)，故当固定 \( a, b \) 时，\( f(x, a, b) \) 只在

\[ x = \frac{a}{2} \] 处达到极值 \( f \left( \frac{a}{2}, a, b \right) \)。当限制 \( 1 \leq x \leq 3 \) 时，只有当 \( 2 \leq a \leq 6 \) 时，\( f(x, a, b) \) 才可能在 \( 1 \leq x \leq 3 \) 内部达到极值。于是，

\[ u(a, b) = \begin{cases} \max \{ f^2(1, a, b), f^2(3, a, b) \}, & 2 \leq a \leq 6; \\ f^2 \left( \frac{a}{2}, a, b \right), & a \leq 2 \text{ 或 } a \geq 6. \end{cases} \]

从上式得知，对一切 \( (a, b) \) 均有 \( u(a, b) \geq 0 \)。

设从上式已解出平面区域 \( \Omega_1, \Omega_2 \) 及 \( \Omega_3 \)，使得
\[ u(a, b) = \begin{cases} 
  f^2(1, a, b) = (1 - a - b)^2, (a, b) \in \Omega_1; \\
  f^2(3, a, b) = (9 - 3a - b)^2, (a, b) \in \Omega_2; \\
  f^2(\frac{a}{2}, a, b) = \left(\frac{a^2}{4} + b\right)^2, (a, b) \in \Omega_3, \\
 2 \leq a \leq 6.
\]

由于 \( u(a, b) \geq 0 \)，不难看出 \( u(a, b) \) 在区域 \( \Omega_i \) \((i = 1, 2, 3) \) 内部均无静止点，再看区域边界的状况。以 \( \Omega_1 \) 及 \( \Omega_3 \) 的边界为例，根据 \( u(a, b) \) 的连续性，即知在边界上

\[ (1 - a - b)^2 = \left(\frac{a^2}{4} + b\right)^2. \]

下面我们求满足条件极值的必要条件的点。设

\[ F(a, b) = (1 - a - b)^2 + \lambda \left( (1 - a - b)^2 - \left(\frac{a^2}{4} + b\right)^2 \right), \]

则

\[ \begin{aligned}
  \frac{\partial F}{\partial a} &= -2(1 + \lambda)(1 - a - b) - \lambda a \left(\frac{a^2}{4} + b\right), \\
  \frac{\partial F}{\partial b} &= -2(1 + \lambda)(1 - a - b) - 2\lambda \left(\frac{a^2}{4} + b\right).
\end{aligned} \]

使 \( \frac{\partial F}{\partial a} = 0 \), \( \frac{\partial F}{\partial b} = 0 \) 且满足条件 \( 1 - a - b \neq 0 \), \( \frac{a^2}{4} + b \neq 0 \) 的点没有。

同法可证：在 \( \Omega_1, \Omega_2 \) 及 \( \Omega_3 \) 的边界上也无静止点。但，\( u(a, b) \) 一定在区域内达到最小值。因此，只能在 \( \Omega_1, \Omega_2, \Omega_3 \) 的边界交点上取得最小值，即在满足方程

\[ (1 - a - b)^2 = (9 - 3a - b)^2 = \left(\frac{a^2}{4} + b\right)^2 \quad (1) \]
的点 \((a, b)\) 上取得最小值。方程 (1) 可转化为下面四组方程

\[
\begin{align*}
1 - a - b = 9 - 3a - b &= -\left( \frac{a^2}{4} + b \right), \quad (2) \\
1 - a - b = 9 - 3a - b &= \frac{a^2}{4} + b, \quad (3) \\
1 - a - b = -(9 - 3a - b) &= -\left( \frac{a^2}{4} + b \right), \quad (4) \\
1 - a - b = -(9 - 3a - b) &= \frac{a^2}{4} + b. \quad (5)
\end{align*}
\]

方程组 (2) 无解。

方程组 (3) 的解为 \(a = 4, b = -\frac{7}{2}\)。对应的 \(\Delta = \frac{1}{2}\)。

方程组 (4) 的解为 \(a = 2, b = 1\)。对应的 \(\Delta = 2\)。

方程组 (5) 的解为 \(a = 6, b = -7\)。对应的 \(\Delta = 2\)。

综上所述，可知：在区间 \((1, 3)\) 内，用线性函数

\[4x - \frac{7}{2}\]

来近似地代替函数 \(x^2\)，即可使绝对偏差 \(\Delta\) 为最小，且 \(\Delta_{\text{min}} = \frac{1}{2}\)。
第七章 带参数的积分

§1．带参数的常义积分

1° 积分的连续性 若函数 $f(x, y)$ 于有界的域 $R(a \leq x \leq A, b \leq y \leq B)$ 内有定义并且是连续的，则

$$F(y) = \int_a^A f(x, y) \, dx$$

是在闭区间 $b \leq y \leq B$ 上的连续函数。

2° 积分符号下的微分法 若除在 1° 中所已指明的条件之外，并且偏导函数 $f'_x(x, y)$ 在区域 $R$ 内连续，则当 $b \leq y \leq B$ 时由拉普拉斯公式

$$\frac{d}{dy} \int_a^A f(x, y) \, dx = \int_a^A f'_x(x, y) \, dx$$

为真。

在更普遍的情况下，当积分的限为参数 $y$ 的可微分函数 $\varphi(y)$ 和 $\psi(y)$ 并且当 $b \leq y \leq B$ 时 $a \leq \varphi(y) \leq A, a \leq \psi(y) \leq A$，有：

$$\frac{d}{dy} \int_{\varphi(y)}^{\psi(y)} f(x, y) \, dx = f(\psi(y), y)\psi'(y) - f(\varphi(y), y)\varphi'(y) + \int_{\varphi(y)}^{\psi(y)} f'_x(x, y) \, dx \quad (b \leq y \leq B)$$
3° 积分符号下的积分法 在1°的条件下有

\[ \int_0^b d\ y \int_0^a f(x, y) \ d\ x = \int_0^a d\ x \int_0^b f(x, y) \ d\ y. \]

3711. 证明：不连续函数 \( f(x, y) = \text{sgn}(x - y) \) 的积分

\[ F(y) = \int_0^1 f(x, y) \ d\ x \]

为连续函数。作出函数 \( u = F(y) \) 的图形。

证 当 \( -\infty < y < 0 \) 时，

\[ F(y) = \int_0^1 1 \ d\ x = 1; \]

当 \( 0 \leq y \leq 1 \) 时，

\[ F(y) = \int_0^y (-1) \ d\ x + \int_y^1 1 \ d\ x = 1 - 2y; \]

当 \( 1 < y \leq +\infty \) 时，

\[ F(y) = \int_0^1 (-1) \ d\ x = -1. \]

由于

\[ \lim_{y \to 0} F(y) = \lim_{y \to 0} (1 - 2y) = 1, \quad \lim_{y \to 0} F(y) = 1 \]

且 \( F(0) = 1 \)，即有

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\[ F(0) = F(-0) = F(0), \]
故 \( u = F(y) \) 当 \( y = 0 \) 时为连续的。

同法可证 \( u = F(y) \) 当 \( y = 1 \) 时为连续的。当 \( y \neq 0, y \neq 1 \) 时，\( u = F(y) \) 显然连续。于是，\( u = F(y) \) 在整个 \( Oy \) 轴上均为连续的。如图7.1所示。

3712. 研究函数

\[ F(y) = \int_{0}^{1} \frac{y f(x)}{x^2 + y^2} \, dx \]

的连续性，其中 \( f(x) \) 在闭区间 \([0, 1]\) 上是正的连续函数。

解  当 \( y \neq 0 \) 时，被积函数是连续的。因此，\( F(y) \) 为连续函数。

当 \( y = 0 \) 时，显然有 \( F(0) = 0 \).

当 \( y > 0 \) 时，设 \( m \) 为 \( f(x) \) 在 \([0, 1]\) 上的最小值，则 \( m > 0 \)。由于

\[ F(y) \geq m \int_{0}^{1} \frac{y}{x^2 + y^2} \, dx = m \arctg \frac{1}{y} \]

及

\[ \lim_{y \to 0} \arctg \frac{1}{y} = \frac{\pi}{2}, \]

故有

\[ \lim_{y \to 0} F(y) \geq \frac{m\pi}{2} > 0. \]

于是，\( F(y) \) 当 \( y = 0 \) 时不连续。

3713. 求；
(a) \( \lim_{a \to 0} \int_{a}^{1 + a} \frac{dx}{1 + x^2 + a^2} \);

(b) \( \lim_{a \to 0} \int_{-1}^{1} \frac{\sqrt{x^2 + a^2}}{d} dx \);

(c) \( \lim_{a \to 0} \int_{0}^{2} x^2 \cos ax dx \);

(r) \( \lim_{a \to \infty} \int_{0}^{1} \frac{dx}{1 + \left(1 + \frac{x}{n}\right)^a} \).

解 (a) 因 \( \frac{1}{1 + x^2 + a^2} \)，\( a, 1 + a \) 都是连续函数，

故含参变量 \( a \) 的积分 \( F(a) = \int_{a}^{1 + a} \frac{dx}{1 + x^2 + a^2} \) 是

\( a \) 在 \( -\infty < a < + \infty \) 上的连续函数，因此

\[ \lim_{a \to 0} \int_{a}^{1 + a} \frac{dx}{1 + x^2 + a^2} = \lim_{a \to 0} F(a) = F(0) = \int_{0}^{1} \frac{dx}{1 + x^2} \]

= \( \arctan x \bigg|_{0}^{1} = \frac{\pi}{4} \).

(6) 同样，\( F(a) = \int_{-1}^{1} \frac{\sqrt{x^2 + a^2}}{d} dx \) 是 \( -\infty < a < + \infty \) 上的连续函数，因此

\[ \lim_{a \to 0} \int_{-1}^{1} \frac{dx}{x^2 + a^2} = \lim_{a \to 0} F(a) = F(0) = \int_{-1}^{1} \frac{dx}{\sqrt{x^2}} \]
\[ = 2 \int_0^1 x \, dx = 1. \]

（b）同样，\( F(a) = \int_0^2 x^2 \cos ax \, dx \) 是 \(-\infty < a < +\infty\) 上的连续函数，故

\[
\lim_{a \to 0} \int_0^2 x^2 \cos ax \, dx
\]

\[ = \lim_{a \to 0} F(a) = F(0) = \int_0^2 x^2 \, dx = \frac{2}{3}. \]

（c）考虑二元函数

\[
f(x, y) = \begin{cases} 
\frac{1}{1 + (1 + xy)^{\frac{1}{2}}} & \text{当 } 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \text{ 时;}
\frac{1}{1 + e^x}, & \text{当 } 0 \leq x \leq 1, \quad y = 0 \text{ 时}.
\end{cases}
\]

由 \( \lim_{u \to +0} (1 + u)^{\frac{1}{2}} = e \) 易知 \( f(x, y) \) 是 \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \) 上的连续函数，从而积分 \( F(y) = \int_0^1 f(x, y) \, dx \) 是 \( 0 \leq y \leq 1 \) 上的连续函数，因此

\[
\lim_{y \to +0} F(y) = F(0),
\]

从而更有

\[
\lim_{n \to +\infty} \int_0^1 \frac{dx}{1 + \left(1 + \frac{x}{n}\right)^n}.
\]
\[ \lim_{n \to \infty} F \left( \frac{1}{n} \right) = F(0) = \int_0^1 f(x, 0) \, dx \]

\[ = \int_0^1 \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} \bigg|_0^1 = \ln \frac{2e}{1 + e}. \]

3714. 设函数 \( f(x) \) 在闭区间 \([a, A]\) 上连续，证明

\[ \lim_{h \to 0} \frac{1}{h} \int_a^x (f(t + h) - f(t)) \, dt = f(x) - f(a) \]

\( (a < x < A). \)

证 由于 \( f(x) \) 在 \([a, A]\) 上连续，故在 \([a, A]\) 上存在原函数，于是，

\[ \lim_{h \to 0} \frac{1}{h} \int_a^x (f(t + h) - f(t)) \, dt \]

\[ = \lim_{h \to 0} \frac{1}{h} \left[ F(x + h) - F(a + h) - F(x) + F(a) \right] \]

\[ = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} - \lim_{h \to 0} \frac{F(a + h) - F(a)}{h} \]

\[ = F'(x) - F'(a) = f(x) - f(a). \]

3715. 在下式中可否于积分符号下完成极限运算

\[ \lim_{y \to 0} \int_0^1 \frac{\sqrt{x}}{y^2} \cdot e^{-\frac{x^2}{y^2}} \, dx. \]

解 不能。事实上，

\[ \lim_{y \to 0} \int_0^1 \frac{\sqrt{x}}{y^2} \cdot e^{-\frac{x^2}{y^2}} \, dx = \lim_{y \to 0} \left( -\frac{1}{2} \cdot e^{-\frac{1}{y^2}} \right) \bigg|_0^1 \]

\[ = \lim_{y \to 0} \left( -\frac{1}{2} - \frac{1}{2} e^{-\frac{1}{y^2}} \right) = -\frac{1}{2}, \]

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而
\[ \int_0^1 \left( \lim_{y \to 0} \frac{x}{y^2} e^{-\frac{x^2}{y^2}} \right) \, dx = \int_0^1 x \, dx = 0. \]

3716. 当 \( y = 0 \) 时，可否根据莱布尼兹法则计算函数

\[ F(y) = \int_0^1 \ln \sqrt{x^2 + y^2} \, dx \]

的导数？

解：不能。事实上，我们有：当 \( y \neq 0 \) 时，

\[
F'(y) = \int_0^1 \ln \sqrt{x^2 + y^2} \, dx = \left[ x \ln \sqrt{x^2 + y^2} \right]_0^1 - \int_0^1 \frac{x^2}{x^2 + y^2} \, dx \\
= \ln \sqrt{1 + y^2} - \int_0^1 \left( 1 - \frac{y^2}{x^2 + y^2} \right) \, dx \\
= \ln \sqrt{1 + y^2} - 1 + y \arctg \frac{1}{y}.
\]

又有

\[ F(0) = \int_0^1 \ln x \, dx = x \ln x \bigg|_0^1 - \int_0^1 dx = -1. \]

由此可知

\[ F'_y(0) = \lim_{y \to 0} \frac{F(y) - F(0)}{y} \]

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\[
\lim_{y \to 0} \left[ \frac{\ln(1+y^2)}{2y} + \arctg \frac{1}{y} \right] = \frac{\pi}{2},
\]

\[
F'(0) = \lim_{y \to 0} \frac{F(y) - F(0)}{y} = \lim_{y \to 0} \left[ \frac{\ln(1+y^2)}{2y} + \arctg \frac{1}{y} \right] = -\frac{\pi}{2},
\]

故 \( F'(0) \) 不存在。

另一方面，当 \( x > 0 \) 时，

\[
\left( \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2} \right) \bigg|_{y=0} = \left. \frac{y}{x^2 + y^2} \right|_{y=0} = 0,
\]

故

\[
\int_0^1 \left( \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2} \right) \bigg|_{y=0} \, dx = 0.
\]

由此可知，当 \( y = 0 \) 时不能在积分号下求导数，就是求右导数或求左导数也不行，因为

\[
F'(0) = \frac{\pi}{2} \neq 0
\]

\[
= \int_0^1 \left( \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2} \right) \bigg|_{y=0} \, dx,
\]

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\[ F'_-(0) = -\frac{\pi}{2} \neq 0 \]
\[ = \int_0^1 \left( \frac{d}{dy} \ln \sqrt{x^2 + y^2} \right) \bigg|_{y=0}^1 \, dx. \]

3717. 若

\[ F(x) = \int_x^{x^2} e^{-xy^2} \, dy, \]
计算 \( F'(x) \).

解 \( F'(x) = \frac{d}{dx} (x^2) \cdot e^{-xy^2} \bigg|_{y=x} \)

\[ = -\frac{dx}{dx} \cdot e^{-xy^2} \bigg|_{y=x} \]

\[ + \int_x^{x^2} \frac{\partial}{\partial x} (e^{-xy^2}) \, dy \]

\[ = 2xe^{-x^2} - e^{-x^2} - \int_x^{x^2} ye^{-xy^2} \, dy. \]

3718. 设:

(a) \[ F(\alpha) = \int_{\sin \alpha}^{\cos \alpha} e^{\frac{\alpha \sqrt{1-x^2}}{x}} \, dx, \]

(b) \[ F(\alpha) = \int_{\alpha \cdot \pi}^{\beta + \alpha} \frac{\sin \alpha x}{x} \, dx, \]

(c) \[ F(\alpha) = \int_0^\alpha \frac{\ln (1+\alpha x)}{x} \, dx, \]

(d) \[ F(\alpha) = \int_0^\alpha f(x+\alpha, x-\alpha) \, dx, \]

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(a) \[ F'(a) = \int_0^{a^2} dx \int_{x-a}^{x+a} \sin(x^2 + y^2 - a^2) dy, \]

求 \( F'(a) \).

解 (a) \[ F'(a) = -\sin a \cdot e^{a \cdot \sin a} - \cos a \cdot e^{a \cdot \cos a} \]

\[ + \int_{\sin a}^{\cos a} \sqrt{1-x^2} e^{\alpha \sqrt{1-x^2}} dx. \]

(6) \[ F'(a) = \frac{\sin a(b+a)}{b+a} - \frac{\sin a(a+a)}{a+a} \]

\[ + \int_a^{b+a} \cos ax dx \]

\[ = \left(\frac{1}{a} + \frac{1}{b+a}\right) \sin a(b+a) \]

\[ - \left(\frac{1}{a} + \frac{1}{a+a}\right) \sin a(a+a). \]

(b) \[ F'(a) = \frac{1}{a} \ln(1+a^2) + \int_0^a \frac{1}{1+ax} dx \]

\[ = \frac{2}{a} \ln(1+a^2). \]

(6) 设 \( u = x + \alpha \), \( v = x - \alpha \)，则

\[ F(a) = \int_0^a f(u, v) dx. \]

于是，

\[ F'(a) = f(2a, 0) + \int_0^a (f_u(u, v) - f_v(u, v)) dx \]

\[ = f(2a, 0) + 2 \int_0^a f_u(u, v) dx \]

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\[- \int_0^a \left( f'_u(u,v) + f'_v(u,v) \right) dx \]
\[= f(2a, 0) + 2 \int_0^a f'_u(u,v) dx \]
\[= f(2a, 0) + 2 \int_0^a f'_u(u,v) dx \]
\[= f(2a, 0) + 2 \int_0^a f'_u(u,v) dx \]
\[- f(x + \alpha, x - \alpha) \bigg|_{x=a}^{x=0} \]
\[= f(2a, 0) + 2 \int_0^a f'_u(u,v) dx \]
\[- [f(2a, 0) - f(\alpha, -\alpha)] \]
\[= f(\alpha, -\alpha) + 2 \int_0^a f'_u(u,v) dx \]

(9) \[ F'(\alpha) = 2a \int_{\alpha^2 - \alpha}^{a^2 + \alpha} \sin(\alpha^4 + y^2 - \alpha^2) dy \]
\[+ \int_0^{a^2} \left[ \frac{\partial}{\partial \alpha} \int_{x - \alpha}^{x + \alpha} \sin(x^2 + y^2 - \alpha^2) dy \right] dx \]
\[= 2a \int_{\alpha^2 - \alpha}^{a^2 + \alpha} \sin(\alpha^4 + y^2 - \alpha^2) dy \]
\[+ \int_0^{a^2} \left\{ \sin(x^2 + (x + \alpha)^2 - \alpha^2) \right\}
- \sin(x^2 + (x - \alpha)^2 - \alpha^2) \cdot (-1) \]
\[
+ \int_{x-a}^{x+a} (-2\alpha \cos(x^2 + y^2 - \alpha^2)) \, dy \Bigg\} \, dx \\
= 2\alpha \int_{a^2-a}^{a^2+a} \sin(\alpha^4 + y^2 - \alpha^2) \, dy \\
+ \int_0^{a^2} \left\{ \sin(2x^2 + 2\alpha x) + \sin(2x^2 - 2\alpha x) \right\} \, dx \\
+ \int_{x-a}^{x+a} (-2\alpha \cos(x^2 + y^2 - \alpha^2)) \, dy \Bigg\} \, dx \\
= 2\alpha \int_{a^2-a}^{a^2+a} \sin(\alpha^4 + y^2 - \alpha^2) \, dy \\
+ 2 \int_0^{a^2} \sin 2x^2 \cos 2\alpha x \, dx \\
- 2\alpha \int_0^{a^2} \, dx \int_{x-a}^{x+a} \cos(x^2 + y^2 - \alpha^2) \, dy. \\
\]

3719. 若

\[ F(x) = \int_0^x (x+y) f(y) \, dy, \]

其中 \( f(x) \) 为可微分的函数，求 \( F''(x) \).

解 \[ F'(x) = 2x f(x) + \int_0^x f(y) \, dy, \]

\[ F''(x) = 2f(x) + 2x f'(x) + f(x) \]

\[ = 3f(x) + 2x f'(x). \]

3720. 设:

\[ F(x) = \int_a^b f(y) |x-y| \, dy, \]

其中 \( a \leq b \) 及 \( f(y) \) 为可微分的函数，求 \( F''(x) \).

解 当 \( x \in (a, b) \) 时，由于
\[ F(x) = \int_x^a (x - y) f(y) \, dy + \int_x^b (y - x) f(y) \, dy, \]

故有

\[ F'(x) = \frac{d}{dx} \int_x^a (x - y) f(y) \, dy \]
\[ - \frac{d}{dx} \int_x^b (y - x) f(y) \, dy \]
\[ = \int_x^a \frac{\partial}{\partial x} [(x - y) f(y)] \, dy \]
\[ - \int_x^b \frac{\partial}{\partial x} [(y - x) f(y)] \, dy \]
\[ = \int_a^x f(y) \, dy + \int_b^x f(y) \, dy, \]

\[ F''(x) = f(x) + f(x) = 2 f(x). \]

当 \( x \in (a, b) \) 时，例如 \( x \leq a \)，则

\[ F(x) = \int_x^a (y - x) f(y) \, dy, \]

故有

\[ F'(x) = \int_x^b \frac{\partial}{\partial x} [(y - x) f(y)] \, dy \]
\[ = - \int_x^b f(y) \, dy, \]

\[ F''(x) = 0; \]

同理，对于 \( x \geq b \) 也可得 \( F''(x) = 0 \)。总之，
\[ F''(x) = \begin{cases} 
2f(x), & \text{当} \ x \in (a, b), \\
0, & \text{当} \ x \notin (a, b). 
\end{cases} \]

3721. 设:

\[ F(x) = \frac{1}{h^2} \int_0^h d\xi \int_0^h f(x + \xi + \eta) d\eta \quad (h \gg 0), \]

其中 \( f(x) \) 为连续函数，求 \( F''(x) \).

解: \[
F(x) = \frac{1}{h^2} \int_0^h d\xi \int_0^h f(x + \xi + \eta) d\eta \\
= \frac{1}{h^2} \int_0^h d\xi \int_{x+\xi}^{x+\xi+h} f(u) du.
\]

于是,

\[ F'(x) = \frac{1}{h^2} \int_0^h \left[ -\frac{\partial}{\partial x} \int_{x+\xi}^{x+\xi+h} f(u) du \right] d\xi \\
= \frac{1}{h^2} \int_0^h \left[ f(x + \xi + h) - f(x + \xi) \right] d\xi \\
= \frac{1}{h^2} \left[ \int_{x+h}^{x+2h} f(u) du - \int_x^{x+h} f(u) du \right],
\]

\[ F''(x) = \frac{1}{h^2} \left[ f(x+2h) - f(x+h) - f(x+h) \\
+ f(x) \right] \\
= -\frac{1}{h^2} \left[ f(x+2h) - 2f(x+h) + f(x) \right].
\]

3722. 设:

\[ F(x) = \int_0^x f(t)(x-t)^{n-1} dt, \]
求 $F^{(n)}(x)$。

解

$$F'(x) = \int_0^x \frac{d}{dx} [f(t)(x-t)^{n-1}] dt$$

$$= (n-1) \int_0^x f(t)(x-t)^{n-2} dt,$$

$$F''(x) = (n-1)(n-2) \int_0^x f(t)(x-t)^{n-3} dt,$$

$$\cdots$$

$$F^{(n-1)}(x) = (n-1)! \int_0^x f(t) dt,$$

最后得

$$F^{(n)}(x) = (n-1)! f(x).$$

3723. 在区间 $1 \leq x \leq 3$ 上用线性函数 $a + bx$ 近似地代替函数 $f(x) = x^2$, 使得

$$\int_1^3 (a + bx - x^2)^2 dx = \min.$$  

解 设 $F(a, b) = \int_1^3 (a + bx - x^2)^2 dx$, 则由于 $F(a, b)$ 是 $a$ 和 $b$ 的二元连续函数，并且易知当 $r = \sqrt{a^2 + b^2} \to +\infty$ 时，$F(a, b) \to +\infty$, 所以 $F(a, b)$ 必在有限处取得最小值。解方程组

$$\left\{ \begin{array}{l}
\frac{\partial F}{\partial a} = 2 \int_1^3 (a + bx - x^2) dx = 4a + 8b - \frac{52}{3} = 0, \\
\frac{\partial F}{\partial b} = 2 \int_1^3 x(a + bx - x^2) dx = 8a + \frac{52}{3} b - 40 = 0
\end{array} \right.$$
得唯一的一组解 $a = -\frac{11}{3}, \ b = 4$.

于是，当 $a = -\frac{11}{3}, \ b = 4$ 时 $F(a, b)$ 达最小值，即所求的线性函数为 $4x - \frac{11}{3}$.

3724. 依条件，函数 $a + bx$ 及 $\sqrt{1 + x^2}$ 在已知区间 $(0, 1)$ 上的平均平方差为最小，求近似公式 $\sqrt{1 + x^2} \approx a + bx$ $(0 \leq x \leq 1)$.

解 按题设，即在区间 $0 \leq x \leq 1$ 上用线性函数 $a + bx$ 近似代替函数 $f(x) = \sqrt{1 + x^2}$，使得

$$\int_0^1 (a + bx - \sqrt{1 + x^2})^2 dx = \min.$$ 

设 $F(a, b) = \int_0^1 (a + bx - \sqrt{1 + x^2})^2 dx$，则 $F(a, b)$ 是 $a$ 和 $b$ 的二元连续函数，并且易知当 $r = \sqrt{a^2 + b^2}$ $\to +\infty$ 时，$F(a, b) \to +\infty$，故 $F(a, b)$ 必在有限处取得最小值，解方程组

$$\begin{align*}
\frac{\partial F}{\partial a} &= 2 \int_0^1 (a + bx - \sqrt{1 + x^2})dx = 2a + b - [\sqrt{2} + \ln(1 + \sqrt{2})] = 0, \\
\frac{\partial F}{\partial b} &= 2 \int_0^1 x(a + bx - \sqrt{1 + x^2})dx = a + \frac{2}{3}b - \frac{2}{3}(2\sqrt{2} - 1) = 0
\end{align*}$$

得唯一的一组解 $a \approx 0.934, \ b \approx 0.427$. 

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于是，当 $a \approx 0.934$, $b \approx 0.427$ 时，$F(a, b)$ 为最小值，即所求的近似公式为

$$\sqrt{1 + x^2} \approx 0.934 + 0.427 x \quad (0 \leq x \leq 1)$$

3725. 求完全椭圆积分

$$E(k) = \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi$$

及

$$F(k) = \int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad (0 \leq k \leq 1)$$

的导函数并以函数 $E(k)$ 和 $F(k)$ 来表示它们。

证明 $E(k)$ 满足微分方程式

$$E''(k) + \frac{1}{k} E'(k) + \frac{E(k)}{1 - k^2} = 0.$$ 

解 $E'(k) = - \int_{0}^{\frac{\pi}{2}} \frac{k \sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \, d\varphi$

$$= \frac{1}{k} \int_{0}^{\frac{\pi}{2}} \frac{(1 - k^2 \sin^2 \varphi) - 1}{\sqrt{1 - k^2 \sin^2 \varphi}} \, d\varphi$$

$$= \frac{1}{k} \left[ \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi - \int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right]$$

$$= \frac{E(k) - F(k)}{k}.$$ 

(1)
$$F'(k) = \int_0^\frac{x}{2} \frac{k \sin^2 \varphi}{(1-k^2 \sin^2 \varphi)^{\frac{3}{2}}} d\varphi$$

$$= -\frac{1}{k} \int_0^\frac{x}{2} \frac{(1-k^2 \sin^2 \varphi)-1}{(1-k^2 \sin^2 \varphi)^{\frac{3}{2}}} d\varphi$$

$$= -\frac{1}{k} \int_0^\frac{x}{2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

$$+ \frac{1}{k} \int_0^\frac{x}{2} \frac{d\varphi}{(1-k^2 \sin^2 \varphi)^{\frac{3}{2}}}.$$  

我们易证

$$(1-k^2 \sin^2 \varphi)^{-\frac{3}{2}} = \frac{1}{1-k^2} (1-k^2 \sin^2 \varphi)^{-\frac{1}{2}}$$

$$- \frac{k^2}{1-k^2} \frac{d}{d\varphi} \left[ \sin \varphi \cos \varphi (1-k^2 \sin^2 \varphi)^{-\frac{1}{2}} \right],$$

故有

$$\int_0^\frac{x}{2} (1-k^2 \sin^2 \varphi)^{-\frac{3}{2}} d\varphi$$

$$= \frac{1}{1-k^2} \int_0^\frac{x}{2} (1-k^2 \sin^2 \varphi)^{\frac{1}{2}} d\varphi.$$  

于是,

$$F'(k) = -\frac{F(k)}{k} + \frac{E(k)}{k(1-k^2)}.$$  \hspace{1cm} (2)

由（1）式，对 $k$ 再求导数，并注意到（2）式，即

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得

\[ E''(k) = \frac{[E'(k) - F'(k)]k - [E(k) - F(k)]}{k^2} \]

\[ = \left[ \frac{E(k) - F(k)}{k} + \frac{F(k)}{k} - \frac{E(k)}{k(1 - k^2)} \right]k - kE'(k) \]

\[ = -\frac{E(k)}{1 - k^2} - \frac{E'(k)}{k}, \]

即

\[ E''(k) + \frac{F'(k)}{k} + \frac{F(k)}{1 - k^2} = 0. \]

3726. 证明：足指数 \( n \) 为整数的贝塞尔函数

\[ J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\varphi - x \sin \varphi) d\varphi \]

满足贝塞尔方程式

\[ x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0. \]

证 \[ J_n'(x) = \frac{1}{\pi} \int_0^\pi \sin \varphi \cdot \sin(n\varphi - x \sin \varphi) d\varphi, \]

\[ J_n''(x) = -\frac{1}{\pi} \int_0^\pi \sin^2 \varphi \cdot \cos(n\varphi - x \sin \varphi) d\varphi. \]

于是，

\[ x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) \]

\[ = -\frac{1}{\pi} \int_0^\pi \left[ (x^2 \sin^2 \varphi + n^2 - x^2) \cos(n\varphi - x \sin \varphi) \right. \]

\[ - x \sin \varphi \cdot \sin(n\varphi - x \sin \varphi)] d\varphi \]
\[ -\frac{1}{x} \int_0^\pi \left[ (n^2 - x^2 \cos^2 \varphi) \cos(n\varphi - x \sin \varphi) 
right. 

\left. - x \sin \varphi \cdot \sin(n\varphi - x \sin \varphi) \right] d\varphi 
\]

本题证

3727. 设

\[ I(\alpha) = \int_0^\alpha \frac{\varphi(x) dx}{\sqrt{\alpha - x}}, \]

其中函数 \( \varphi(x) \) 及其导函数 \( \varphi'(x) \) 在闭区间 \( 0 \leq x \leq \alpha \) 上连续。

证明：当 \( 0 < \alpha < x \) 时有

\[ I'(\alpha) = \frac{\varphi(0)}{\sqrt{\alpha}} + \int_0^\alpha \frac{\varphi'(x)}{\sqrt{\alpha - x}} dx. \]

证 当 \( x = \alpha \) 时，一般说来被积函数变成无穷，所以

我们不能直接在积分号下求导数。设 \( x = \alpha t \)，则此积

分变成以下形式

\[ I(\alpha) = \sqrt{\alpha} \int_0^1 \frac{\varphi(\alpha t)}{\sqrt{1-t}} dt. \]

由于 \( \frac{1}{\sqrt{1-t}} \) 在 \( [0, 1] \) 上绝对可积，故可利用积分

号下求导数的公式。于是，

\[ I'(\alpha) = \frac{1}{2} \frac{1}{\sqrt{\alpha}} \int_0^1 \frac{\varphi(\alpha t)}{\sqrt{1-t}} dt \]
\sqrt{\alpha} \int_0^1 \frac{t \varphi'(at)}{\sqrt{1-t}} \, dt.

再将 \( x = at \) 代入上式，得
\[
I'(a) = \frac{1}{2\alpha} \int_0^a \frac{\varphi(x)}{\sqrt{\alpha - x}} \, dx
\]
\[
+ \frac{1}{\alpha} \int_0^a \frac{x \varphi'(x)}{\sqrt{\alpha - x}} \, dx. \tag{1}
\]

利用分部积分法可得
\[
\frac{1}{\alpha} \int_0^a \frac{\varphi(x)}{\sqrt{\alpha - x}} \, dx
\]
\[
= \frac{2}{\sqrt{\alpha}} \varphi(0) + \frac{2}{\alpha} \int_0^a \sqrt{\alpha - x} \varphi'(x) \, dx. \tag{2}
\]

另一方面，又有
\[
\int_0^a \frac{x \varphi'(x)}{\sqrt{\alpha - x}} \, dx
\]
\[
= - \int_0^a \sqrt{\alpha - x} \varphi'(x) \, dx
\]
\[
+ \alpha \int_0^a \frac{\varphi'(x)}{\sqrt{\alpha - x}} \, dx. \tag{3}
\]

将 (2) 式及 (3) 式代入 (1) 式，最后得
\[
I'(a) = \frac{\varphi(0)}{\sqrt{\alpha}} + \int_0^a \frac{\varphi'(x)}{\sqrt{\alpha - x}} \, dx.
\]

3728. 设有函数
\[
u(x) = \int_0^1 K(x, y) v(y) \, dy,
\]

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其中

\[
K(x, y) = \begin{cases} 
x(1-y), & \text{若 } x \leq y, \\
y(1-x), & \text{若 } x > y,
\end{cases}
\]

及 \( v(y) \) 都是连续的，证明已知函数满足方程式

\[
u''(x) = -v(x) \quad (0 \leq x \leq 1).
\]

证 由题设得

\[
u(x) = \int_0^x y(1-x)v(y)dy + \int_x^1 x(1-y)v(y)dy.
\]

于是，求导数即得

\[
u'(x) = x(1-x)v(x) - \int_0^x yv(y)dy - x(1-x)v(x) + \int_x^1 (1-y)v(y)dy = -\int_0^x yv(y)dy + \int_x^1 (1-y)v(y)dy,
\]

\[
v''(x) = -xv(x) - (1-x)v(x) = -v(x),
\]
所以，函数 \( u(x) \) 满足方程

\[
u''(x) = -v(x) \quad (0 \leq x \leq 1).
\]

3729. 设

\[
F(x, y) = \int_y^x (x-yz)f(z)dz,
\]
其中 \( f(z) \) 为可微分的函数，求 \( F''_{xy}(x, y) \).
解 \( F'_x(x, y) = y(x - xy^2)f(xy) + \int_{\frac{x}{y}}^x f(z)dz, \)

\[
F''_{xx}(x, y) = (x - xy^2)f(xy) \\
+ y(-2xy)f(xy) \\
+ y(x - xy^2)f'(xy) \\
+ xf(xy) + \frac{x}{y^2}f\left(\frac{x}{y}\right) \\
= x(2 - 3y^2)f(xy) \\
+ x^2y(1 - y^2)f'(xy) \\
+ \frac{x}{y^2}f\left(\frac{x}{y}\right).
\]

3730. 设 \( f(x) \) 为可微分两次的函数及 \( F(x) \) 为可微分的函数。证明：函数

\[
u(x, t) = \frac{1}{2}\left[f(x - at) + f(x + at)\right]
\]

\[
+ \frac{1}{2a} \int_{x-at}^{x+at} F(z)dz
\]

满足弦振动的方程式

\[
\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}
\]

及初值条件：\( u(x, 0) = f(x), \ u_x(x, 0) = F(x) \).

证 \( \frac{\partial u}{\partial t} = \frac{1}{2}\left[-af'(x - at) + af'(x + at)\right] \)

\[
+ \frac{1}{2} F(x + at) + \frac{1}{2} F(x - at),
\]
\[
\frac{-\partial^2 u}{\partial t^2} = \frac{1}{2} \left[ a^2 f''(x-at) + a^2 f''(x+at) \right] \\
+ \frac{a}{2} F'(x+at) - \frac{a}{2} F'(x-at), \quad (1)
\]

\[
\frac{\partial u}{\partial x} = \frac{1}{2} \left[ f'(x-at) + f'(x+at) \right] \\
+ \frac{1}{2a} F(x+at) - \frac{1}{2a} F(x-at),
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \left[ f''(x-at) + f''(x+at) \right] \\
+ \frac{1}{2a} F'(x+at) - \frac{1}{2a} F'(x-at). \quad (2)
\]

比较 (1) 式及 (2) 式，即得

\[
\frac{-\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.
\]

此外，还有

\[
u(x, 0) = \frac{1}{2} \left[ f(x-0 \cdot t) + f(x+0 \cdot t) \right]
+ \frac{1}{2a} \int_{x-0 t}^{x+0 t} F(z) dz = f(x),
\]

\[
u(x, 0) = \frac{1}{2} \left[ -a f'(x) + a f'(x) \right]
\]

\[+ \frac{1}{2} F(x) + \frac{1}{2} F(x) = F(x).
\]

本题完成。
3731. 证明：若函数 \( f(x) \) 在闭区间 \((0, l)\) 上连续及当 \(0 \leq \xi \leq l\) 时 \((x-\xi)^2 + y^2 + z^2 \neq 0\)，则函数

\[
u(x, y, z) = \int_0^l \frac{f(\xi) \, d\xi}{\sqrt{(x-\xi)^2 + y^2 + z^2}}
\]

满足拉普拉斯方程式

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
\]

证 利用积分号下的求导法则，得

\[
\frac{\partial u}{\partial x} = -\int_0^l \frac{2(x-\xi)f(\xi) \, d\xi}{2[(x-\xi)^2 + y^2 + z^2]^{3/2}}
\]

\[
= -\int_0^l \frac{(x-\xi)f(\xi) \, d\xi}{[(x-\xi)^2 + y^2 + z^2]^{3/2}},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \int_0^l \frac{f(\xi) \cdot [2(x-\xi)^2 - y^2 - z^2]}{[(x-\xi)^2 + y^2 + z^2]^{5/2}} \, d\xi. \quad (1)
\]

同法可得

\[
\frac{\partial^2 u}{\partial y^2} = \int_0^l \frac{f(\xi) \cdot [-(x-\xi)^2 + 2y^2 - z^2]}{[(x-\xi)^2 + y^2 + z^2]^{5/2}} \, d\xi, \quad (2)
\]

\[
\frac{\partial^2 u}{\partial z^2} = \int_0^l \frac{f(\xi) \cdot [-(x-\xi)^2 - y^2 + 2z^2]}{[(x-\xi)^2 + y^2 + z^2]^{5/2}} \, d\xi. \quad (3)
\]

将 (1)、(2)、(3) 三式相加，即证得

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
\]
应用对参数的微分法，计算下列积分：

$$3732. \int_0^\frac{\pi}{2} \ln(a^2\sin^2x + b^2\cos^2x) \, dx.$$  

解：将 $b$ 视为常数，$a$ 视为参变量。令  

$$I(a) = \int_0^\frac{\pi}{2} \ln(a^2\sin^2x + b^2\cos^2x) \, dx.$$  

先设 $a > 0, b > 0$。我们有  

$$I'(a) = \int_0^\frac{\pi}{2} \frac{2a \sin^2x}{a^2\sin^2x - b^2\cos^2x} \, dx.$$  

若 $a = b$，有  

$$I'(b) = \frac{2}{b} \int_0^\frac{\pi}{2} \sin^2x \, dx = \frac{\pi}{2b}.$$  

若 $a \neq b$，则作代换 $t = \tan x$，得  

$$I'(a) = \frac{2}{a} \int_0^\infty \frac{t^2dt}{(t^2 + 1)(t^2 + \frac{b^2}{a^2})}.$$  

$$= \frac{2}{a} \left( \frac{a^2}{a^2 - b^2} \cdot \arctan t \right)_{\infty}^{0} - \frac{b^2}{a^2 - b^2} \cdot \frac{a}{b} \cdot \arctan \frac{a}{b} \cdot \frac{\pi}{2b}.$$  

$$= \frac{\pi}{a + b}.$$  

因此  

$$I'(a) = \frac{\pi}{a + b} \quad (0 \leq a \leq +\infty).$$
积分之．得
\[ I(a) = \pi \ln (a+b) + C \quad (0 \leq a \leq +\infty) , \]
其中 \( C \) 为某常数，令 \( a = b \)，得
\[ I(b) = \pi \ln 2b + C, \]
而
\[ I(b) = \int_0^\pi \ln b^2 \, dx = \pi \ln b, \]
代入，解之，得
\[ C = \pi \ln \frac{1}{2} . \]
于是，
\[ I(a) = \pi \ln (a+b) + \pi \ln \frac{1}{2} \]
\[ = \pi \ln \frac{a+b}{2} \quad (0 \leq a \leq +\infty) . \]
若 \( a \leq 0 \) 或 \( b \leq 0 \)，则可化为 \( a > 0 \) 且 \( b > 0 \) 的情形，得
\[ I(a) = \int_0^{\frac{\pi}{2}} \ln (a^2 \sin^2 x + b^2 \cos^2 x) \, dx \]
\[ = \int_0^{\frac{\pi}{2}} \ln (|a|^2 \sin^2 x + |b|^2 \cos^2 x) \, dx \]
\[ = I(|a|) = \pi \ln \frac{|a| + |b|}{2} . \]
于是，不论 \( a, b \) 是正是负，在任何情形，均有
\[ \int_0^{\frac{\pi}{2}} \ln (a^2 \sin^2 x + b^2 \cos^2 x) \, dx = \pi \ln \frac{|a| + |b|}{2} . \]
\[ 3733. \int_0^{\frac{\pi}{2}} \ln (1 - 2a \cos x + a^2) \, dx . \]

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解 设 \( I(a) = \int_0^\pi \ln(1 - 2a \cos x - a^2) \, dx \)。当 \(|a| \leq 1\) 时，由于 \(1 - 2a \cos x + a^2 \geq 1 - 2|a| + a^2 = (1 - |a|)^2 > 0\)，故 \(\ln(1 - 2a \cos x - a^2)\) 为连续函数且具有连续导数，从而可在积分号下求导数。将 \( I(a) \) 对 \(a\) 求导数，得

\[
I'(a) = \int_0^\pi \frac{-2 \cos x + 2a}{1 - 2a \cos x + a^2} \, dx
\]

\[
= \frac{1}{a} \int_0^\pi \left( 1 + \frac{a^2 - 1}{1 - 2a \cos x + a^2} \right) \, dx
\]

\[
= \frac{\pi}{a} - \frac{1 - a^2}{a(1 + a^2)} \int_0^\pi \frac{dx}{1 + \left( \frac{-2a}{1 + a^2} \right) \cos x}
\]

\[
= \frac{\pi}{a} - \frac{2}{a} \arctg \left( \frac{1 + a}{1 - a} \frac{x}{2} \right) \bigg|_0^\pi
\]

\[
= \frac{\pi}{a} - \frac{2}{a} \cdot \frac{\pi}{2} = 0.
\]

于是，当 \(|a| \leq 1\) 时，\( I(a) = C \)（常数）。但是，\( I(0) = 0 \)，故 \(C = 0\)，从而 \( I(a) = 0 \)。

当 \(|a| \geq 1\) 时，令 \( b = \frac{1}{a} \)，则 \(|b| \leq 1\)，并有

\( I(b) = 0 \).

于是，我们有
\[ I(a) = \int_0^x \ln \left( \frac{b^2 - 2b \cos x + 1}{b^2} \right) dx \]

\[ = I(b) - 2\pi \ln |b| \]

\[ = -2\pi \ln |b| = 2\pi \ln |a|. \]

当 \( |a| = 1 \) 时，

\[ I(1) = \int_0^x 1n \ 2 \ (1 - \cos x) dx \]

\[ = \int_0^x \left( \ln 4 + 2 \ln \sin \frac{x}{2} \right) dx \]

\[ = 2\pi \ln 2 + 4 \int_0^{\frac{\pi}{2}} \ln \sin t \ dt \]

\[ = 2\pi \ln 2 + 4 \left( -\frac{\pi}{2} \ln 2 \right) ** \]

\[ = 0 ; \]

同法可求得 \( I(-1) = 0 \).

综上所述，故知

\[ \int_0^x \ln(1 - 2a \cos x + a^2) dx \]

\[ = \begin{cases} 
0, & \text{当} |a| \leq 1; \\
2\pi \ln |a|, & \text{当} |a| > 1.
\end{cases} \]

*）利用2028题(a)的结果。

**）利用2353题(a)的结果。

3734. \[ \int_0^x \frac{\frac{\pi}{2} \arctg(a \tan x)}{\tan x} \ dx. \]

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解。令 \( I(a) = \int_0^\frac{\pi}{2} f(x, a) \, dx \)，其中 \( f(x, a) = \frac{\arctan(a \tan x)}{\tan x} \)。本来 \( f(x, a) \) 在 \( x = 0 \) 和 \( x = \frac{\pi}{2} \) 时无定义，但因 \( \lim_{x \to 0} f(x, a) = a, \lim_{x \to \frac{\pi}{2}^-} f(x, a) = 0 \)。故若补充定义 \( f(0, a) = a, f\left(\frac{\pi}{2}, a\right) = 0 \)，则 \( f(x, a) \) 为 \( 0 \leq x \leq \frac{\pi}{2}, -\infty < a < +\infty \) 上的连续函数。

又当 \( 0 < x < \frac{\pi}{2}, -\infty < a < +\infty \) 时，

\[
f'_a(x, a) = \frac{1}{\tan x} \cdot \frac{\tan x}{1 + a^2 \tan^2 x} = \frac{1}{1 + a^2 \tan^2 x}.
\]

而按规定 \( f(0, a) = a, f\left(\frac{\pi}{2}, a\right) = 0 \)，故

\[
f'_a(0, a) = 1, \quad f'_a\left(\frac{\pi}{2}, a\right) = 0.
\]

由此可知

\[
\begin{cases}
\frac{1}{1 + a^2 \tan^2 x}, & \text{当 } 0 \leq x < \frac{\pi}{2}, -\infty < a < +\infty \\
0, & \text{当 } x = \frac{\pi}{2}, -\infty < a < +\infty \text{时}.
\end{cases}
\]
显然 \( f_1(x, a) \) 在 \( 0 \leq x \leq \frac{\pi}{2} \)，\( 0 \leq a = +\infty \) 上连续，

在 \( 0 \leq x \leq \frac{\pi}{2} \)，\( -\infty < a < 0 \) 上也连续（注意，在点
\( x = \frac{\pi}{2} \)，\( a = 0 \) 不连续），故积分号下求导数法则知

\[
I'(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + a^2 \tan^2 x}
\]

\( (0 < a < +\infty \) 或 \( -\infty < a < 0 \) )。

作代换 \( \tan x = t \)，得（当 \( a^2 \neq 1 \) 时）

\[
\int_0^{\frac{\pi}{2}} \frac{dx}{1 + a^2 \tan^2 x} = \int_0^{+\infty} \frac{dt}{(1 + t^2)(1 + a^2 t^2)}
\]

\[
= \frac{1}{1 - a^2} \int_0^{+\infty} \left( \frac{1}{1 + t^2} - \frac{a^2}{a^2 t^2 + 1} \right) dt
\]

\[
= \frac{\pi}{2(1 + |a|)}.
\]

若 \( a^2 = 1 \)，则

\[
\int_0^{\frac{\pi}{2}} \frac{dx}{1 + a^2 \tan^2 x} = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) \, dx = \frac{\pi}{4}.
\]

总之，有

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\[ I'(a) = \frac{\pi}{2(1 + |a|)} \]

\( (0 \leq a \leq +\infty \) 或 \(-\infty \leq a \leq 0 \) 。

积分之，得

\[ I(a) = \frac{\pi}{2} \ln(1 + a) + C_1 \quad (0 \leq a \leq +\infty) , \]

\[ I(a) = -\frac{\pi}{2} \ln(1 - a) + C_2 \quad (-\infty \leq a \leq 0) , \]

其中 \( C_1, C_2 \) 是两个常数。由于上面已述 \( f(x, a) \) 在

\[ 0 \leq x \leq \frac{\pi}{2}, \quad -\infty \leq a \leq +\infty \] 上连续, 故 \( I(a) \) 在 \(-\infty \leq a \leq +\infty \) 上连续,因此

\[ \lim_{a \to +\infty} I(a) = \lim_{a \to -\infty} I(a) = I(0) \; , \]

但 \( I(0) = 0 \) 。

\[ \lim_{a \to -0+} I(a) = C_1, \quad \lim_{a \to +0-} I(a) = C_2 \; , \]

故 \( C_1 = C_2 = 0 \) 。于是, 最后得

\[ I(a) = \frac{\pi}{2} \operatorname{sgn} a \ln(1 + |a|) \quad (-\infty \leq a \leq +\infty) . \]

3735. \[ \int_0^{\frac{\pi}{2}} \ln\frac{1 + \alpha \cos x}{1 - \alpha \cos x} \cdot \frac{dx}{\cos x} \quad (|\alpha| \leq 1) . \]

解 一

设 \( I(a) = \int_0^{\frac{\pi}{2}} \ln\frac{1 + \alpha \cos x}{1 - \alpha \cos x} \cdot \frac{dx}{\cos x} \) 。由于

\[ \frac{1 + \alpha \cos x}{1 - \alpha \cos x} = \frac{1 - \alpha^2 \cos^2 x}{1 - 2\alpha \cos x + \alpha^2 \cos^2 x} \]

\[ \geq \frac{1 - \alpha^2}{1 + 2|\alpha| + \alpha^2} \]
\[
\frac{1-a^2}{(1+|a|)^2} > 0,
\]

故 \(\ln \frac{1+a \cos x}{1-a \cos x}\) 为连续函数。又由于

\[
\lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} \cdot \ln \frac{1+a \cos x}{1-a \cos x}
\]

\[
= \lim_{t \to 0} \frac{\ln(1+at)-\ln(1-at)}{t}
\]

\[
= \lim_{t \to 0} \frac{a}{1+at} - \frac{-a}{1-at} = 2a,
\]

今补充被积函数在 \(x = \frac{\pi}{2}\) 处的值为 2a，即知被积函数为连续函数，且它对 \(a\) 有连续导数，从而可在积分号下求导数，得

\[
I'(a) = \int_{0}^{\frac{\pi}{2}} \left( \frac{1}{1+a \cos x} + \frac{1}{1-a \cos x} \right) dx
\]

\[
= \frac{2}{\sqrt{1-a^2}} \left[ \arctg \left( \sqrt{\frac{1-a}{1+a}} \cdot \frac{x}{2} \right) \right]_{0}^{\frac{\pi}{2}}
\]

\[
= \frac{\pi}{\sqrt{1-a^2}},
\]

从而 \(I(a) = \pi \arcsin a + C (|a| < 1)\)。又 \(I(0) = 0\)，故 \(C = 0\)。
于是，
\[
\int_0^\frac{\pi}{2} \ln \frac{1+a \cos x}{1-a \cos x} \cdot \frac{dx}{\cos x} = \pi \arcsin \alpha \quad (|a| < 1).
\]

*) 利用2038题(a)的结果。

解法二
把被积函数表成下述积分形式
\[
\frac{1}{\cos x} \cdot \ln \frac{1+a \cos x}{1-a \cos x} = 2a \int_0^1 \frac{dy}{1-a^2 y^2 \cos^2 x}.
\]

注意，此式当 \( x = \frac{\pi}{2} \) 时也成立，此时左端应理解为其
极限值
\[
\lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} \cdot \ln \frac{1+a \cos x}{1-a \cos x} = 2a.
\]

于是，当 \( a \neq 0 \) 时，
\[
\int_0^\frac{\pi}{2} \ln \frac{1+a \cos x}{1-a \cos x} \cdot \frac{dx}{\cos x} = 2a \int_0^1 \frac{dy}{1-a^2 y^2 \cos^2 x}.
\]

\[
= 2a \int_0^1 dy \int_0^1 \frac{dx}{1-a^2 y^2 \cos^2 x}.
\]

\[
= 2a \int_0^1 dy \int_0^\frac{\pi}{2} \frac{dx}{1-a^2 y^2 \cos^2 x} \quad (**)
\]

\[
= 2a \int_0^1 \frac{\frac{\pi}{2}}{2 \sqrt{1-a^2 y^2}} dy = \pi a \cdot \frac{1}{a} \arcsin \alpha \bigg|_0^1 = \pi \arcsin \alpha.
\]
当 \( a = 0 \) 时，原积分显然为零。因此，

\[
\int_0^\frac{\pi}{2} \ln \frac{1 + a \cos x}{1 - a \cos x} \cdot \frac{dx}{\cos x} = \pi \arcsin a \quad (|a| \ll 1).
\]

\[\text{**} \] 利用2028题(a)的结果，即得

\[
\int_0^\frac{\pi}{2} \frac{dx}{1 - a^2 y^2 \cos^2 x}
\]

\[
= \frac{1}{2} \int_0^\frac{\pi}{2} \left( \frac{1}{1 + ay \cos x} + \frac{1}{1 - ay \cos x} \right) dx
\]

\[
= \frac{1}{2} \cdot \frac{2}{\sqrt{1 - a^2 y^2}} \left[ \arctg \left( \frac{\sqrt{1 - ay} \tan \frac{x}{2}}{1 + ay} \right) \right]_0^\frac{\pi}{2}
\]

\[
= \frac{1}{2} \cdot \frac{2}{\sqrt{1 - a^2 y^2}} \cdot \frac{\pi}{2} = \frac{\pi}{2} \sqrt{1 - a^2 y^2}.
\]

3736. 利用公式

\[
\arctg \frac{x}{y} = \int_0^1 \frac{dy}{1 + x^2 y^2},
\]

计算积分

\[
\int_0^1 \frac{\arctg \frac{x}{y}}{x} \cdot \frac{dx}{\sqrt{1 - x^2}}.
\]

解

\[
\int_0^1 \frac{\arctg \frac{x}{y}}{x} \cdot \frac{dx}{\sqrt{1 - x^2}} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \int_0^1 \frac{dy}{1 + x^2 y^2}.
\]

由于函数 \( \frac{1}{1 + x^2 y^2} \) 在 \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) 上连续

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续，且 $\frac{1}{\sqrt{1-x^2}}$ 在 $[0, 1]$ 上绝对可积，故上述积分分号可交换

$$
\int_0^1 \frac{\arctg x}{x} \cdot \frac{dx}{\sqrt{1-x^2}} = \int_0^1 dy \int_0^1 \frac{dx}{\sqrt{1-x^2}(1+x^2y^2)}.
$$

作代换 $x = \cos t$，可得

$$
\int_0^1 \frac{dx}{\sqrt{1-x^2}(1+x^2y^2)} = \int_0^{\frac{\pi}{2}} \frac{dt}{1+y^2 \cos^2 t} = \frac{1}{\sqrt{1+y^2}} \arctg \left( \frac{\tg t}{\sqrt{1+y^2}} \right) \bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{2 \sqrt{1+y^2}}.
$$

于是，由 (1) 式及 (2) 式即得

$$
\int_0^1 \frac{\arctg x}{x} \cdot \frac{dx}{\sqrt{1-x^2}} = \int_0^1 \frac{\pi dy}{2 \sqrt{1+y^2}} = \frac{\pi}{2} \ln(y + \sqrt{1+y^2}) \bigg|_0^1 = \frac{\pi}{2} \ln(1 + \sqrt{2}).
$$

3737. 应用积分符号下的积分法，计算积分

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\[
\int_0^1 \frac{x^b - x^a}{\ln x} \, dx \quad (a \geq 0, \ b > 0).
\]

解  首先注意，因为

\[
\lim_{x \to 0} \frac{x^b - x^a}{\ln x} = 0,
\]

\[
\lim_{x \to 1^-} \frac{x^b - x^a}{\ln x} = \lim_{x \to 1^-} \frac{bx^{b-1} - ax^{a-1}}{x^{-1}} = \lim_{x \to 1^-} (bx^b - ax^a) = b - a,
\]

故 \( \int_0^1 \frac{x^b - x^a}{\ln x} \, dx \) 不是广义积分，并且，如果补充定义被积函数在 \( x = 0 \) 时的值为 0，在 \( x = 1 \) 时的值为 \( b - a \)，则可理解为 \([0, 1]\) 上连续函数的积分。由于

\[
\frac{x^b - x^a}{\ln x} = \int_a^b x^y \, dy \quad (0 \leq x \leq 1)
\]

(注意，\( x = 0 \) 时左端规定为 0，\( x = 1 \) 时左端规定为 \( b - a \))，而函数 \( x^y \) 在 \( 0 \leq x \leq 1, \ a \leq y \leq b \) 上连续（不妨设 \( a < b \)），故有

\[
\int_0^1 \frac{x^b - x^a}{\ln x} \, dx
\]

\[
= \int_0^1 dx \int_a^b x^y \, dy = \int_a^b dy \int_0^1 x^y \, dx
\]

\[
= \int_a^b \frac{dy}{1 + y} = \ln \frac{1 + b}{1 + a}.
\]

3738. 计算积分：
(a) \[ \int_0^1 \sin \left( \ln \frac{1}{x} \right) \frac{x^b - x^a}{\ln x} \, dx; \]

(b) \[ \int_0^1 \cos \left( \ln \frac{1}{x} \right) \frac{x^b - x^a}{\ln x} \, dx \quad (a > 0, \quad b > 0) \]

解 (a) 不妨设 \( c \leq b \).

\[
\int_0^1 \sin \left( \ln \frac{1}{x} \right) \frac{x^b - x^a}{\ln x} \, dx \\
= \int_0^1 \sin \left( \ln \frac{1}{x} \right) \, dx \int_a^b x^y \, dy \\
= \int_a^b \, dy \int_0^1 \sin \left( \ln \frac{1}{x} \right) x^y \, dx,
\]

这里，当 \( x = 0 \) 时，\( \sin \left( \ln \frac{1}{x} \right) x^y \) 理解为零，从而 \( \sin \left( \ln \frac{1}{x} \right) x^y \) 在 \( 0 \leq x \leq 1, \quad a \leq y \leq b \) 上连续，故可应用积分号下的积分法交换积分次序。

作代换 \( x = e^{-t} \)，可得

\[
\int_0^1 \sin \left( \ln \frac{1}{x} \right) x^y \, dx \\
= \int_0^{+\infty} e^{- (y+1) t} \sin t \, dt \\
= \frac{1}{1+(1+y)^2} \left[ -(y+1) \sin t - \cos t \right] e^{- (y+1) t} \bigg|_0^{+\infty}
\]
\[
\frac{1}{1+(1+y)^2}.
\]
于是，最后得
\[
\int_0^1 \sin \left( \ln \frac{1}{x} \right) \frac{x^b-x^a}{\ln x} \, dx
= \int_a^b dy \frac{dy}{1+(1+y)^2} = \arctg (1+y) \bigg|_a^b
= \arctg (1+b) - \arctg (1+a)
= \arctg \frac{b-a}{1+(1+b)(1+a)}.
\]

(b) 同(a)并利用1828题的结果易得
\[
\int_0^1 \cos \left( \ln \frac{1}{x} \right) \frac{x^b-x^a}{\ln x} \, dx
= \int_a^b dy \int_0^1 \cos \left( \ln \frac{1}{x} \right) x^y \, dx
= \int_a^b dy \frac{1+y}{1+(1+y)^2} \, dx = \frac{1}{2} \ln \left[ 1+(1+y)^2 \right] \bigg|_a^b
= \frac{1}{2} \ln \frac{b^2+2b+2}{a^2+2a+2}.
\]

*) 利用1829题的结果。

3739. 设 \( F(k) \) 和 \( E(k) \) 为完全椭圆积分（参阅问题3725）。
证明公式
\[
(a) \quad \int_0^\tau F(k) \, dk = E(k) - k^2 F(k),
\]

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(6) \[
\int_0^k E(k) k \, dk = \frac{1}{3} \left[ (1 + k^2) E(k) - k_1^2 F(k) \right].
\]
其中 \( k_1^2 = 1 - k^2 \).

证 (a) 利用第25题的结果，可得
\[
\left( E(k) - k_1^2 F(k) \right) = E'(k) + 2k F(k) - (1 - k^2) F'(k)
\]
\[
= \frac{E(k) - F(k)}{k} + 2k F(k)
\]
\[
- (1 - k^2) \left[ \frac{E(k)}{k(1 - k^2)} - \frac{F(k)}{k} \right]
\]
\[
= k F(k).
\]
于是，
\[
E(k) - k_1^2 F(k) = \int_0^k k F(k) \, dk + C,
\]
其中 \( C \) 为常数。但当 \( k = 0 \) 时，上式左端为 \( E(0) - F(0) = \frac{\pi}{2} - \frac{\pi}{2} = 0 \)，而右端等于 \( C \)，故 \( C = 0 \)。最后证得
\[
\int_0^k k F(k) \, dk = E(k) - k_1^2 F(k).
\]

(6) 由于
\[
\frac{1}{3} \left[ (1 + k^2) E(k) - k_1^2 F(k) \right] = \frac{1}{3} \left[ 2k E(k) + (1 + k^2) E'(k) + 2k F(k) \right]
\]
\[-(1-k^2)F'(k)\]
\[= \frac{1}{3} \left\{ 2k E(k) + (1+k^2) \cdot \frac{E(k) - F(k)}{k} \right\}
+ 2k F(k) - (1-k^2) \cdot \left[ \frac{E(k)}{k(1-k^2)} - \frac{F(k)}{k} \right] \]
\[= k E(k), \]

故
\[\frac{1}{3} ((1+k^2)E(k) - k^2 F(k)) = \int_0^1 k E(k) \, dk + C,\]

以 \(k = 0\) 代入上式，得 \(C = 0\). 于是，最后证得
\[\int_0^1 k E(k) \, dk = \frac{1}{3} ((1+k^2)E(k) - k^2 F(k)).\]

3740. 证明公式
\[\int_0^x x J_0(x) \, dx = x J_1(x),\]

其中 \(J_0(x)\) 及 \(J_1(x)\) 为足指数是 0 与 1 的贝塞尔函数
（参阅问题 3726）.

证
\[\int_0^x u J_0(u) \, du = \frac{1}{\pi} \int_0^x u \, du \int_0^\pi \cos(-u \sin \varphi) \, d\varphi\]
\[= \frac{1}{\pi} \int_0^x u \, du \int_0^\pi \cos(\varphi - u \sin \varphi) \cos \varphi \]
\[+ \sin(\varphi - u \sin \varphi) \sin \varphi \, d\varphi\]
\[= \frac{1}{\pi} \int_0^x u \, du \int_0^\pi u \cos(\varphi - u \sin \varphi) \cos \varphi \, d\varphi\]
\[+ \frac{1}{\pi} \int_0^x u \, du \int_0^\pi u \sin(\varphi - u \sin \varphi) \sin \varphi \, d\varphi\]

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\[
\begin{aligned}
&= \frac{1}{\pi} \int_0^\pi d\varphi \int_0^\pi \cos(\varphi - u \sin \varphi) d(u \sin \varphi) \\
&\quad + \frac{1}{\pi} \int_0^\pi d\varphi \int_0^\pi u \sin(\varphi - u \sin \varphi) d(u \sin \varphi - \varphi) \\
&= \frac{1}{\pi} \int_0^\pi du \int_0^\pi \cos(\varphi - u \sin \varphi) d(u \sin \varphi - \varphi) \\
&\quad + \frac{1}{\pi} \int_0^\pi du \int_0^\pi \cos(\varphi - u \sin \varphi) d\varphi \\
&\quad + \frac{1}{\pi} \int_0^\pi d\varphi \int_0^\pi u d \cos(\varphi - u \sin \varphi) \\
&= \frac{1}{\pi} \int_0^\pi du \int_0^\pi \cos(\varphi - u \sin \varphi) d\varphi \\
&\quad + \frac{1}{\pi} \int_0^\pi du \int_0^\pi \cos(\varphi - x \sin \varphi) d\varphi \\
&\quad - \frac{1}{\pi} \int_0^\pi d\varphi \int_0^\pi \cos(\varphi - u \sin \varphi) du \\
&= \frac{1}{\pi} \int_0^\pi du \int_0^\pi \cos(\varphi - u \sin \varphi) d\varphi \\
&\quad + \frac{1}{\pi} \int_0^\pi du \int_0^\pi x \cos(\varphi - x \sin \varphi) d\varphi \\
&\quad - \frac{1}{\pi} \int_0^\pi d\varphi \int_0^\pi \cos(\varphi - u \sin \varphi) d\varphi \\
&= \frac{1}{\pi} \int_0^\pi x \cos(\varphi - x \sin \varphi) d\varphi = x J_1(x),
\end{aligned}
\]

上述各式中的被积函数显然为 \( u \) 及 \( \varphi \) 的二元连续函数，因此，交换积分顺序是合理的。本题获证。
§ 2．带参数的广义积分。积分的一致收敛性

1° 一致收敛性的定义 若对于任何的 $\varepsilon > 0$，都存在有数 $B = B(\varepsilon)$，使得在 $b \geq B$ 的条件下有

$$\left| \int_{b}^{+\infty} f(x, y) \, dx \right| \leq \varepsilon \quad (\gamma_1 \leq y \leq \gamma_2),$$

则称广义积分

$$\int_{a}^{+\infty} f(x, y) \, dx$$

（其中函数 $f(x, y)$ 于域 $a \leq x \leq +\infty, \gamma_1 \leq y \leq \gamma_2$ 内是连续的）在区间 $(\gamma_1, \gamma_2)$ 内一致收敛。

积分 (1) 的一致收敛与形状如下的一切级数

$$\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x, y) \, dx$$

（其中 $a = a_0 < a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$ 且 $\lim_{n \to \infty} a_n = +\infty$）的一致收敛等价。

若积分 (1) 在区间 $(\gamma_1, \gamma_2)$ 中一致收敛，则在这个区间内它是参数 $y$ 的连续函数。

2° 哥白判别法则 积分 (1) 在区间 $(\gamma_1, \gamma_2)$ 内一致收敛的充分而且必要的条件为，对于任何的 $\varepsilon > 0$ 便存在有数 $B = B(\varepsilon)$，使得只要是 $b' \geq B$ 及 $b'' \geq B$ 则

当 $\gamma_1 \leq y \leq \gamma_2$ 时

$$\left| \int_{b'}^{b''} f(x, y) \, dx \right| \leq \varepsilon.$$ 

3° 外尔什特拉斯判别法 对于积分 (1) 一致收敛的
充分条件为，与参数 \( y \) 无关的强函数 \( F(x) \) 存在，使得

1. 当 \( a \leq x < +\infty \) 时 \( f(x, y) \leq F(x) \)
及

2. \( \int_{-\infty}^{+\infty} F(x) \, dx \leq +\infty \).

4° 对于不连续函数的广义积分有类似的定理。

求积分的收敛域：

3741. \( \int_{0}^{+\infty} \frac{e^{-ax}}{1+x^2} \, dx \).

解 当 \( a \geq 0 \) 时，

\[
\frac{e^{-ax}}{1+x^2} \leq \frac{1}{1+x^2}.
\]

而积分

\[
\int_{0}^{+\infty} \frac{dx}{1+x^2} = \arctan x \bigg|_{0}^{+\infty} = \frac{\pi}{2},
\]

故原积分收敛。

当 \( a < 0 \) 时，原积分显然发散。于是，积分

\[
\int_{0}^{+\infty} \frac{e^{-ax}}{1+x^2} \, dx
\]

的收敛域为 \( a \geq 0 \) 的一切 \( a \) 值。

3742. \( \int_{\infty}^{-\infty} \frac{x \cos x}{x^2 + x^2} \, dx \).

解 首先注意

\[
\left( \frac{x}{x^2 + x^2} \right)' = \frac{(1-p)x^2 + (1-q)x^2}{(x^2 + x^2)^2}.
\]
若 $\max(p, q) \geq 1$，则显然当 $x$ 充分大时，$\left(\frac{x}{x^p + x^q}\right)' \leq 0$，从而当 $x$ 充分大时函数 $\frac{x}{x^p + x^q}$ 是递减的，并且很明显，这时

$$
\lim_{x \to +\infty} \frac{x}{x^p + x^q} = 0.
$$

又因 $\left| \int_{\pi}^{A} \cos x \, dx \right| = |\sin A| \leq 1$ (对任何 $A \geq \pi$)，故知 $\int_{\pi}^{+\infty} \frac{x \cos x}{x^p + x^q} \, dx$ 收敛。

若 $\max(p, q) \leq 1$，则恒有 $\left(\frac{x}{x^p + x^q}\right)' \geq 0$，函数 $\frac{x}{x^p + x^q}$ 在 $x \geq \pi$ 上是递增的。于是，对于任何正整数 $n$，有

$$
\int_{2n\pi}^{2n\pi + \pi} \frac{x \cos x}{x^p + x^q} \, dx
\geq \frac{\sqrt{2}}{2} \int_{2n\pi}^{2n\pi + \pi} \frac{x}{x^p + x^q} \, dx
\geq \frac{\sqrt{2}}{2} \cdot \frac{\pi}{\pi^p + \pi^q} \cdot \frac{\pi}{4}
= \frac{\pi^2 \sqrt{2}}{8(\pi^p + \pi^q)} = \text{常数} \geq 0,
$$

故不满足柯西收敛准则，因此积分 $\int_{x}^{+\infty} \frac{x \cos x}{x^p + x^q} \, dx$
3743. \( \int_{0}^{+\infty} \frac{\sin x^2}{x^p} \, dx \).

解 若 \( q = 0 \)，则由于积分 \( \int_{a}^{+\infty} \frac{1}{x^q} \, dx \) 仅当 \( p > 1 \) 时收敛，而积分 \( \int_{0}^{+\infty} \frac{1}{x^p} \, dx \) 仅当 \( p < 1 \) 时收敛，故积分 \( \int_{0}^{+\infty} \frac{\sin x^2}{x^p} \, dx \) 对于任何的 \( p \) 值及 \( q = 0 \) 发散。

若 \( q \neq 0 \)，则积分
\[
\int_{0}^{+\infty} \frac{\sin x^2}{x^p} \, dx = \int_{0}^{+\infty} x^{-q} \sin x^2 \, dx,
\]

利用 2380 题的结果即知：当 \( \frac{1 - p}{q} \) 时，原积分收敛。

3744. \( \int_{0}^{2} \frac{dx}{|\ln x|^p} \).

解 考虑积分
\[
\int_{0}^{1} \frac{dx}{|\ln x|^p} = \int_{0}^{1} \frac{dx}{\ln(\frac{1}{x})^p} = \int_{0}^{1} \ln^{-p}(\frac{1}{x}) \, dx,
\]

利用 2362 题的结果即知：它当 \( -p > -1 \) 或 \( p < 1 \) 时收敛。

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再考虑积分
\[ \int_1^2 \frac{dx}{|\ln x|^p} = \int_1^2 \frac{dx}{|\ln x|^p}. \]

由于
\[ \lim_{x \to 1^+} (x-1)^p \cdot \frac{1}{\ln x} = \left[ \lim_{x \to 1^+} \frac{x-1}{\ln x} \right]^p = \left[ \lim_{x \to 1^+} \frac{1}{x-1} \right]^p = 1, \]

故积分 \( \int_1^2 \frac{dx}{|\ln x|^p} \) 与积分 \( \int_1^2 \frac{dx}{(x-1)^p} \) 具有相同的敛散性。而后者显然当 \( p < 1 \) 时收敛，\( p \geq 1 \) 时发散，从而前者亦然。

于是，仅当 \( p < 1 \) 时，积分
\[ \int_0^1 \frac{dx}{|\ln x|^p} \]
收敛。

3745. \( \int_0^1 \frac{\cos \frac{1}{1-x^2}}{\sqrt{1-x^2}} \, dx \).

解 \( \int_0^1 \frac{\cos \frac{1}{1-x^2}}{\sqrt{1-x^2}} \, dx = \int_0^1 \frac{\cos \frac{1}{1-x^2}}{\sqrt{1-x} \cdot \sqrt{1+x}} \, dx \).

由于当 \( 0 \leq x \leq 1 \) 时，对于任意的 \( n, \frac{1}{\sqrt{1+x}} \) 与
\[ \frac{1}{\sqrt{1+x}} \]
都是单调有界函数，故原积分与积分

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\[
\int_0^1 \frac{\cos \frac{1}{\sqrt{1-x^2}}}{\sqrt{1-x^2}} \, dx
\]

同敛散，对此积分作代换 \( t = \frac{1}{1-x^2} \)，则得

\[
\int_0^1 \frac{\cos \frac{1}{\sqrt{1-x^2}}}{\sqrt{1-x^2}} \, dx = \int_1^{+\infty} \frac{\cos t}{t^{2-\frac{1}{n}}} \, dt.
\]

易知积分 \( \int_1^{+\infty} \frac{\cos t}{t^{2-\frac{1}{n}}} \, dt \) 仅当 \( \alpha > 0 \) 时收敛。事实上，
当 \( \alpha > 0 \) 时它显然收敛。当 \( \alpha = 0 \) 时它显然发散。当 \( \alpha < 0 \) 时，令 \( \beta = -\alpha (\beta > 0) \)，则对于正整数 \( n \) 有

\[
\int_{2\pi}^{2\pi+\frac{\pi}{4}} \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} + \infty (n \to \infty),
\]

故积分 \( \int_1^{+\infty} t^{\cos t} \, dt \) 发散。

于是，积分

\[
\int_0^1 \frac{\cos \frac{1}{\sqrt{1-x^2}}}{\sqrt{1-x^2}} \, dx
\]

仅当 \( 2 - \frac{1}{n} > 0 \) 时收敛，即仅当 \( n < 0 \) 或 \( n > \frac{1}{2} \) 时收敛。
3746. \( \int_0^{+\infty} \frac{\sin x}{x^p + \sin x} \, dx \) (\( p \gg 0 \)).

解 因为

\[
\lim_{x \to 0} \frac{\sin x}{x^p + \sin x} = \lim_{x \to 0} \frac{\sin x}{x^{p-1}} = \begin{cases} 
1, & \text{当 } p \gg 1 \text{ 时;} \\
\frac{1}{2}, & \text{当 } p = 1 \text{ 时;} \\
0, & \text{当 } 0 \leq p \leq 1 \text{ 时,}
\end{cases}
\]

故 \( x = 0 \) 不是积分 \( \int_0^{+\infty} \frac{\sin x}{x^p + \sin x} \, dx \) 的瑕点，因此，只要讨论积分 \( \int_2^{+\infty} \frac{\sin x}{x^p + \sin x} \, dx \) (\( p \gg 0 \)) 的敛散性。

由于

\[
\frac{\sin x}{x^p + \sin x} = \frac{\sin x}{x^p} - \frac{\sin^2 x}{x^p(x^p + \sin x)},
\]

而 \( \int_2^{+\infty} \sin x \, dx \) 收敛 (当 \( p \gg 0 \))，故只要讨论

\[
\int_2^{+\infty} \frac{\sin^2 x}{x^p(x^p + \sin x)} \, dx
\]

的敛散性。但当 \( p \gg 0, \ x \gg 2 \) 时，

\[
0 \leq \frac{1}{2} \left[ \frac{1}{x^p(x^p + 1)} - \frac{\cos 2x}{x^p(x^p + 1)} \right] = \frac{\sin^2 x}{x^p(x^p + 1)} \leq \frac{\sin^2 x}{x^p(x^p + \sin x)}
\]

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\[
\frac{\sin^2 x}{x^p(x^p-1)} \leq \frac{1}{x^p(x^p-1)}.
\]

而易知 \( \int_2^{+\infty} \frac{\cos 2x}{x^p(x^p+1)} \, dx \) 恒收敛 (当 \( p \geq 0 \) 时)，积分
\[
\int_2^{+\infty} \frac{dx}{x^p(x^p+1)}
\]
当 \( 0 \leq p \leq \frac{1}{2} \) 时发散，积分
\[
\int_2^{+\infty} \frac{dx}{x^p(x^p-1)}
\]
当 \( p > \frac{1}{2} \) 时收敛，故积分
\[
\int_2^{+\infty} \frac{\sin^2 x}{x^p(x^p+\sin x)} \, dx
\]
当 \( p > \frac{1}{2} \) 时收敛，当 \( 0 < p \leq \frac{1}{2} \) 时发散。由此可知，积分
\[
\int_0^{+\infty} \frac{\sin x}{x^p+\sin x} \, dx
\]
（\( p > 0 \)）仅当 \( p > \frac{1}{2} \) 时收敛。

利用与级数比较的方法研究下列积分的收敛性：

3747. \( \int_0^{+\infty} \frac{\cos x}{x+a} \, dx \)

解 设 \( a > 0 \)。我们证明：对任何数列
\[
0 = a_0 < a_1 < a_2 < \cdots < a_n < \cdots \quad (a_n \to +\infty),
\]
级数
\[
\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \frac{\cos x}{x+a} \, dx
\]
都收敛。事实上，有
\[
\int_{a_n}^{a_{n+1}} \frac{\cos x}{x+a} \, dx
\]
\[
= \frac{\sin x}{x+a} \bigg|_{a_n}^{a_{n+1}} + \int_{a_n}^{a_{n+1}} \frac{\sin x}{(x+a)^2} \, dx,
\]
\[\text{574}\]
故
\[
\sum_{n=m}^{m+p-1} \int_{a_n}^{a_{n+1}} \frac{\cos x}{x+a} dx = \frac{\sin a_{m+p}}{a_{n+p} + a} - \frac{\sin a_m}{a_{m} + a} + \int_{a_m}^{a_{m+p}} \frac{\sin x}{(x+a)^2} dx,
\]

从而
\[
\left| \sum_{n=m}^{m+p-1} \int_{a_n}^{a_{n+1}} \frac{\cos x}{x+a} dx \right| \leq \frac{1}{a_{n+p} + a} + \frac{1}{a_{m} + a} + \int_{a_m}^{a_{m+p}} \frac{dx}{(x+a)^2} = \frac{1}{a_{n+p} + a} + \frac{1}{a_{m} + a} + \left( \frac{1}{a_{m} + a} - \frac{1}{a_{m+p} + a} \right) = \frac{2}{a_{n} + a},
\]

由此可知，满足柯西收敛准则，从而级数
\[
\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} \frac{\cos x}{x+a} dx
\]
收敛，因此，积分\(\int_{0}^{\infty} \frac{\cos x}{x+a} dx\)收敛。

若\(a = 0\)，显然瑕积分\(\int_{0}^{\frac{\pi}{2}} \frac{\cos x}{x} dx\)发散，故广义积分\(\int_{0}^{\infty} \frac{\cos x}{x} dx\)发散。

下设\(a<0\)，若\(a = -(n+\frac{1}{2})\pi\) (\(n = 0, 1, 2, \ldots\))，则

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\[
\int_{0}^{\infty} \frac{\cos x}{x+a} \, dx \\
= \int_{0}^{(n+1)\pi} \frac{\cos x}{x+a} \, dx + \int_{(n+1)\pi}^{\infty} \frac{\cos x}{x+a} \, dx \\
= \int_{0}^{(n+1)\pi} \frac{\cos x}{x+a} \, dx + (-1)^{n+1} \int_{0}^{\infty} \frac{\cos x}{x+a} \, dx.
\]

由上所证，右端第二个积分收敛；又由于
\[
\lim_{x \to (n+\frac{1}{2})\pi} \frac{\cos x}{x+a} = (-1)^{n+1},
\]
故右端第一个积分收敛（它不是广义积分，补充定义被积函数在 \(x=(n+\frac{1}{2})\pi\) 时的值为 \((-1)^{n+1}\) 后即为连续函数的积分）；从而，此时积分 \(\int_{0}^{\infty} \frac{\cos x}{x+a} \, dx\) 收敛。

若 \(a < 0\) 但 \(a \neq -(n+\frac{1}{2})\pi\) \((n=0, 1, 2, \ldots)\)，此时 \(\cos(-a) \neq 0\)。由连续性，可取 \(\delta > 0\)，使当 \(-a \leq x \leq -a + \delta\) 时 \(\cos x\) 保持定号且
\[
|\cos x| \geq \frac{1}{2} |\cos(-a)|.
\]

于是，
\[
\left| \int_{-\infty}^{-a+\delta} \frac{\cos x}{x+a} \, dx \right| \\
\geq \frac{1}{2} |\cos(-a)| \cdot \int_{-a}^{-a+\delta} \frac{dx}{x+a} = +\infty.
\]
由此可知，瑕积分 \( \int_{-a}^{a} \frac{\cos x}{x+a} \, dx \) 发散。从而积分

\[ \int_{0}^{+\infty} \frac{\cos x}{x+a} \, dx \] 更是发散。

综上所述，积分

\[ \int_{0}^{+\infty} \frac{\cos x}{x+a} \, dx \]

仅当 \( a > 0 \) 及 \( a = -(n+\frac{1}{2})\pi \) \((n = 0, 1, 2, \ldots)\)
时收敛。

3748. \[ \int_{0}^{+\infty} \frac{x \, dx}{1 + x^n \sin^2 x} \quad (n > 0) \]

解 由于被积函数非负，故只要考虑化为一种特殊的（正项）级数即可。我们有

\[ \int_{0}^{+\infty} \frac{x \, dx}{1 + x^n \sin^2 x} \, dx \]

\[ = \int_{0}^{\frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \]

\[ + \sum_{k=1}^{\infty} \int_{(k-1)\frac{\pi}{4}}^{\frac{\pi}{4} + \frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \]

\[ + \sum_{k=1}^{\infty} \int_{(k-1)\frac{\pi}{4}}^{\frac{\pi}{4} + \frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \]

又积分

\[ 0 < \int_{(k-1)\frac{\pi}{4}}^{\frac{\pi}{4} + \frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \]
\[ A \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \frac{h \pi \, dx}{1 + [(k-1)\pi]^n \sin^2 x}, \]
\[ = \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \left( \frac{(k-1)\pi \, dx}{1 + [(k-1)\pi]^n \sin^2 x} \right), \]
\[ = A \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x}, \]
\[ = A \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \frac{(k+1)\pi \, dx}{1 + [(k+1)\pi]^n \sin^2 x}. \]

\[ H \]
\[ \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \frac{dx}{1 + a^2 \sin^2 x} = \frac{-1}{\sqrt{1 + a^2}} \arctg \left( \frac{\ctg x}{\sqrt{1 + a^2}} \right) \bigg|_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \]
\[ = \frac{2}{\sqrt{1 + a^2}} \arctg \frac{1}{\sqrt{1 + a^2}} = \frac{2}{\sqrt{1 + a^2}} \cdot \frac{\pi}{4} \]
\[ = \frac{\pi}{2 \sqrt{1 + a^2}}, \]
\[ \int_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \frac{dx}{1 + a^2 \sin^2 x} = \frac{1}{\sqrt{1 + a^2}} \arctg \left( \sqrt{1 + a^2} \tan x \right) \bigg|_{(k-1)\pi + \frac{\pi}{4}}^{ik+\frac{\pi}{4}} \]
\[ = \frac{2}{\sqrt{1 + a^2}} \arctg \sqrt{1 + a^2}. \]
由于

\[ \frac{\pi}{4} < \arctg \sqrt{1 + a^2} \leq \frac{\pi}{2} \]

从而

\[ \frac{\pi}{2 \sqrt{1 + a^2}} < \int_{x = -\frac{\pi}{4}}^{x = \frac{\pi}{4}} \frac{dx}{1 + a^2 \sin^2 x} \leq \frac{\pi}{\sqrt{1 + a^2}} \]

于是，

\[ 0 < \int_{(k-1) \pi + \frac{\pi}{4}}^{(k+1) \pi - \frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \leq \frac{(k+1) \pi^2}{2 \sqrt{1 + (k-1) \pi^2}^n} \]

\[ \frac{(k-1) \pi^2}{2 \sqrt{1 + (k+1) \pi^2}^n} \]

由于当 \( n \geq 4 \) 时，级数 \( \sum_{i=1}^{\infty} \frac{k \pi^2}{2 \sqrt{1 + [(k-1) \pi]^n}} \) 及

\[ \sum_{i=1}^{\infty} \frac{(k+1) \pi^2}{\sqrt{1 + [(k-1) \pi]^n}} \] 收敛；而当 \( n \leq 4 \) 时，级数

\[ \sum_{i=1}^{\infty} \frac{(k-1) \pi^2}{2 \sqrt{1 + [(k+1) \pi]^n}} \] 发散，故级数

\[ \sum_{i=1}^{\infty} \int_{(k-1) \pi + \frac{\pi}{4}}^{(k+1) \pi - \frac{\pi}{4}} \frac{x \, dx}{1 + x^n \sin^2 x} \]

当 \( n \geq 4 \) 时收敛，而级数

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\[
\sum_{i=1}^{\infty} \int_{t\frac{1}{4}}^{t\frac{1}{4}+\frac{1}{4}} \frac{x \, dx}{1+x^n \sin^2 x}
\]

仅当 \( n \gg 4 \) 时收敛。

因此，积分

\[
\int_0^{+\infty} \frac{x \, dx}{1+x^n \sin^2 x}
\]

仅当 \( n \gg 4 \) 时收敛。

3749. \[ \int_x^{+\infty} \frac{dx}{x^\frac{1}{3} \sqrt[3]{\sin^2 x}}. \]

解 由于被积函数非负，故只要考虑化为一种特殊的（正项）级数即可。我们有

\[
\int_x^{+\infty} \frac{dx}{x^\frac{1}{3} \sqrt[3]{\sin^2 x}} = \sum_{i=1}^{\infty} \int_{x}^{(n+1)x} \frac{dx}{x^\frac{1}{3} \sqrt[3]{\sin^2 x}}
\]

\[
= \sum_{i=1}^{\infty} \int_0^{(n+1)x} \frac{dx}{(x+n\pi)^\frac{1}{3} \sqrt[3]{\sin^2 x}}.
\]

于是，

\[
\int_0^{x} \frac{dx}{\sqrt[3]{\sin^2 x}} \cdot \sum_{n=1}^{\infty} \frac{1}{(n+1)^p \pi^p}
\]

\[
\int_x^{+\infty} \frac{dx}{x^\frac{1}{3} \sqrt[3]{\sin^2 x}}.
\]

\[
\int_0^{+\infty} \frac{dx}{\sqrt[3]{\sin^2 x}} \cdot \sum_{n=1}^{\infty} \frac{1}{n^p \pi^p}.
\]

易证积分

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\[
\int_0^\infty \frac{dx}{\sqrt[3]{\sin^2 x}}
\]
收敛，且级数
\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]
当 \( p > 1 \) 时收敛；当 \( p \leq 1 \) 时发散。因此，原积分仅当 \( p > 1 \) 时收敛。
3750. \[ \int_0^{+\infty} \frac{\sin(x+x^2)}{x^n} \] dx.
解 我们有
\[
\int_0^{+\infty} \frac{\sin(x+x^2)}{x^n} \] dx
\[
= \int_0^1 \frac{\sin(x+x^2)}{x^n} \] dx + \int_1^{+\infty} \frac{\sin(x+x^2)}{x^n} \] dx.
易知右端第一个积分 (x=0可能是瑕点) 当 \( n < 2 \) 时收敛，当 \( n \geq 2 \) 时发散。下面研究右端第二个积分。先设 \( n \geq -1 \)。对任何序列
\[
1 = a_0 < a_1 < \cdots < a_k < \cdots (a_k \to +\infty),
\]
\[
\int_{a_k}^{a_{k+1}} \frac{\sin(x+x^2)}{x^n} \] dx
\[
= - \int_{a_k}^{a_{k+1}} \frac{d[\cos(x+x^2)]}{x^n(1+2x)}
\]
\[
= - \frac{\cos(x+x^2)}{x^n(1+2x)} \bigg|_{a_k}^{a_{k+1}}
\]
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\[- \int_{a_k}^{a_{k+1}} \frac{2(n+1)x+n}{x^{n+1}(1+2x^2)} \cos(x+x^2) \, dx, \]

故

\[
\sum_{k=m}^{m+p-1} \int_{a_k}^{a_{k+1}} \frac{\sin(x+x^2)}{x^n} \, dx
\]

\[- \frac{\cos(x+x^2)}{x^n(1+2x)} \bigg|_{a_m}^{a_{m+p}}
\]

\[- \int_{a_m}^{a_{m+p}} \frac{2(n+1)x+n}{x^{n+1}(1+2x^2)} \cos(x+x^2) \, dx, \]

从而

\[
\left| \sum_{k=m}^{m+p-1} \int_{a_k}^{a_{k+1}} \frac{\sin(x+x^2)}{x^n} \, dx \right|
\]

\[
\leq \frac{1}{2a_{n+1}^2} + \frac{1}{2(a_{n+1})^2} + \int_{a_m}^{a_{m+p}} \frac{2(n+1)x+n}{x^{n+1}(1+2x^2)} \, dx.
\]

易知积分 \( \int_1^{+\infty} \frac{2(n+1)x+n}{x^{n+1}(1+2x^2)} \, dx \) 收敛（因为

\[
\lim_{x \to +\infty} \frac{2(n+1)x+n}{x^{n+1}(1+2x^2)} = \frac{n+1}{2} \geq 0,
\]

\( n+2 \geq 1 \).

由此可知，对任给的 \( \varepsilon > 0 \)，必存在 \( N \)，使当 \( n \geq N \) 时，对 \( p = 1, 2, 3, \ldots \)，均有

\[
\left| \sum_{k=m}^{m+p-1} \int_{a_k}^{a_{k+1}} \frac{\sin(x+x^2)}{x^n} \, dx \right| \leq \varepsilon.
\]

于是，根据柯西收敛准则，级数

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\[
\sum_{k=0}^{\infty} \int_{\xi_k}^{\eta_k} \frac{\sin(x+x^2)}{x^n} \, dx
\]
收敛，从而积分 \( \int_{1}^{+\infty} \frac{\sin(x+x^2)}{x^n} \, dx \) 收敛。

再设 \( n \leq -1 \)。令 \( \xi_k \) 和 \( \eta_k \) 分别表示方程 \( x^2 + x = 2k\pi + \frac{\pi^k}{4} \) 和 \( x^2 + x = 2k\pi + \frac{\pi}{2} \) 的（唯一）正根，其中 \( k = 1, 2, 3, \ldots \)；即令

\[
\xi_k = \frac{1}{2} (\sqrt{1 + 3k\pi + \pi} - 1),
\]

\[
\eta_k = \frac{1}{2} (\sqrt{1 + 3k\pi + 2\pi} - 1).
\]

于是 \( \eta_k \gg \xi_k \to +\infty \)（当 \( k \to \infty \) 时），我们有（注意 \( -n \gg 1 \)）

\[
\int_{\xi_k}^{\eta_k} \frac{\sin(x+x^2)}{x^n} \, dx
\]

\[
\geq \frac{1}{\sqrt{2}} \int_{\xi_k}^{\eta_k} x^{-n} \, dx \geq \frac{1}{\sqrt{2}} \int_{\xi_k}^{\eta_k} x \, dx
\]

\[
\geq \frac{1}{\sqrt{2}} \xi_k (\eta_k - \xi_k)
\]

\[
= \frac{\pi}{4\sqrt{2}} \cdot \frac{\sqrt{1 + 8k\pi + \pi} - 1}{\sqrt{1 + 8k\pi + 2\pi} + \sqrt{1 + 8k\pi + \pi}}
\]

\[
\to \frac{\pi}{8\sqrt{2}} \quad \text{（当} \ k \to \infty \text{时）。}
\]

由此可知，此时积分 \( \int_{1}^{+\infty} \frac{\sin(x+x^2)}{x^n} \, dx \) 发散。
综上所述，积分
\[ \int_0^{+\infty} \frac{\sin(x+x^2)}{x^n} \, dx \]
仅当 \(-1 < n < 2\) 时收敛。

3751. 在肯定的意义上表达出来，甚至是积分
\[ \int_a^{+\infty} f(x, y) \, dx \]
在已知区间 \((y_1, y_2)\) 内不一致收敛？
解 若对于某个正数 \(\varepsilon_0\)，不论 \(B\) 取得多大，恒存在 \(b_0 \geq B\) 以及 \(y_0 \in (y_1, y_2)\) (\(b_0\) 与 \(y_0\) 都依赖于 \(B\))，使得
\[ \left| \int_{b_0}^{+\infty} f(x, y_0) \, dx \right| \geq \varepsilon_0, \]
则 \(\int_a^{+\infty} f(x, y) \, dx\) 在区间 \((y_1, y_2)\) 内不一致收敛。

3752. 证明：若 1）积分
\[ \int_a^{+\infty} f(x) \, dx \]
收敛，2）函数 \(\phi(x, y)\) 有界并关于 \(x\) 是单调的，则积分
\[ \int_a^{+\infty} f(x) \phi(x, y) \, dx \]
一致收敛（在对应的域内）。
证 设 \(|\phi(x, y)| \leq L\)，则由题设 1）知：对于任给的 \(\varepsilon > 0\)，总存在数 \(B = B(\varepsilon)\)，使当 \(A' \geq A \geq B\) 时，就
有不等式

\[
\left| \int_A^{A'} f(x) \, dx \right| \leq \frac{\epsilon}{2L}.
\]  \hspace{1cm} (1)

由积分中值定理知：存在 \( \xi \in (A, A') \)，使有下述等式

\[
\int_A^{A'} f(x) \varphi(x, y) \, dx = \varphi(A + 0, y) \cdot \int_A^\xi f(x) \, dx + \varphi(A' - 0, y) \cdot \int_\xi^{A'} f(x) \, dx. \]  \hspace{1cm} (2)

由 (1) 式，得

\[
\left| \int_A^\xi f(x) \, dx \right| \leq \frac{\epsilon}{2L}, \quad \left| \int_\xi^{A'} f(x) \, dx \right| \leq \frac{\epsilon}{2L}.
\]

于是，由 (2) 式，可得

\[
\left| \int_A^{A'} f(x) \varphi(x, y) \, dx \right|
\]

\[\leq L \cdot \frac{\epsilon}{2L} + L \cdot \frac{\epsilon}{2L} = \epsilon, \]

即积分 \( \int_a^{+\infty} f(x) \varphi(x, y) \, dx \) 在对应的 \( y \) 域内一致收敛。

3753. 证明：一致收敛的积分

\[
I = \int_1^{+\infty} e^{-\frac{1}{y^2}} \left( x - \frac{1}{y} \right)^2 \, dx \quad (0 \leq y \leq 1)
\]
不能以与参数无关的收敛积分为强函数。
证 任给 \( \varepsilon > 0 \)。取 \( A_0 > 1 \) 充分大，使

\[
\int_{A_0}^{+\infty} \frac{\sqrt{x}}{e} e^{-u^2} du < \varepsilon.
\]

下证：当 \( A > A_0 \) 时，对一切 \( 0 < y < 1 \)，均有

\[
\int_{A}^{+\infty} e^{-\frac{1}{y^2}} (x - \frac{1}{y})^2 dx < \varepsilon.
\]

事实上，当 \( \frac{\varepsilon}{\sqrt{\pi}} < y < 1 \) 时，

\[
\int_{A}^{+\infty} e^{-\frac{1}{y^2}} (x - \frac{1}{y})^2 dx < \int_{A}^{+\infty} e^{-(x - \frac{1}{y})^2} dx
\]

\[
= \int_{A}^{+\infty} e^{-u^2} du \leq \int_{A}^{+\infty} \frac{\sqrt{\pi}}{e} e^{-u^2} du
\]

\[
\leq \int_{A_0}^{+\infty} \frac{\sqrt{\pi}}{e} e^{-u^2} du < \varepsilon.
\]

当 \( 0 < y < \frac{\varepsilon}{\sqrt{\pi}} \) 时，

\[
\int_{A}^{+\infty} e^{-\frac{1}{y^2}} (x - \frac{1}{y})^2 dx
\]

\[
= \int_{1}^{+\infty} e^{-\frac{1}{y^2}} (x - \frac{1}{y})^2 dx
\]

\[
= \int_{1}^{\frac{1}{y}} e^{-\frac{1}{y^2}} (x - \frac{1}{y})^2 dx
\]

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\[
+ \int_{\frac{1}{y}}^{+\infty} e^{-\frac{1}{y^2}(x - \frac{1}{y})^2} \, dx
\]

\[
= \int_0^{\frac{1}{y^2}} e^{-\frac{1}{y^2}t^2} \, dt + \int_{\frac{1}{y^2}}^{+\infty} e^{-\frac{1}{y^2}t^2} \, dt
\]

\[
\leq 2 \int_0^{+\infty} e^{-t^2} \, dt = 2y \int_0^{+\infty} e^{-u^2} \, du
\]

\[
= 2y \cdot \frac{\sqrt{\pi}}{2} = \varepsilon.
\]

由此可知，积分 \( \int_1^{+\infty} e^{-\frac{1}{y^2}(x - \frac{1}{y})^2} \, dx \) 在 \( 0 \leq y \leq 1 \) 上一致收敛。

最后证明，不存在这样的函数 \( \varphi(x) \) \((x \geq 1)\)，使

\[
0 \leq e^{-\frac{1}{y^2}(x - \frac{1}{y})^2} \leq \varphi(x)
\]

\((x \geq 1, \ 0 \leq y \leq 1)\) ，（1）

并且 \( \int_1^{+\infty} \varphi(x) \, dx \) 收敛。用反证法。假定有这样的函数 \( \varphi(x) \) 存在，则由 \( \int_1^{+\infty} \varphi(x) \, dx \) 的收敛性可知，必存在点 \( x_0 \geq 1 \) 使 \( \varphi(x_0) < 1 \) 。于是，令 \( y_0 = \frac{1}{x_0} \)，则 \( 0 < y_0 < 1 \) 且

\[
e^{-\frac{1}{y_0^2} \left( x_0 - \frac{1}{y_0} \right)^2} = 1 \Rightarrow \varphi(x_0),
\]

此显然与（1）式矛盾。由此可知，一致收敛的积分
I 的被积函数不能以与参数 \( y \) 无关的具收敛积分的函数为强函数，证毕。

3754. 证明：积分

\[ I = \int_{0}^{+\infty} \alpha e^{-\alpha x} dx \]

1）任何区间 \( 0 \leq a \leq b \) 内一致收敛；2）区间 \( 0 \leq a \leq b \) 内非一致收敛。

证 显然，积分 \( I \) 对于每一个定值 \( \alpha \geq 0 \) 是收敛的。

事实上，当 \( \alpha = 0 \) 时，\( \int_{0}^{+\infty} \alpha e^{-\alpha x} dx = 0 \)；当 \( \alpha > 0 \)
时，\( \int_{0}^{+\infty} \alpha e^{-\alpha x} dx = -e^{-\alpha x} \bigg|^{+\infty}_{0} = 1 \)。

1）如果 \( 0 \leq a \leq b \)，则由于

\[ 0 \leq \int_{A}^{+\infty} \alpha e^{-\alpha x} dx = e^{-\alpha A} \leq e^{-\alpha A} \]

故对于任给的 \( \varepsilon > 0 \)，可以找到不依赖于 \( \alpha \) 的数

\[ A_{0} = \frac{1}{\alpha} \ln \frac{1}{e} \]

使当 \( A > A_{0} \) 时，就有

\[ \int_{A}^{+\infty} \alpha e^{-\alpha x} dx < e^{-\alpha A_{0}} = \varepsilon \]

于是，在区间 \( 0 \leq a \leq b \) 上积分 \( I \) 一致收敛。

2）如果 \( 0 \leq a \leq b \)，则不存在这样的数 \( A_{0} \)。事实上，取 \( 0 \leq \varepsilon \leq 1 \) 就办不到。由于当 \( \alpha \to +0 \) 时，

\( e^{-\alpha x} \to 1 \)，故对于足够小的 \( \alpha \) 值，\( e^{-\alpha} \) 就比任意一个
小于 1 的数 \( \varepsilon \) 为大。因此，在区间 \( 0 \leq a \leq b \) 上，积
分1对\(\alpha\)的收敛是不一致的。

5755. 证明题里把积分

\[ I = \int_{0}^{+\infty} \frac{\sin \alpha x}{x} \, dx \]

1）在每一个不含数值 \(\alpha = 0\) 的闭区间\([a, b]\)上一致收敛，2）在含数值 \(\alpha = 0\) 的每一个闭区间\([a, b]\)上非一致收敛。

证 不失一般性，我们只考虑\(\alpha\)的正值。

1）由于积分

\[ \int_{0}^{+\infty} \frac{\sin z}{z} \, dz = \frac{\pi}{2} \]

是收敛的，故对于任给的 \(\varepsilon > 0\)，存在数 \(A_0\)，使当 \(A > A_0\)时，恒有

\[ \left| \int_{A}^{+\infty} \frac{\sin z}{z} \, dz \right| < \varepsilon. \]

当 \(\alpha > 0\) 时，由于

\[ \int_{A}^{+\infty} \frac{\sin \alpha x}{x} \, dx = \int_{A\alpha}^{+\infty} \frac{\sin z}{z} \, dz, \]

故取 \(A > A_0 \alpha\)，对于 \(\alpha > a > 0\)，就有

\[ \left| \int_{A}^{+\infty} \frac{\sin \alpha x}{x} \, dx \right| < \varepsilon. \]

于是，在区间 \(0 < a < \alpha < b\)上，积分 \(I\)是一致收敛的。

2）对于任何的 \(A > 0\)，当 \(a \to +0\) 时,
\[
\int_{\alpha}^{+\infty} \frac{\sin \alpha x}{x} \, dx = \int_{\frac{\alpha}{\pi}}^{+\infty} \frac{\sin z}{z} \, dz \to \int_{0}^{+\infty} \frac{\sin z}{z} \, dz = \frac{\pi}{2}.
\]

因此，当 \( \alpha \to 0 \) 且充分小时，有

\[
\int_{\alpha}^{+\infty} \frac{\sin \alpha x}{x} \, dx \approx \frac{\pi}{4}.
\]

于是，在区间 \( 0 \leq a \leq b \ (b > 0) \) 上，积分 \( I \) 不一致收敛。

研究下列积分在所指定区间内的一致收敛性：

3756. \( \int_{0}^{+\infty} e^{-\alpha x} \sin x \, dx \ \ (0 \leq \alpha_0 \leq \alpha \leq +\infty) \).

解 由于当 \( 0 \leq \alpha_0 \leq \alpha \leq +\infty \) 时，

\[
|e^{-\alpha x} \sin x| \leq e^{-\alpha_0 x},
\]

且积分 \( \int_{0}^{+\infty} e^{-\alpha x} \, dx = -\frac{1}{\alpha} \) 收敛，故积分

\[
\int_{0}^{+\infty} e^{-\alpha x} \sin x \, dx
\]

在区间 \( 0 \leq \alpha_0 \leq \alpha \leq +\infty \) 上一致收敛。

3757. \( \int_{1}^{+\infty} x^a e^{-x} \, dx \ (a \leq a \leq b) \).

解 当 \( a \leq \alpha \leq b \) 且 \( x \geq 1 \) 时，

\[0 \leq x^a e^{-x} \leq x^b e^{-x}.
\]

由于
\[
\lim_{x \to +\infty} x^2 \cdot x^b e^{-s} = \lim_{x \to +\infty} \frac{x^{b+2}}{e^x} = 0,
\]

故积分 \(\int_1^{+\infty} x^b e^{-s} \, dx\) 收敛，从而积分
\[
\int_1^{+\infty} x^a e^{-s} \, dx
\]

在区间 \(a \leq a \leq b\) 上一致收敛。

3758. \(\int_{-\infty}^{+\infty} \frac{\cos \alpha x}{1 + x^2} \, dx\) \((-\infty < a < +\infty)\).

解 由于 \(|\frac{\cos \alpha x}{1 + x^2}| \leq \frac{1}{1 + x^2}\)，且积分 \(\int_{-\infty}^{+\infty} \frac{d x}{1 + x^2}\)

= \pi 收敛，故积分
\[
\int_{-\infty}^{+\infty} \frac{\cos \alpha x}{1 + x^2} \, dx
\]

在 \(-\infty < a < +\infty\) 上一致收敛。

3759. \(\int_0^{+\infty} \frac{d x}{(x + a)^2 + 1} \) \((0 \leq a \leq +\infty)\).

解 由于 \(0 \leq \frac{1}{(x + a)^2 + 1} \leq \frac{1}{1 + x^2} \) \((0 \leq a \leq +\infty)\)，

且积分 \(\int_0^{+\infty} \frac{d x}{1 + x^2} = \frac{\pi}{2}\) 收敛，故积分
\[
\int_0^{+\infty} \frac{d x}{(x + a)^2 + 1}
\]

在 \(0 \leq a \leq +\infty\) 上一致收敛。
3760. \( \int_{0}^{+\infty} \frac{\sin x}{x} e^{-ax} \, dx \) \( (0 \leq a \leq +\infty) \).

解 首先注意，因为

\[
\lim_{x \to 0} \frac{\sin x}{x} e^{-ax} = 1,
\]

故 \( x = 0 \) 不是瑕点。

证法一

由于 \( \left| \int_{0}^{t} \sin x \, dx \right| = \left| 1 - \cos t \right| \leq 2 \) ，而当 \( 0 \leq a \leq +\infty \) 时，函数 \( \frac{e^{-ax}}{x} \) 在 \( x \to 0 \) 关于 \( x \) 递减，并且当 \( x \to +\infty \) 时它关于 \( a \) 一致趋于零（因为 \( 0 \leq a \leq +\infty \)，\( x \gg 0 \) 时，\( 0 \leq \frac{e^{-ax}}{x} \leq \frac{1}{x} \)），故由

蝶里黑里判别法知积分 \( \int_{0}^{+\infty} \frac{\sin x}{x} e^{-ax} \, dx \) 在 \( 0 \leq a \leq +\infty \) 上一致收敛。

证法二

由积分学第二中值定理知：当 \( A' > A > 0 \) 时，

\[
\left| \int_{A}^{A'} \frac{\sin x}{x} e^{-ax} \, dx \right| = \frac{1}{A} \int_{A'}^{A} e^{-ax} \sin x \, dx
\]

其中 \( A < \xi < A' \)。我们知道 \( e^{-ax} \sin x \) 的原函数是

\[
F(x) = -\frac{\alpha \sin x + \cos x}{1 + \alpha^2} e^{-ax},
\]

显然，当 \( a > 0 \)，\( x > 0 \) 时，
\[
|F_{\xi}(\alpha)| \leq \frac{1+\alpha}{1+\alpha^2} \leq \frac{2\alpha}{1+\alpha^2} + \frac{1}{1+\alpha^2} \leq 2,
\]

故当 \(A' \gg A \gg 0, \ 0 \leq \alpha \leq +\infty\) 时，

\[
\left|\int_{A}^{A'} \frac{\sin x}{x} e^{-ax} \, dx\right| = \left|\frac{1}{A} (F_{\xi}(\xi) - F_{\xi}(A))\right| \leq \frac{4}{A}.
\]

由此，利用一致收敛的柯西收敛准则，即知积分

\[
\int_{0}^{+\infty} \sin \frac{x}{x^p} e^{-ax} \, dx
\]

在 \(0 \leq \alpha \leq +\infty\) 上一致收敛。证毕。

3761. \(\int_{1}^{+\infty} e^{-ax} \frac{\cos x}{x^p} \, dx \quad (0 \leq \alpha \leq +\infty)\)，其中 \(p \geq 0\)

是常数。

解 由于

\[
\left|\int_{1}^{A} \cos x \, dx\right| = |\sin A - \sin 1| \leq 2,
\]

而当 \(0 \leq \alpha \leq +\infty\) 时，函数 \(\frac{e^{-ax}}{x^p}\) 在 \(x \geq 1\) 关于 \(x\) 递减且当 \(x \to +\infty\) 时关于 \(x\) 一致趋于零（因为 \(0 \leq \alpha \leq +\infty, x \geq 1\) 时，\(0 \leq \frac{e^{-ax}}{x^p} \leq \frac{1}{x^2}\))，

故由迪里黑里判别法即知 \(\int_{1}^{+\infty} e^{-ax} \frac{\cos x}{x^p} \, dx\) 在 \(0 \leq \alpha \leq +\infty\) 上一致收敛。

注意，也可仿3760题证法二，利用积分学第二中
值定理来证明。

3762. \( \int_0^{+\infty} \sqrt{\alpha} e^{-\alpha x^2} dx \) (0 \( \leq \alpha \leq +\infty \)).

解  此积分是收敛的。事实上，当 \( \alpha = 0 \) 时，积分值为
\( \alpha > 0 \) 时，设 \( \sqrt{\alpha} x = t \)，则得
\[
\int_0^{+\infty} \sqrt{\alpha} e^{-\alpha x^2} dx = \int_0^{+\infty} e^{-t^2} dt = \sqrt{\frac{\pi}{2}}.
\]

但是，此积分却不一致收敛。事实上，对于任何的 \( \alpha > 0 \)，由于
\[
\lim_{\alpha \to 0} \int_0^{+\infty} \sqrt{\alpha} e^{-\alpha x^2} dx = \lim_{\alpha \to 0} \int_{\sqrt{\alpha} A}^{+\infty} e^{-t^2} dt
= \int_0^{+\infty} e^{-t^2} dt = \sqrt{\frac{\pi}{2}},
\]
故对于 \( 0 < \varepsilon_0 < \sqrt{\frac{\pi}{2}} \)，必存在 \( \alpha_0 > 0 \)，使有
\[
\int_0^{+\infty} \sqrt{\alpha_0} e^{-\alpha_0 x^2} dx \geq \varepsilon_0,
\]
即此积分不是一致收敛的。

3763. \( \int_{-\infty}^{+\infty} e^{-(x-a)^2} dx \) (a) \( a < x < b \);

(6) \(-\infty < a < +\infty \).

解  显然，对任何固定的 \( a \)，积分 \( \int_{-\infty}^{+\infty} e^{-(x-a)^2} dx \)
都收敛，并且（作代换 \( x-a = t \)）
\[
\int_{-\infty}^{+\infty} e^{-(x-a)^2} dx = \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}.
\]
（a）取正数 $R$ 充分大，使 $-R \leq a < b \leq R$，显然，当 $|x| \geq R$ 时，对一切 $a < x < b$，有
\[
0 < e^{-(x-a)^2} < e^{-(|x|-R)^2},
\]
显然积分 $\int_{-\infty}^{+\infty} e^{-(|x|-R)^2} \, dx = 2 \int_{0}^{+\infty} e^{-(x-R)^2} \, dx$
收敛，故积分 $\int_{-\infty}^{+\infty} e^{-(x-a)^2} \, dx$ 对 $a < x < b$ 一致收敛。

（b）对任何 $A \to 0$，有
\[
\lim_{A \to +\infty} \int_{A}^{+\infty} e^{-(x-a)^2} \, dx = \lim_{A \to +\infty} \int_{A-a}^{+\infty} e^{-t^2} \, dt = \int_{-\infty}^{+\infty} e^{-t^2} \, dt = \sqrt{\pi},
\]
故当 $\alpha$ 充分大时，$\int_{A}^{+\infty} e^{-(x-a)^2} \, dx \to \frac{\sqrt{\pi}}{2}$；由此可知 $\int_{0}^{+\infty} e^{-(x-a)^2} \, dx$ 在 $-\infty < a < +\infty$ 上非一致收敛，当然 $\int_{-\infty}^{+\infty} e^{-(x-a)^2} \, dx$ 在 $-\infty < a < +\infty$ 上更非一致收敛。

3764. $\int_{0}^{+\infty} e^{-x^2(1+y^2)} \sin x \, dy (-\infty < x < +\infty)$。

解 此积分对任一固定的 $x$ 值，显然是收敛的，且当 $x \to 0$ 时，
\[
\int_{0}^{+\infty} e^{-x^2(1+y^2)} \sin x \, dy = \frac{\sin x}{x} e^{-x^2} \cdot \frac{\sqrt{\pi}}{2}.
\]
但是，它对 $-\infty < x < +\infty$ 却不是一致收敛的。事实上，对于任何的 $A > 0$，当 $x \to 0$ 时，

$$
\int_{A}^{+\infty} e^{-x^2(1+y^2)} \sin x \, dy
= \frac{\sin x}{x} e^{-x^2} \cdot \int_{Ax}^{+\infty} e^{-t^2} \, dt \to \int_{0}^{+\infty} e^{-t^2} \, dt
= \frac{\sqrt{\pi}}{2} \quad (x \to +0),
$$

由此可知积分不一致收敛。

3765. $\int_{0}^{+\infty} \frac{\sin(x^2)}{1+x^p} \, dx \quad (p \geq 0)$. 

解 由3300题易知积分

$$
\int_{0}^{+\infty} \sin(x^2) \, dx
$$

收敛，又 $\frac{1}{1+x^p} \quad (p \geq 0)$ 在 $x \geq 0$ 上对 $x$ 单调递减且一致有界：

$$
0 \leq \frac{1}{1+x^p} \leq 1 \quad (p \geq 0, \ x \geq 0),
$$

故由亚伯利判别法知积分

$$
\int_{0}^{+\infty} \frac{\sin(x^2)}{1+x^p} \, dx
$$

对 $p \geq 0$ 一致收敛。

3766. $\int_{0}^{1} x^{p-1} \ln x \frac{1}{x} \, dx, \quad (a) \ p \geq p_0 \geq 0$, 

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(6) $p \gg 0$ ($q \gg -1$).

解 首先注意，$x = 0$ 和 $x = 1$ 都可能是瑕点. 作代换 $x = e^{-t}$，得

$$
\int_0^1 x^{p-1} \ln^q x \, dx = \int_0^\infty e^{-(p-1)t} t^q e^{-t} \, dt
$$

$$
= \int_0^\infty e^{-qt} t^q \, dt,
$$

右端的积分当 $p \gg 0$ ($q \gg -1$) 时是收敛的，从而左端的积分此时也收敛. 更由于 $(e, \varepsilon \gg 0$ 很小)

$$
\int_{\varepsilon}^{1/\varepsilon} x^{p-1} \ln^q x \, dx = \int_{\ln 1/\varepsilon}^{\ln 1} e^{-qt} t^q \, dt,
$$

故 $\int_0^1 x^{p-1} \ln^q x \, dx$ 的一致收敛性等价于 $\int_0^\infty e^{-qt} t^q \, dt$ 的一致收敛性.

(a) 当 $p \gg p_0 \gg 0$ 时，由于

$$
0 \leq e^{-qt} t^q \leq e^{-p_0 t} t^q \quad (0 < t < +\infty),
$$

而积分 $\int_0^\infty e^{-p_0 t} t^q \, dt$ 收敛，故积分 $\int_0^\infty e^{-qt} t^q \, dt$ 一致收敛 (对于 $p \gg p_0 \gg 0$). 从而原积分 $\int_0^1 x^{p-1} \ln^q x \, dx$ 当 $p \gg p_0 \gg 0$ 时一致收敛.

(6) 对任何 $A > 0$, $p \gg 0$. 作代换 $pt = s$, 则

$$
\int_A^{\infty} e^{-p t} t^q \, dt = \frac{1}{p^{q+1}} \int_A^{\infty} s^q e^{-s} ds.
$$

由于 $q \gg -1$, 故积分 $\int_0^{\infty} s^q e^{-s} ds$ 收敛，且显然

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\[ 0 \leq \int_0^{+\infty} s^4 e^{-s} ds < +\infty, \]

于是，有
\[ \lim_{t \to +\infty} \int_0^t e^{-t^2} dt = +\infty, \]

由此即知积分 \( \int_0^{+\infty} e^{-t^2} dt \) 在 \( p > 0 \) 上非一致收敛。

从而原积分 \( \int_0^1 x^{p-1} \ln x \frac{1}{x} dx \) 当 \( p > 0 \) 时非一致收敛。

* 利用2361题的结果（在其中作代换 \( pt = s \)）。

3767. \[ \int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx \quad (0 \leq n < +\infty). \]

解 注意，当 \( x = 1 \) 是渐点，由于当 \( 0 \leq x < 1 \) 时，有
\[ 0 \leq \frac{x^n}{\sqrt{1-x^2}} < \frac{1}{\sqrt{1-x^2}} \quad (0 \leq n < +\infty), \]

而积分 \( \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x \bigg|_0^1 = \frac{\pi}{2} \) 收敛，故由

外氏判别法知积分 \( \int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx \) 当 \( 0 \leq n < +\infty \) 时一致收敛。

3768. \[ \int_0^1 \sin \frac{1}{x} \cdot \frac{dx}{x^n} \quad (0 \leq n \leq 2). \]

解 作代换 \( \frac{1}{x} = t \)，则
\[ \int_0^1 \sin \frac{1}{x} \cdot \frac{dx}{x^n} = \int_1^{+\infty} t^{n-2} \sin t \, dt, \]

并且，很明显，\( \int_0^1 \sin \frac{1}{x} \cdot \frac{dx}{x^n} \)的一致收敛相当于
\( \int_1^{+\infty} t^{n-2} \sin t \, dt \)的一致收敛。显然，当\( n < 2 \)时，积分
\( \int_1^{+\infty} t^{n-2} \sin t \, dt \)是收敛的。下证：当\( 0 \leq n < 2 \)时，
它不一致收敛。事实上，当\( 0 < n < 2 \)时，对任何正整数\( m \)，有
\[
\int_{2m\pi + \frac{\pi}{4}}^{2m\pi + \frac{\pi}{2}} t^{n-2} \sin t \, dt = \frac{\sqrt{2}}{2} \int_{2m\pi + \frac{\pi}{4}}^{2m\pi + \frac{\pi}{2}} t^{2-n} \, dt
\]
\[
= \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} \cdot \frac{1}{(2m\pi + \frac{\pi}{2})^{2-n}}.
\]

由于
\[
\lim_{n \to 0} \frac{1}{(2m\pi + \frac{\pi}{2})^{2-n}} = 1,
\]
故当\( n \)在\( 0 \leq n < 2 \)

内且与2充分接近时，必有
\[
\frac{1}{(2m\pi + \frac{\pi}{2})^{2-n}} > \frac{1}{2}.
\]

于是，这时
\[
\int_{2m\pi + \frac{\pi}{4}}^{2m\pi + \frac{\pi}{2}} t^{n-2} \sin t \, dt = \frac{\sqrt{2} \pi}{16} = \text{常数} \Rightarrow 0,
\]

故\( \int_1^{+\infty} t^{n-2} \sin t \, dt \)在\( 0 \leq n < 2 \)上非一致收敛。
3769. \( \int_0^2 \frac{x^\alpha dx}{\sqrt{(x-1)(x+2)^2}} \quad (|\alpha| < \frac{1}{2}). \)

解 首先注意 \( x = 1 \), \( x = 2 \) 是根点; \( x = 0 \) 可能是根点。将积分分成在 \((0, 1)\) 及 \((1, 2)\) 上的两个积分。

当 \( 0 < x < 1 \) 且 \(|\alpha| < \frac{1}{2}\) 时，

\[
\left| \frac{x^\alpha}{\sqrt{(x-1)(x+2)^2}} \right| \leq \frac{1}{x^{\frac{1}{2}} (1-x)^{\frac{1}{4}} (x-2)^{\frac{3}{8}}}.
\]

当 \( 1 < x < 2 \) 且 \(|\alpha| < \frac{1}{2}\) 时，

\[
\left| \frac{x^\alpha}{\sqrt{(x-1)(x+2)^2}} \right| \leq \frac{\sqrt{2}}{(x-1)^{\frac{1}{4}} (x-2)^{\frac{3}{8}}}.
\]

易知上述两个不等式右端的函数分别在区间 \((0, 1)\) 及 \((1, 2)\) 上的积分收敛，故此积分判别法知积分

\[
\int_0^2 \frac{x^\alpha dx}{\sqrt{(x-1)(x+2)^2}}
\]

对 \(|\alpha| < \frac{1}{2}\) 一致收敛。

3770. \( \int_0^1 \frac{\sin \alpha x}{\sqrt{|x-\alpha|}} dx \quad (0 \leq \alpha \leq 1). \)

解 \( \int_0^1 \frac{\sin \alpha x}{\sqrt{|x-\alpha|}} dx \)

\[
= \int_0^\alpha \frac{\sin \alpha x}{\sqrt{\alpha-x}} dx + \int_\alpha^1 \frac{\sin \alpha x}{\sqrt{x-\alpha}} dx.
\]
对于积分 \( \int_{0}^{a} \frac{\sin \alpha x}{\sqrt{a-x}} \, dx \)，由于

\[
\left| \int_{a-\eta}^{a} \frac{\sin \alpha x}{\sqrt{a-x}} \, dx \right| \leq \int_{a-\eta}^{a} \frac{dx}{\sqrt{a-x}}.
\]

\[
= 2 \sqrt{\eta},
\]

故对于任给的 \( \epsilon > 0 \)，只要取 \( 0 < \eta \leq \frac{\epsilon^2}{4} \)，即有

\[
\left| \int_{a-\eta}^{a} \frac{\sin \alpha x}{\sqrt{a-x}} \, dx \right| \leq \epsilon.
\]

因此，对 \( 0 \leq \alpha \leq 1 \) 它是一致收敛的。

对于积分 \( \int_{1}^{x} \frac{\sin \alpha x}{\sqrt{x-\alpha}} \, dx \)，由于

\[
\left| \int_{a}^{a+\eta} \frac{\sin \alpha x}{\sqrt{x-\alpha}} \, dx \right| \leq \int_{a}^{a+\eta} \frac{dx}{\sqrt{x-\alpha}}.
\]

\[
= 2 \sqrt{\eta},
\]

故对于任给的 \( \epsilon > 0 \)，只要取 \( 0 < \eta \leq \frac{\epsilon^2}{4} \)，即有

\[
\left| \int_{a}^{a+\eta} \frac{\sin \alpha x}{\sqrt{x-\alpha}} \, dx \right| \leq \epsilon.
\]

因此，对 \( 0 \leq \alpha \leq 1 \) 它是一致收敛的。

于是，积分

\[
\int_{0}^{1} \frac{\sin \alpha x}{\sqrt{|x-\alpha|}} \, dx
\]

对 \( 0 \leq \alpha \leq 1 \) 一致收敛。
3771. 若积分在参数的已知值的某邻域内一致收敛，则称此积分对参数的已知值一致收敛。证明积分
\[ I = \int_0^{+\infty} \frac{\alpha \, dx}{1 + \alpha^2 x^2} \]
在每一个 \( \alpha \neq 0 \) 的值一致收敛，而在 \( \alpha = 0 \) 非一致收敛。
证 设 \( \alpha_0 \) 为任一不为零的数，不妨设 \( \alpha_0 \geq 0 \)。今取
\[ \delta \gg 0, \quad \text{使} \quad \alpha_0 - \delta \gg 0. \]
下面证明积分 \( I \) 在 \((\alpha_0 - \delta, \alpha_0 + \delta)\) 内一致收敛。事实上，当 \( \delta \in (\alpha_0 - \delta, \alpha_0 + \delta) \)
时，由于
\[ 0 \leq \frac{\alpha}{1 + \alpha^2 x^2} \leq \frac{\alpha_0 + \delta}{1 + (\alpha_0 - \delta)^2 x^2}, \]
且积分
\[ \int_0^{+\infty} \frac{\alpha_0 + \delta}{1 + (\alpha_0 - \delta)^2 x^2} \, dx \]
收敛，故由外氏判别法知积分
\[ \int_0^{+\infty} \frac{\alpha \, dx}{1 + \alpha^2 x^2} \]
在 \((\alpha_0 - \delta, \alpha_0 + \delta)\) 内一致收敛，从而在 \( \alpha_0 \) 点一致收敛。由 \( \alpha_0 \) 的任意性知积分 \( I \) 在每一个 \( \alpha \neq 0 \) 的值一致收敛。
其次，我们证明积分 \( I \) 在 \( \alpha = 0 \) 非一致收敛。事实上，对原点的任何邻域 \((-\delta, \delta)\) 均有下述结果：对任何的 \( \delta > 0 \)，有
$$\int_{-\infty}^{+\infty} \frac{\alpha \, dx}{1 + \alpha^2 x^2} = \int_{-\infty}^{+\infty} \frac{dt}{1 + t^2} \quad (\alpha > 0).$$

由于

$$\lim_{\alpha \to 0} \int_{-\infty}^{+\infty} \frac{dt}{1 + t^2} = \int_{0}^{+\infty} \frac{dt}{1 + t^2} = \frac{\pi}{2},$$

故取 $0 < \varepsilon_0 < \frac{\pi}{2}$, 在 $(-\delta, \delta)$ 中必存在某一个 $\alpha_0 > 0$, 使有

$$\left| \int_{\alpha_0}^{+\infty} \frac{dt}{1 + t^2} \right| > \varepsilon_0,$$

即

$$\left| \int_{A}^{+\infty} \frac{\alpha_0 \, dx}{1 + \alpha_0^2 x^2} \right| > \varepsilon_0.$$

因此，积分 $I$ 在 $\alpha = 0$ 点的任一邻域 $(-\delta, \delta)$ 内非一致收敛，从而积分 $I$ 在 $\alpha = 0$ 时非一致收敛。

3772. 在下式中

$$\lim_{\alpha \to 0} \int_{0}^{+\infty} a e^{-ax} \, dx$$

把极限移到积分符号内合理吗？

解 不合理。事实上，由3754题2）的结果知，积分

$$\int_{0}^{+\infty} a e^{-ax} \, dx$$

对 $0 < a < b (b > 0)$ 的收敛并非一致，故一般不能应用积分符号与极限符号的交换定理。对于本题，实际上也不能交换，这是由于
\[
\int_0^{+\infty} \left( \lim_{a \to +0} ae^{-ax} \right) dx = 0,
\]

而
\[
\lim_{a \to +0} \int_0^{+\infty} ae^{-ax} dx = \lim_{a \to +0} (-e^{-ax}) \bigg|_0^{+\infty} = 1,
\]

故得
\[
\lim_{a \to +0} \int_0^{+\infty} ae^{-ax} dx \neq \int_0^{+\infty} \left( \lim_{a \to +0} ae^{-ax} \right) dx.
\]

377.3. 函数 \( f(x) \) 在区间 \((0, +\infty)\) 内可积分，证明公式
\[
\lim_{a \to +0} \int_0^{+\infty} e^{-ax} f(x) dx = \int_0^{+\infty} f(x) dx.
\]
证 容许有限个瑕点。为叙述简单起见，例如，设只有 一个瑕点 \( x = 0 \)。已知积分 \( \int_0^{+\infty} f(x) dx \) 收敛且被积函数中不含有 \( a \)，故它关于 \( a \) 一致收敛。又因函数 \( e^{-ax} \) 对于固定的 \( 0 \leq a \leq 1 \)，关于 \( x \) (\( x > 0 \)) 是递减的，而且一致有界。\( 0 \leq e^{-ax} \leq 1 \) \((0 \leq a \leq 1, x > 0)\)，故根据亚伯尔判别法知积分 \( \int_0^{+\infty} e^{-ax} f(x) dx \) 在 \( 0 \leq a \leq 1 \)
上一致收敛，于是，对于任给的 \( \varepsilon > 0 \)，可取 \( \eta > 0 \)，\( A_0 > 0 \) \((\eta < A_0)\)，使
\[
\left| \int_0^\eta e^{-ax} f(x) dx \right| \leq \frac{\varepsilon}{5},
\]
\[
\left| \int_{A_0}^{+\infty} e^{-ax} f(x) dx \right| \leq \frac{\varepsilon}{5} \quad (0 \leq a \leq 1).
\]
由于 \( f(x) \) 在 \([\eta, A_0]\) 上常义可积，故有界，即存在常数

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$M_0$，使 $|f(x)| \leq M_0$ ($\eta \leq x \leq A_0$)。再根据二元函数 $e^{-ax}$ 在 $0 \leq a \leq 1$，$\eta \leq x \leq A_0$ 上的一致连续性知，必存在 $\delta > 0$ （$\delta = 1$），使当 $0 < a < \delta$ 时，对一切 $\eta \leq x \leq A_0$，均有

$$0 \leq 1 - e^{-ax} \leq \frac{e}{5A_0M_0}.$$ 于是，当 $0 < a < \delta$ 时，恒有

$$\left| \int_0^{+\infty} e^{-ax} f(x) dx - \int_0^{+\infty} f(x) dx \right|$$

$$= \left| \int_{\eta}^{A_0} (e^{-ax} - 1) f(x) dx + \int_{A_0}^{+\infty} e^{-ax} f(x) dx - \int_0^{+\infty} f(x) dx \right|$$

$$\leq M_0 A_0 \cdot \frac{e}{5A_0M_0} + \frac{e}{5} + \frac{e}{5} + \frac{e}{5} + \frac{e}{5} = \varepsilon.$$ 由此可知

$$\lim_{a \to +0} \int_0^{+\infty} e^{-ax} f(x) dx = \int_0^{+\infty} f(x) dx.$$ 3774. 若 $f(x)$ 在区间 $(0, +\infty)$ 内绝对可积分，证明

$$\lim_{n \to +\infty} \int_0^{+\infty} f(x) \sin nx \, dx = 0.$$ 证 由 $f(x)$ 在区间 $(0, +\infty)$ 内的绝对可积性可知：对于任给的 $\varepsilon > 0$，存在 $\delta > 0$，使有

$$\int_0^{+\infty} |f(x)| \, dx \leq \frac{\varepsilon}{3}.$$
于是，
\[
\left| \int_0^\alpha f(x) \sin nx \, dx \right| \leq \left| \int_0^\alpha f(x) \sin nx \, dx \right| + \frac{c}{3}.
\]
先设 \( f(x) \) 在 \( [0, \alpha] \) 中无瑕点，我们在 \( [0, \alpha] \) 中插入分点 \( 0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = \alpha \)，并设 \( f(x) \) 在 \( [t_{k-1}, t_k] \) 上的下确界为 \( m_k \)，则有
\[
\int_0^\alpha f(x) \sin nx \, dx = \sum_{i=1}^m \int_{t_{k-1}}^{t_k} f(x) \sin nx \, dx
\]
\[
= \sum_{i=1}^m \int_{t_{k-1}}^{t_k} (f(x) - m_k) \sin nx \, dx
\]
\[
+ \sum_{i=1}^m m_k \int_{t_{k-1}}^{t_k} \sin nx \, dx,
\]
从而有
\[
\left| \int_0^\alpha f(x) \sin nx \, dx \right| \leq \sum_{i=1}^m w_k \Delta t_k + \sum_{i=1}^m |m_k| \cdot \frac{|\cos nt_{i-1} - \cos nt_i|}{n}
\]
\[
\leq \sum_{i=1}^m w_k \Delta t_k + \frac{2}{n} \sum_{i=1}^m |m_k|,
\]
其中 \( w_k \) 为 \( f(x) \) 在区间 \( [t_{i-1}, t_i] \) 上的振幅，\( \Delta t_i = t_i - t_{i-1} \)。

由于 \( f(x) \) 在 \( [0, \alpha] \) 上可积，故可取某一分法，使有
\[ \left| \sum_{k=1}^{n} w_k \Delta t_k \right| \leq \frac{e}{3}. \]

对于这样固定的分法，\( \sum_{i=1}^{m} |m_i| \)为一定值，因而存在\( N \)，使当\( n \gg N \)时，恒有

\[ \frac{2}{n} \sum_{i=1}^{m} |m_i| \leq \frac{e}{3}. \]

于是，对于上述所选取的\( N \)，当\( n \gg N \)时，

\[ \left| \int_0^{+\infty} f(x) \sin nx \, dx \right| \leq \left| \int_0^{A} f(x) \sin nx \, dx \right| + \left| \int_{A}^{+\infty} f(x) \sin nx \, dx \right| \leq \sum_{k=1}^{n} w_k \Delta t_k + \frac{2}{n} \sum_{k=1}^{m} |m_i| + \int_{A}^{+\infty} |f(x)| \, dx \leq \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e, \]

即

\[ \lim_{n \to \infty} \int_0^{+\infty} f(x) \sin nx \, dx = 0. \]

其次，设\( f(x) \)在区间\( (0, A) \)中有瑕点。为简便起见，不妨设只有一个瑕点，且为\( 0 \)。于是，对于任给的\( e > 0 \)，存在\( n \gg 0 \)，使有

\[ \int_0^{n} |f(x)| \, dx \leq \frac{e}{3}. \]
但是，\( f(x) \)在\([n, A]\)上无瑕点，故应用上述结果可知存在\( N \)，使当\( n \gg N \)时，恒有

\[
\left| \int_0^A f(x) \sin nx \, dx \right| \leq \frac{\varepsilon}{3}.
\]

于是，当\( n \gg N \)时，有

\[
\left| \int_0^{+\infty} f(x) \sin nx \, dx \right|
\]

\[
\leq \int_0^n \left| f(x) \right| \, dx + \left| \int_n^A f(x) \sin nx \, dx \right|
\]

\[
+ \int_A^{+\infty} \left| f(x) \right| \, dx
\]

\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

即

\[
\lim_{n \to \infty} \int_0^{+\infty} f(x) \sin nx \, dx = 0.
\]

总之，当\( f(x) \)在\((0, +\infty)\)内绝对可积，不论\( f(x) \)在\((0, +\infty)\)内有无瑕点，均可证得

\[
\lim_{n \to \infty} \int_0^{+\infty} f(x) \sin nx \, dx = 0.
\]

3775. 证明：若（1）在每一个有穷区间\((a, b)\)内

\[
f(x, y) \equiv f(x, y_0) \equiv f(x, y), \quad (2) \quad \left| f(x, y) \right| \leq F(x),
\]

其中

\[
\int_a^b F(x) \, dx \leq +\infty,
\]

则
\[
\lim_{y \to y_0} \int_a^b f(x, y) \, dx = \int_a^b \lim_{y \to y_0} f(x, y) \, dx.
\]

证  条件 (1) 表示当 $y \to y_0$ 时，当 $x$ 在任何有穷区间 $(a, b)$ 上，$f(x, y)$ 都一致趋于 $f(x, y_0)$。于是，有

\[
\lim_{y \to y_0} \int_a^b f(x, y) \, dx = \int_a^b f(x, y_0) \, dx
\]

（对任何 $b \geq a$）。

又在不等式 $|f(x, y)| \leq F(x)$ 中令 $y \to y_0$（任意固定 $x$），得 $|f(x, y_0)| \leq F(x)$，故 $\int_a^b f(x, y_0) \, dx$ 收敛。

任给 $\varepsilon > 0$，由于 $\int_a^+ \infty F(x) \, dx \leq + \infty$，故可取定某 $b > a$，使 $\int_a^b F(x) \, dx \leq \frac{\varepsilon}{3}$。对此 $b$，又可取 $\delta > 0$，使当 $0 < |y - y_0| < \delta$ 时，恒有

\[
\left| \int_a^b f(x, y) \, dx - \int_a^b f(x, y_0) \, dx \right| \leq \frac{\varepsilon}{3}.
\]

于是，当 $0 < |y - y_0| < \delta$ 时，恒有

\[
\left| \int_a^b f(x, y) \, dx - \int_a^+ \infty f(x, y_0) \, dx \right|
\]

\[
\leq \left| \int_a^b f(x, y) \, dx - \int_a^b f(x, y_0) \, dx \right|
\]

\[
+ \int_b^+ \infty |f(x, y)| \, dx + \int_b^+ \infty |f(x, y_0)| \, dx
\]

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\[ \leq \frac{e}{3} + \int_{a}^{\infty} F(x) \, dx + \int_{b}^{\infty} F(x) \, dx. \]
\[ \leq \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e. \]

由此可知

\[ \lim_{y \to y_0} \int_{a}^{\infty} f(x, y) \, dx. \]

\[ = \int_{a}^{\infty} f(x, y_0) \, dx = \int_{a}^{\infty} \lim_{y \to y_0} f(x, y) \, dx. \]
证毕。

注。本题中应假定，对任何 \( b > a \)，\( f(x, y) \) 关于 \( x \) 在 \([a, b]\) 上可积。

3776. 利用积分符号与极限号互换，计算积分

\[ \int_{a}^{\infty} e^{-x^2} \, dx = \int_{a}^{\infty} \lim_{n \to \infty} \left[ \left(1 + \frac{x^2}{n}\right)^{-n} \right] \, dx. \]

解 先证积分符号与极限号能互换。事实上，(1) 函数 \( \left(1 + \frac{x^2}{n}\right)^{-n} \) 在 \( 0 \leq x \leq A \) 上连续 (任何 \( A > 0 \))，故它在 \([0, A]\) 上可积，(2) 又 \( \left(1 + \frac{x^2}{n}\right)^{-n} \) 在 \([0, A]\) 上关于 \( n \) 为单调减小的，且

\[ \lim_{n \to \infty} \left(1 + \frac{x^2}{n}\right)^{-n} = e^{-x^2} \]

为连续函数，故按狄尼定理，当 \( n \to \infty \) 时，函数
\[
(1 + \frac{x^2}{n})^{-n} \text{ 在 } [0, A] \text{ 上一致趋向于 } e^{-x^2}, \quad (3)
\]
由于 \(0 < (1 + \frac{x^2}{n})^{-n} \leq \frac{1}{1 + x^2} \quad (n = 1, 2, \cdots)\)，且
\[
\int_0^{+\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2} \leq +\infty, \quad \text{ 故积分 } \int_0^{+\infty} (1 + \frac{x^2}{n})^{-n} dx
\]
关于 \(n\) 一致收敛。因此，我们可以应用积分符号与极限号的互换定理 *)，从而得
\[
\int_0^{+\infty} e^{-x^2} dx = \lim_{n \to +\infty} \int_0^{+\infty} \frac{dx}{(1 + \frac{x^2}{n})^n}.
\]
而
\[
\int_0^{+\infty} \frac{dx}{(1 + \frac{x^2}{n})^n} = \frac{n}{\sqrt{n}} \int_0^{+\infty} \frac{dt}{(1 + t^2)^n} = \frac{1}{\sqrt{n}} I_n,
\]
又由于
\[
I_{n-1} = \int_0^{+\infty} \frac{dt}{(1 + t^2)^{n-1}}
\]
\[
= \frac{t}{(1 + t^2)^{n-1}} \bigg|_0^{+\infty} + 2(n-1) \int_0^{+\infty} \frac{t^2}{(1 + t^2)^n} dt
\]
\[
= 2(n-1)I_{n-1} - 2(n-1)I_n,
\]
故得
\[
I_n = \frac{2n-3}{2n-2} I_{n-1}.
\]
又因 \(I_1 = \int_0^{+\infty} \frac{dt}{1 + t^2} = \frac{\pi}{2}\)，将上式递推即得

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于是，
\[
\int_0^{+\infty} e^{-x^2} \, dx = \lim_{n \to \infty} \frac{(2n-3)!!}{(2n-2)!!} \cdot \frac{\pi}{2} \cdot \frac{\sqrt{n}}{2}.
\]

根据瓦里斯公式，我们有
\[
\frac{\pi}{2} = \lim_{n \to \infty} \frac{[(2n)!!]^2}{(2n+1)[(2n-1)!!]^2}
= \lim_{n \to \infty} \frac{[(2n-2)!!]^2}{(2n-1)[(2n-3)!!]^2}.
\]

最后得
\[
\int_0^{+\infty} e^{-x^2} \, dx = \frac{\pi}{2} \lim_{n \to \infty} \frac{(2n-3)!! \sqrt{n}}{(2n-2)!!}
= \frac{\pi}{2} \lim_{n \to \infty} \frac{(2n-3)!! \sqrt{2n-1}}{(2n-2)!!} \cdot \sqrt{\frac{n}{2n-1}}
= \frac{\pi}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}.
\]

*) 参看菲金哥尔茨著《微积分学教程》第二卷480页定理1。

3777. 证明：积分
\[
F(a) = \int_0^{+\infty} e^{-(x-a)^2} \, dx
\]
是参数a的连续函数。
证 \[ F(a) = \int_{0}^{+\infty} e^{-(x-a)^2} \, dx = \int_{-\infty}^{+\infty} e^{-x^2} \, dx. \]

\[ = \int_{-\infty}^{0} e^{-x^2} \, dx + \int_{0}^{+\infty} e^{-x^2} \, dx \]

\[ = \int_{0}^{a} e^{-x^2} \, dx - \frac{\sqrt{\pi}}{2}. \]

由变上限积分的性质可知积分 \( \int_{0}^{a} e^{-x^2} \, dx \) 是 \( a (-\infty \leq a \leq +\infty) \) 的连续函数，故 \( F(a) \) 也是 \( a (-\infty \leq a \leq +\infty) \) 的连续函数。

3778. 求函数

\[ F(a) = \int_{0}^{+\infty} \frac{\sin(1-a^2)x}{x} \, dx \]

的不连续点。作出函数 \( y = F(a) \) 的图形。

解 当 \( 1-a^2 \gg 0 \) 即 \( |a| \ll 1 \) 时，

\[ F'(a) = \int_{0}^{+\infty} \frac{\sin(1-a^2)x}{(1-a^2)x} \, d[(1-a^2)x] \]

\[ = \int_{0}^{+\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}. \]

当 \( 1-a^2 \ll 0 \) 即 \( |a| \gg 1 \) 时，

\[ F'(a) = -\int_{0}^{+\infty} \frac{\sin(1-a^2)x}{(1-a^2)x} \, d[(1-a^2)x] \]

\[ = -\int_{0}^{+\infty} \frac{\sin t}{t} \, dt = -\frac{\pi}{2}. \]
当 $1 - a^2 = 0$
即 $|a| = 1$ 时，

$$F(a) = 0.$$  

于是，$a = \pm 1$ 为 $F(a)$ 的不连续点。如图 7-2 所示。

研究下列函数在所指定区间内的连续性：

3779. $F(a) = \int_0^{+\infty} \frac{x \, dx}{2 + x^a} \forall |a| > 2$.

解 对于积分 $\int_1^{+\infty} \frac{x \, dx}{2 + x^a}$，由于当 $x \geq 1$ 时，

$$0 < \frac{x}{2 + x^a} \leq \frac{x}{x^a} \leq \frac{1}{x^{a_0} - 1},$$

其中 $a \geq a_0 > 2$，且积分

$$\int_1^{+\infty} \frac{dx}{x^{a_0} - 1}$$

收敛，故积分

$$\int_1^{+\infty} \frac{x \, dx}{2 + x^a}$$

对 $a \geq a_0$ 一致收敛，从而积分

$$\int_0^{+\infty} \frac{x \, dx}{2 + x^a}$$
对 $a \geq a_0$ 一致收敛。因此，$F(a)$ 当 $a \geq a_0$ 时连续。由于 $a_0 \geq 2$ 的任意性，故知 $F(a)$ 当 $a \geq 2$ 时连续。

3780. $F(a) = \int_1^{+\infty} \frac{\cos x}{x^a} \, dx \quad \text{当} a > 0.$

解  对于任何 $A \geq 1$，均有

$$\left| \int_1^{A} \cos x \, dx \right| \leq 2.$$ 

而函数 $\frac{1}{x^a}$ 在 $x \geq 1$, $a \geq 0$ 时关于 $x$ 单调递减，且

$$0 < \frac{1}{x^a} \leq \frac{1}{x_0^a} \quad (x \geq 1, \, a \geq a_0 \geq 0)$$

知：当 $x \to +\infty$ 时 $\frac{1}{x^a}$ 在 $a \geq a_0$ 时一致趋于零。因此，由费里哈里判别法知积分

$$\int_1^{+\infty} \frac{\cos x}{x^a} \, dx$$

对 $a \geq a_0 \geq 0$ 一致收敛。于是，函数 $F(a)$ 当 $a \geq a_0$ 时连续。由于 $a_0 \geq 0$ 的任意性，故知 $F(a)$ 当 $a \geq 0$ 时连续。

3781. $F(a) = \int_0^{\pi} \frac{\sin x}{x^a (\pi - x)^a} \, dx \quad \text{当} 0 < a < 2.$

解  $F(a) = \int_0^{\pi/2} \frac{\sin x}{x^a (\pi - x)^a} \, dx$
$$+ \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{x^a (\pi - x)^a} \, dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{x^a (\pi - x)^a} \, dx$$

$$- \int_{0}^{\frac{\pi}{2}} \frac{\sin(\pi - t)}{\pi (\pi - t)^a} \, dt$$

$$= 2 \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{x^a (\pi - x)^a} \, dx.$$
一致收敛。从而 $F(\alpha)$ 在 $\alpha_0 \leq \alpha \leq \alpha_1$ 上连续。由 $0 \leq \alpha_0 \leq \alpha_1 \leq 2$ 的任意性即知 $F(\alpha)$ 在 $0 \leq \alpha \leq 2$ 上连续。

$$3782. \quad F(\alpha) = \int_0^{+\infty} \frac{e^{-x}}{|\sin x|^\alpha} \, dx \quad \text{当} \quad 0 \leq \alpha \leq 1.$$ 

解：

$$F(\alpha) = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{e^{-x}}{|\sin x|^\alpha} \, dx$$ 

$$= \sum_{n=0}^{\infty} \int_0^{\pi} e^{-x} \frac{1}{\sin^\alpha t} \, dt.$$ 

当 $0 \leq \alpha \leq \alpha_0 \leq 1$ 时，

$$\int_0^{\pi} e^{-\frac{(n+1)}{\sin^\alpha t}} \, dt \leq e^{-\pi} \int_0^{\pi} \frac{1}{\sin^{\alpha_0 t}} \, dt.$$ 

显然，积分

$$\int_0^{\pi} \frac{dt}{\sin^{\alpha_0 t}} = 2 \int_0^{\frac{\pi}{2}} \frac{dt}{\sin^{\alpha_0 t}}$$ 

且 $\lim_{t \to 0^+} \frac{1}{\sin^{\alpha_0 t}} = 1$，故它是收敛的。而级数

$$\sum_{n=0}^{\infty} e^{-x}$$ 为公比等于 $e^{-\pi} < 1$ 的几何级数，它也收敛。

于是，由外氏判别法知级数

$$\sum_{n=0}^{\infty} \int_0^{\pi} e^{-\frac{(n+1)}{\sin^{\alpha_0 t}}} \, dt.$$ 

对 $0 \leq \alpha \leq \alpha_0$ 一致收敛。从而，注意到被积函数是正的，即知积分

$$\int_0^{+\infty} \frac{e^{-x}}{|\sin x|^\alpha} \, dx$$ 

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对 \(0 < a \leq a_0\) 一致收敛。因此，\(F(a)\) 在 \(0 < a \leq a_0\) 上连续，由 \(a_0 < 1\) 的任意性知 \(F(a)\) 当 \(0 < a < 1\) 时连续。

3783. \(F(a) = \int_0^{\infty} ae^{-x^2} dx \) 当 \(-\infty < a < +\infty\)。

解 当 \(a \neq 0\) 时，

\[
F(a) = -\frac{1}{\alpha} e^{-x^2} \bigg|_0^{+\infty} = -\frac{1}{\alpha},
\]

显然它是连续的。

当 \(a = 0\) 时，

\[
F(0) = \int_0^{+\infty} 0 \cdot e^0 dx = 0.
\]

于是，显见 \(F(a)\) 当 \(a = 0\) 时不连续。

§3．广义积分中的变量代换。广义积分号下
微分法及积分法

1° 对参数的微分法 若 1）函数 \(f(x, y)\) 于域 \(a \leq x < +\infty, y_1 < y < y_2\) 内是连续的并对参数 \(y\) 可微分，

2） \(\int_x^{+\infty} f(x, y) dx\) 收敛； 3） \(\int_x^{+\infty} f_y(x, y) dx\) 于区间 \((y_1, y_2)\) 内一致收敛，则当 \(y_1 \leq y \leq y_2\) 时

\[
-\frac{d}{dy} \int_x^{+\infty} f(x, y) dx = \int_x^{+\infty} f_y(x, y) dx
\]

（莱布尼兹法则）。

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2° 对参数积分的公式 若 1）函数 \( f(x, y) \) 当 \( x \geq a \) 及 \( y_1 \leq y \leq y_2 \) 时是连续的，2） \( \int_{a}^{+\infty} f(x, y) \, dx \) 在有穷的区间 \((y_1, y_2)\) 内一致收敛，则

\[
\int_{y_1}^{y_2} dy \int_{a}^{+\infty} f(x, y) \, dx = \int_{a}^{+\infty} dx \int_{y_1}^{y_2} f(x, y) \, dy. \tag{1}
\]

若 \( f(x, y) \geq 0 \)，则公式（1）在假定等式（1）的一端有意义时，对于无穷的区间 \((y_1, y_2)\) 也正确。

3784. 利用公式

\[
\int_{0}^{1} x^{n-1} \, dx = \frac{1}{n} \quad (n > 0).\]

计算积分

\[
I = \int_{0}^{1} x^{n-1} \ln x \, dx, \quad \text{其中 } m \text{ 为自然数}.
\]

解 \( \frac{dx^{n-1}}{dn} = x^{n-1} \ln x \) \((n > 0 \text{ 为任意实数})\)。积分

\[
\int_{0}^{1} x^{n-1} \ln x \, dx \quad (1)
\]

对于 \( n \geq n_0 > 0 \) 为一致收敛。事实上，当 \( 0 < x < 1 \)，\( n \geq n_0 > 0 \) 时，

\[
|x^{n-1} \ln x| \leq -x^{n_0-1} \ln x,
\]

而积分 \( \int_{0}^{1} x^{n_0-1} \ln x \, dx \) 显然收敛。因此，由外氏
判别法即知积分（1）对 \( n \geq n_0 \geq 0 \) — 致收敛。于是，积分

\[
\int_0^1 x^{n-1} \, dx
\]

对参数 \( n \geq n_0 \) 求导数时，积分号与导数符号可交换，即

\[
\frac{d}{dn} \int_0^1 x^{n-1} \, dx = \int_0^1 \frac{dx^{n-1}}{dn} \, dx
\]

\[= \int_0^1 x^{n-1} \ln x \, dx.
\]

由 \( n_0 \geq 0 \) 的任意性知，上式对任意 \( n \geq 0 \) 均成立。

同理对 \( n \) 逐次求导数，也可在积分号下求导数，即

\[
\frac{d^2}{dn^2} \int_0^1 x^{n-1} \, dx = \int_0^1 \frac{d}{dn} \left( x^{n-1} \ln x \right) \, dx
\]

\[= \int_0^1 x^{n-1} \ln^2 x \, dx,
\]

……

由数学归纳法，可得

\[
\frac{d^n}{dn^n} \int_0^1 x^{n-1} \, dx = \int_0^1 x^{n-1} \ln^n x \, dx.
\]

但是，\( \int_0^1 x^{n-1} \, dx = \frac{1}{n} \quad (n \geq 0) \)，故有

\[
\frac{d^n}{dx^n} \int_0^1 x^{n-1} \, dx = \frac{(-1)^{n+1}}{n^{n+1}}.
\]
从而得
\[ \int_{0}^{1} x^{n-1} \ln x \, dx = \frac{(-1)^{n+1}}{n^{n+1}}. \]

*) 利用2362题的结果。

3785. 利用公式
\[ \int_{0}^{+\infty} \frac{dx}{x^2 + a} = \frac{\pi}{2\sqrt{a}} \quad (a > 0), \]

计算积分
\[ I = \int_{0}^{+\infty} \frac{dx}{(x^2 + a)^{n+1}}, \]
其中 \( n \) 为自然数。

解
\[ \frac{\partial}{\partial a} \left( \frac{1}{x^2 + a} \right) = -\frac{1}{(x^2 + a)^2}. \]
积分
\[ \int_{0}^{+\infty} \frac{dx}{(x^2 + a)^2} \quad (1) \]

对 \( a \gg a_0 \gg 0 \) 一致收敛。事实上，当 \( x \gg 0, a \gg a_0 \gg 0 \) 时，
\[ \frac{1}{(x^2 + a)^2} \ll \frac{1}{(x^2 + a_0)^2}, \]

而积分 \( \int_{0}^{+\infty} \frac{dx}{(x^2 + a_0)^2} \) 显然收敛。因此，由外氏判别
法知积分 (1) 当 \( a \gg a_0 \gg 0 \) 时一致收敛。于是，利用莱布尼兹法则，即得
\[ \frac{d}{da} \int_{0}^{+\infty} \frac{dx}{x^2 + a} = \int_{0}^{+\infty} \frac{\partial}{\partial a} \left( \frac{1}{x^2 + a} \right) \, dx \]
\[ = - \int_0^{+\infty} \frac{dx}{(x^2 + 1)^2}. \]

由 \( a_0 \gg 0 \) 的任意性知，上式对一切 \( a \gg 0 \) 均成立。

同理对积分 \( \int_0^{+\infty} \frac{dx}{x^2 + a} \) 逐次求导数，得

\[ \frac{d^n}{da^n} \int_0^{+\infty} \frac{dx}{x^2 + a} = (-1)^n n! \int_0^{+\infty} \frac{dx}{(x^2 + a)^{n+1}}. \]

但是，

\[ \frac{d}{da} \int_0^{+\infty} \frac{dx}{x^2 + a} = \frac{d}{da} \left( \frac{\pi}{2 \sqrt{a}} \right). \]

\[ = - \frac{\pi}{2^2} \cdot \frac{1}{\sqrt{a}}, \]

\[ \frac{d^2}{da^2} \int_0^{+\infty} \frac{dx}{x^2 + a} = \frac{d}{da} \left( - \frac{\pi}{2^2} \cdot \frac{1}{\sqrt{a}^3} \right). \]

\[ = \frac{1 \cdot 3 \pi}{2^3} \cdot \frac{1}{\sqrt{a^5}}, \]

......

由数学归纳法，可得

\[ \frac{d^n}{da^n} \int_0^{+\infty} \frac{dx}{x^2 + a} = \frac{(2n-1) \cdot 1 \cdot \pi}{2^{n+1}} \cdot (-1)^n \cdot a^{-(n+\frac{1}{2})}, \]

最后得

\[ I = \frac{\pi}{2} \cdot \frac{(2n-1) \cdot 1 \cdot a^{-(n+\frac{1}{2})}}{(2n)!}. \]

3786. 证明迪里黑里积分

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\[ I(\alpha) = \int_0^{+\infty} \frac{\sin \alpha x}{x} \, dx \]

当 \( \alpha \neq 0 \) 时有导函数，但不能利用莱布尼兹法则来求它。

证 当 \( \alpha \geq 0 \) 时，令 \( \alpha x = y \)，得

\[ I(\alpha) = \int_0^{+\infty} \frac{\sin y}{y} \, dy = \frac{\pi}{2} . \]

当 \( \alpha < 0 \) 时，\( I(\alpha) = -I(-\alpha) = -\frac{\pi}{2} \)，于是，\( I' (\alpha) = 0 \)。

但是，如果利用莱布尼兹法则来求，即得错误的结果。事实上，积分

\[ \int_0^{+\infty} \frac{\sin \alpha x}{x} \frac{dx}{d\alpha} = \int_0^{+\infty} \cos \alpha x \, dx = \int_0^{+\infty} \cos \alpha x \, dx \]

发散，而 \( I' (\alpha) = 0 \) (\( \alpha \neq 0 \)) 存在，因此，本题不能应用莱布尼兹法则求 \( I' (\alpha) \)。

3787. 证明：函数

\[ I(\alpha) = \int_0^{+\infty} \frac{\cos x}{1 + (x + \alpha)^2} \, dx \]

在区域 \(-\infty \leq \alpha \leq +\infty\) 内连续并可微分。

证 设 \( \alpha_0 \) 为 \((-\infty, +\infty)\) 内任意一点。记 \( M = \max (|\alpha_0 - 1|, |\alpha_0 + 1|) \)，则当 \( x \gg M \)，\( \alpha \in (\alpha_0 - 1, \alpha_0 + 1) \) 时，恒有

\[ \left| \frac{\cos x}{1 + (x + \alpha)^2} \right| \leq \frac{1}{1 + (x - M)^2} . \]
由于积分 $\int_{0}^{+\infty} \frac{dx}{1+(x-M)^2}$ 收敛，故积分

$$\int_{0}^{+\infty} \frac{\cos x}{1+(x+\alpha)^2} dx$$

及

$$\int_{0}^{+\infty} \frac{d}{d\alpha} \left[ \frac{\cos x}{1+(x+\alpha)^2} \right] dx$$

在$(a_0-1, a_0+1)$内一致收敛，从而 $F(\alpha)$ 在$(a_0-1, a_0+1)$内连续且可微分，且可在积分号下求导数。由 $\alpha_0$ 的任意性，即知 $F(\alpha)$ 在$(-\infty, +\infty)$ 内连续且可微分。

3788．从等式

$$\frac{e^{-ax}-e^{-bx}}{x} = \int_{a}^{b} e^{-xy} dy$$

出发，计算积分

$$\int_{0}^{+\infty} \frac{e^{-ax}-e^{-bx}}{x} dx \quad (a > 0, \ b > 0).$$

解 不妨设 $a < b$，注意到 $e^{-ix}$ 在域：$x > 0, a < y < b$ 上连续。又积分 $\int_{0}^{+\infty} e^{-xy} dx$ 对 $a < y < b$ 是一致收敛的。事实上，当 $x > 0, a < y < b$ 时，

$$0 < e^{-y} \leq e^{-ax}.$$
但积分 \( \int_{0}^{+\infty} e^{-ax} dx \) 收敛，故积分 \( \int_{0}^{+\infty} e^{-bx} dx \) 是一致收敛的。于是，利用对参数的积分公式，即得

\[
\int_{0}^{+\infty} dx \int_{x}^{b} e^{-sy} dy = \int_{x}^{b} dy \int_{0}^{+\infty} e^{-sy} dx.
\]

上式左端为 \( \int_{0}^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} dx \)，右端为 \( \int_{x}^{b} \frac{dy}{y} = \ln \frac{b}{a} \)，从而得

\[
\int_{0}^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \ln \frac{b}{a} \quad (a \gg 0, \ b \gg 0).
\]

3789. 证明傅里叶公式

\[
\int_{0}^{+\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a} \quad (a \gg 0, \ b \gg 0),
\]

式中 \( f(x) \) 为连续函数及积分 \( \int_{A}^{+\infty} \frac{f(x)}{x} dx \) 对任何的 \( A \gg 0 \) 都有意义。

证 对任意的 \( A \gg 0 \)，有

\[
\int_{A}^{+\infty} \frac{f(ax) - f(bx)}{x} dx = \int_{A}^{+\infty} \frac{f(ax)}{x} dx - \int_{A}^{+\infty} \frac{f(bx)}{x} dx
\]

\[
= \int_{A}^{+\infty} \frac{f(t)}{t} dt - \int_{A}^{+\infty} \frac{f(t)}{t} dt
\]

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\[
\begin{align*}
&= \int_{A_0}^{A_b} \frac{f(t)}{t} dt = f(\xi) \int_{A_0}^{A_b} \frac{dt}{t} \\
&= f(\xi) \ln \frac{b}{a} (A_0 \leq \xi \leq A_b).
\end{align*}
\]

当 \( A \rightarrow +0 \) 时，\( \xi \rightarrow +0 \)。由 \( f(x) \) 在 \( x = 0 \) 点的连续性，即得

\[
\int_{0}^{+\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}.
\]

利用傅立叶公式，计算积分；

3790. \( \int_{0}^{+\infty} \frac{\cos ax - \cos bx}{x} dx \) (\( a \gg 0, \ b \gg 0 \)).

解 由于 \( \cos x \) 在 \( (0, +\infty) \) 内连续，且对任何 \( A \gg 0 \)，积分 \( \int_{A}^{+\infty} \frac{\cos x}{x} dx \) 存在，故由傅立叶公式，有

\[
\int_{0}^{+\infty} \frac{\cos ax - \cos bx}{x} dx = \cos 0 \cdot \ln \frac{b}{a} = \ln \frac{b}{a}.
\]

3791. \( \int_{0}^{+\infty} \frac{\sin ax - \sin bx}{x} dx \) (\( a \gg 0, \ b \gg 0 \)).

解 同3790题，由于 \( \sin 0 = 0 \)，故

\[
\int_{0}^{+\infty} \frac{\sin ax - \sin bx}{x} dx = 0.
\]

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解 令 \( f(x) = \frac{\pi}{2} - \text{arctg} \, x \)，则 \( f(x) \) 在 \( 0 < x < +\infty \) 上连续。

由于 \( f(x) > 0 \) 且 [利用洛必达法则]

\[
\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} x^{-1} = \lim_{x \to +\infty} 1 - \frac{1}{1 + x^2} = 1,
\]

故对任何 \( A > 0 \)，积分 \( \int_{A}^{+\infty} \frac{f(x)}{x} \, dx \) 都收敛。因此

由傅茹兰公式，有

\[
\int_{0}^{+\infty} \frac{\left(\frac{\pi}{2} - \text{arctg} \, ax\right) - \left(\frac{\pi}{2} - \text{arctg} \, bx\right)}{x} \, dx = \frac{\pi}{2} \ln \frac{b}{a},
\]

故

\[
\int_{0}^{+\infty} \frac{\text{arctg} \, ax - \text{arctg} \, bx}{x} \, dx = \frac{\pi}{2} \ln \frac{a}{b}.
\]

利用对参数的微分法计算下列积分：

3793. \( \int_{0}^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} \, dx \) (\( a > 0 \), \( b > 0 \)).
解 由于
\[
\lim_{x \to +0} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} = \lim_{x \to +0} \frac{-2\alpha x e^{-\alpha x^2} + 2\beta x e^{-\beta x^2}}{1} = 0,
\]
故 \(x = 0\) 不是瑕点。又由于
\[
\lim_{x \to +\infty} x^2 \cdot \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} = \lim_{x \to +\infty} \left( \frac{x}{e^{\alpha x^2}} - \frac{x}{e^{\beta x^2}} \right) = 0,
\]
故对任何 \(\alpha > 0, \beta > 0\) 积分 \(\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx\)
都收敛。今将 \(\beta > 0\) 固定，而把所求积分视为含参变量 \(\alpha (\alpha > 0)\) 的积分，即令
\[
I(\alpha) = \int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx \quad (\alpha > 0).
\]
而
\[
\int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} \right) dx = -\int_0^{+\infty} x e^{-\alpha x^2} dx.
\]
下证右端积分在 \(\alpha \geq \alpha_0 > 0\) 时一致收敛。事实上，当 \(\alpha \geq \alpha_0, 0 \leq x \leq +\infty\) 时，
\[
0 \leq x e^{-\alpha x^2} \leq x e^{-\alpha_0 x^2},
\]
而积分 \(\int_0^{+\infty} x e^{-\alpha_0 x^2} dx = \frac{1}{2 \alpha_0}\) 收敛，故积分

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\[
\int_0^{+\infty} x e^{-\alpha x^2} \, dx \text{ 在 } \alpha \geq \alpha_0 \text{ 时一致收敛。因此，当 } \alpha \geq \alpha_0 \text{ 时，可在积分号下对参数求导数：}
\]
\[
I'(\alpha) = -\int_0^{+\infty} x e^{-\alpha x^2} \, dx = -\frac{1}{2\alpha}.
\]

由 \( \alpha_0 > 0 \) 的任意性知，上式对一切 \( \alpha > 0 \) 皆成立。积分之，得
\[
I(\alpha) = -\frac{1}{2} \ln \alpha + C \quad (0 < \alpha < +\infty),
\]
其中 \( C \) 为待定的常数。在此式中令 \( \alpha = \beta \)，并注意到
\[
I(\beta) = \int_0^{+\infty} \frac{e^{-\beta x^2} - e^{-\beta x^2}}{x} \, dx = 0,
\]
即得
\[
0 = I(\beta) = -\frac{1}{2} \ln \beta + C,
\]
由此知 \( C = \frac{1}{2} \ln \beta \)。于是，
\[
I(\alpha) = -\frac{1}{2} \ln \alpha + \frac{1}{2} \ln \beta = \frac{1}{2} \ln \frac{\beta}{\alpha} \quad (\alpha > 0),
\]
即
\[
\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} \, dx = \frac{1}{2} \ln \frac{\beta}{\alpha} \quad (\alpha > 0, \beta > 0).
\]

注，本题中，实际应考察积分 \( I(\alpha) = \int_0^{+\infty} f(x, \alpha) \, dx \)，
其中 \( f(x, \alpha) = \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} \)。当 \( 0 < x < +\infty \) 时，
\[
\frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x}, \quad \text{当 } x = 0 \text{ 时}.
\]

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易知 \( f(x, \alpha) \) 是 \( 0 \leq x < +\infty, \quad 0 < \alpha < +\infty \) 上的连续函数（\( \beta > 0 \) 固定）。我们证明：

\[
f'_x(x, \alpha) = -x e^{-\alpha x^2} \quad (0 \leq x < +\infty, \quad 0 < \alpha < +\infty).
\]

事实上，当 \( 0 < x < +\infty \) 时，此式显然成立。由于 \( f(0, \alpha) = 0 \)（\( 0 < \alpha < +\infty \)），故 \( f'_x(0, \alpha) = 0 \)（\( 0 < \alpha < +\infty \)）。因此，上式当 \( x = 0 \) 时也成立。

\( f'_x(x, \alpha) \) 显然是 \( 0 \leq x < +\infty, \quad 0 < \alpha < +\infty \) 上的连续函数。

在以下许多题中，我们都应作此理解，但不必写

出 \( f(x, \alpha) \)。函数

\[
\frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x}
\]

（\( x = 0 \) 时规定其函数值为其极限值 \( 0 \)），而公式

\[
\frac{\partial}{\partial \alpha} \left( \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} \right) = -x e^{-\alpha x^2}
\]

当 \( x = 0 \) 时也成立（如上述）。这样，才严格符合莱

布尼兹法则（积分号下求导数）的条件。

另外，本题若利用逐次积分来作可更简单一些。

今作如下：易知（不妨设 \( \alpha \leq \beta \)）

\[
\frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} = \int_{0}^{\beta} x e^{-y x^2} \, dy,
\]

而积分 \( \int_{0}^{+\infty} x e^{-y x^2} \, dx \) 当 \( 0 \leq y \leq \beta \) 时一致收敛（因为

\( 0 \leq x e^{-y x^2} \leq x e^{-\alpha x^2} \)，而 \( \int_{0}^{+\infty} x e^{-\alpha x^2} \, dx \) 收敛），

故可交换积分次序，得

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$$\int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \, dx$$

$$= \int_0^{+\infty} dx \int_0^{b} x e^{-y x^2} \, dy$$

$$= \int_0^{b} dy \int_0^{+\infty} x e^{-y x^2} \, dx$$

$$= \int_0^{b} \frac{dy}{2y} = \frac{1}{2} \ln \beta .$$

3794. \( \int_0^{+\infty} \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \right)^2 \, dx \) \((\alpha \to 0, \beta \to 0)\)

解 由于

$$\lim_{x \to +0} \frac{e^{-\alpha x} - e^{-\beta x}}{x}$$

$$= \lim_{x \to +0} \frac{-\alpha e^{-\alpha x} + \beta e^{-\beta x}}{1} = \beta - \alpha ,$$

故 \( x = 0 \) 不是瑕点，又由于

$$\lim_{x \to +\infty} x^2 \cdot \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \right)^2 = 0 ,$$

故积分 \( \int_0^{+\infty} \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \right)^2 \, dx \) 收敛 \((\alpha \to 0, \beta \to 0)\).

同样，将 \( \beta \to 0 \) 固定，考虑含参变量 \( \alpha \) 的积分:

$$I(\alpha) = \int_0^{+\infty} \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \right)^2 \, dx \) \ (\alpha \to 0) ,$$

由于

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\[
\int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{e^{-ax} - e^{-bx}}{x} \right)^2 \, dx = -2 \int_0^{+\infty} \frac{e^{-2ax} - e^{-\left(\alpha + \beta\right)x}}{x} \, dx
\]
\[
= -2 \ln \frac{\alpha + \beta}{2 \alpha} \quad (\alpha > 0).
\]

而当 \(\alpha \gg \alpha_0 > 0, \quad 1 \leq x \leq +\infty\) 时,
\[
\left| \frac{e^{-2ax} - e^{-\left(\alpha + \beta\right)x}}{x} \right| \leq \frac{2e^{-\alpha_0 x}}{x},
\]
且 \(\int_0^{+\infty} \frac{e^{-\alpha_0 x}}{x} \, dx\) 收敛（因为 \(\lim_{x \to +\infty} \frac{e^{-\alpha_0 x}}{x} = 0\)
故 \(\int_1^{+\infty} \frac{e^{-2ax} - e^{-\left(\alpha + \beta\right)x}}{x} \, dx\) 当 \(\alpha \gg \alpha_0\) 时一致收敛，
从而 \(\int_0^{+\infty} \frac{e^{-2ax} - e^{-\left(\alpha + \beta\right)x}}{x} \, dx\) 当 \(\alpha \gg \alpha_0\) 时一致收敛
（注意，因为 \(\lim_{x \to +0} \frac{e^{-2ax} - e^{-\left(\alpha + \beta\right)x}}{x} = \beta - \alpha\)，故 \(x = 0\) 不是端点）．因此，根据莱布尼兹法则，当 \(\alpha \gg \alpha_0\) 时可
在积分号下求导数：
\[
I'(\alpha) = \int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{e^{-ax} - e^{-bx}}{x} \right)^2 \, dx
\]
\[
=-2 \ln \frac{\alpha + \beta}{2 \alpha}.
\]

由 \(\alpha_0 > 0\) 的任意性知，上式对一切 \(\alpha > 0\) 皆成立。
积分之，并注意到

\[ \int \ln \frac{\alpha + \beta}{2\alpha} \, d\alpha = \alpha \ln \frac{\alpha + \beta}{2\alpha} + \beta \ln (\alpha + \beta) + C, \]

即得

\[ I(\alpha) = -2\alpha \ln \frac{\alpha + \beta}{2\alpha} - 2\beta \ln (\alpha + \beta) + C_1, \]

其中 \( C_1 \) 是待定常数。令 \( \alpha = \beta \)，则由于 \( I(\beta) = 0 \)，得

\[ 0 = -2\beta \ln \frac{2\beta}{2\beta} - 2\beta \ln 2\beta + C_1, \]

故 \( C_1 = 2\beta \ln 2 \beta \)。于是，得

\[ I(\alpha) = \ln \left( \frac{2\alpha}{\alpha + \beta} \right)^2 - 2\beta \ln (\alpha + \beta) + 2\beta \ln 2\beta \]

\[ = \ln \frac{(2\alpha)^2 (2\beta)^2}{(\alpha + \beta)^2 + 2\beta}, \]

即

\[ \int_0^{+\infty} \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \right)^2 \, dx \]

\[ = \ln \frac{(2\alpha)^2 (2\beta)^2}{(\alpha + \beta)^2 + 2\beta} (\alpha > 0, \beta > 0). \]

*）利用3783题的结果。

3795. \( \int_0^{+\infty} \frac{e^{-ax} - e^{-\beta x}}{x} \sin mx \, dx (\alpha > 0, \beta > 0) \).

解  当 \( m = 0 \) 时，
\[ \int_{0}^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin mx \, dx = 0, \]

故下设 \( m \neq 0 \). 由于

\[ \lim_{x \to 0} \frac{e^{-ax} - e^{-bx}}{x} \sin mx = 0, \]

故 \( x = 0 \) 不是零点，从而被积函数在域：\( 0 \leq x < +\infty \)及 \( \alpha > 0 \), \( \beta > 0 \)内连续 (\( x = 0 \) 时的函数值理解为极限值). 又由于

\[
\left| \frac{e^{-ax} - e^{-bx}}{x} \sin mx \right| \leq \frac{e^{-ax} + e^{-bx}}{x} \quad (x \to 0),
\]

而积分 \( \int_{1}^{+\infty} \frac{e^{-ax} + e^{-bx}}{x} \, dx \) 收敛，故积分

\[ \int_{1}^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} \sin mx \, dx \]

收敛。当 \( \alpha > \alpha_0 \to 0 \) 时，积分

\[ \int_{0}^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{e^{-ax} - e^{-bx}}{x} \sin mx \right) \, dx \]

\[ = - \int_{0}^{+\infty} e^{-ax} \sin mx \, dx \]

是一致收敛的。事实上，

\[ |e^{-ax} \sin mx| \leq e^{-\alpha_0 x} \quad (x \to 0), \]

而积分 \( \int_{0}^{+\infty} e^{-\alpha_0 x} \, dx = \frac{1}{\alpha_0} \) 收敛。于是，对于积分
$$I(\alpha) = \int_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin mx \, dx$$

当 $\alpha > \alpha_0$ 时可应用莱布尼兹法则，得

$$I'(\alpha) = - \int_{0}^{\infty} e^{-\alpha x} \sin mx \, dx = - \frac{m}{\alpha^2 + m^2}.$$  (*)

由 $\alpha_0 > 0$ 的任意性知，上式对一切 $\alpha > 0$ 均成立。从而

$$I(\alpha) = - \int_{0}^{\infty} \frac{m}{\alpha^2 + m^2} \, d\alpha = - \arctg \frac{\alpha}{m} + C,$$

其中 $C$ 是定常数。令 $\alpha = \beta$，则得

$$I(\beta) = 0 = - \arctg \frac{\beta}{m} + C,$$

故 $C = \arctg \frac{\beta}{m}$. 最后得

$$\int_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin mx \, dx$$

$$= \arctg \frac{\beta}{m} - \arctg \frac{\alpha}{m} \quad (m \neq 0).$$

(*) 利用1829题的结果。

3796. $\int_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx \, dx \quad (\alpha > 0, \beta > 0)$.

解 同3795题，我们可证明：当 $\alpha \geqslant \alpha_0 > 0$ 时，对积分

$$I(\alpha) = \int_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx \, dx$$
可应用莱布尼兹法则，得

\[ I''(\alpha) = \int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx \right) dx \]

\[ = \int_0^{+\infty} e^{-\alpha x} \cos mx \, dx = -\frac{\alpha}{\alpha^2 + m^2}. \]

由 \( \alpha \geq 0 \) 的任意性知，上式对一切 \( \alpha > 0 \) 均成立。

从而

\[ I(\alpha) = -\int \frac{\alpha \, d\alpha}{\alpha^2 + m^2} = -\frac{1}{2} \ln(\alpha^2 + m^2) + C, \]

其中 \( C \) 是常定常数。令 \( \alpha = \beta \)，则得

\[ I(\beta) = 0 = -\frac{1}{2} \ln(\beta^2 + m^2) + C, \]

故 \( C = \frac{1}{2} \ln(\beta^2 + m^2) \)。最后得

\[ \int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx \, dx = \frac{1}{2} \ln \frac{\beta^2 + m^2}{\alpha^2 + m^2} \quad (\alpha \geq 0, \beta > 0). \]

*）利用1828题的结果。

计算下列积分：

![3797.](image)

解

由于

\[ \lim_{x \to 0} \frac{\ln(1 - \alpha^2 x^2)}{x^2 \sqrt{1 - x^2}} \]

\[ \lim_{x \to 0} \frac{\ln(1 - \alpha^2 x^2)}{x^2} = \lim_{x \to 0} \frac{\ln(1 - \alpha^2 x^2)}{x^2} \]

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$$\lim_{x \to +0} \frac{2\alpha^4 x}{1 - \alpha^2 x^2} = -\alpha^4,$$

故 $x = 0$ 不是瑕点。从而被积函数在域：$0 \leq x \leq 1$ 及 $|\alpha| \leq 1$ 内连续（$x = 0$ 时的函数值理解为极限值）。又由于当 $|\alpha| \leq 1$ 时，

$$\left| \frac{\ln \left( \frac{1 - \alpha^2 x^2}{\sqrt{1 - x^2}} \right)}{\frac{1 - x^2}{x^2 \sqrt{1 - x^2}}} \right| \leq \frac{\ln \left( \frac{1 - x^2}{\sqrt{1 - x^2}} \right)}{x^2 \sqrt{1 + x}} \quad (0 \leq x \leq 1),$$

面积分 $\int_0^1 \frac{\ln \left( \frac{1 - x^2}{\sqrt{1 - x^2}} \right)}{x^2 \sqrt{1 - x^2}} d\alpha$ 收敛（因为 $\lim_{x \to 1-0} \frac{\ln \left( \frac{1 - x^2}{\sqrt{1 - x^2}} \right)}{x^2 \sqrt{1 + x}} = 0$），

故积分

$$\int_0^1 \frac{\ln \left( \frac{1 - \alpha^2 x^2}{\sqrt{1 - x^2}} \right)}{x^2 \sqrt{1 - x^2}} d\alpha$$

对 $|\alpha| \leq 1$ 一致收敛。从而为 $\alpha$ 的连续函数（$-1 \leq \alpha \leq 1$）。另一方面，易知积分

$$\int_0^1 \frac{\partial}{\partial \alpha} \left[ \frac{\ln \left( \frac{1 - \alpha^2 x^2}{\sqrt{1 - x^2}} \right)}{x^2 \sqrt{1 - x^2}} \right] d\alpha$$

$$= -2\alpha \int_0^1 \frac{d\alpha}{(1 - \alpha^2 x^2) \sqrt{1 - x^2}}$$

对 $|\alpha| \leq \alpha_0 \leq 1$ 一致收敛。事实上，

$$\left| \frac{-2\alpha}{(1 - \alpha^2 x^2) \sqrt{1 - x^2}} \right|$$

$$\leq \frac{2}{1 - \alpha^6} \cdot \frac{1}{\sqrt{1 - x^2}} \quad (0 \leq x \leq 1),$$
而积分
\[ \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \]
收敛。于是，对积分
\[ I(\alpha) = \int_0^1 \frac{\ln (1-\alpha^2 x^2)}{x^2 \sqrt{1-x^2}} \, dx \]
当 \( |\alpha| \ll \alpha_0 \) 时可应用莱布尼兹法则，得
\[ I'(\alpha) = -2\alpha \int_0^1 \frac{dx}{(1-\alpha^2 x^2) \sqrt{1-x^2}} \cdot \]
由 \( \alpha_0 \ll 1 \) 的任意性知，上式对一切 \( |\alpha| \ll 1 \) 均成立。
先求不定积分
\[ I_1 = \int \frac{dx}{(1-\alpha^2 x^2) \sqrt{1-x^2}} \]
作代换 \( x = \sin t \)，易得
\[ I_1 = \int \frac{dt}{1-\alpha^2 \sin^2 t} = \frac{1}{2} \left( \int \frac{dt}{1-\alpha \sin t} + \int \frac{dt}{1+\alpha \sin t} \right) \cdot \]
再对右端两个积分作代换 \( u = \tan \frac{t}{2} \)，可得
\[ \int \frac{dt}{1-\alpha \sin t} = \frac{2}{\sqrt{1-\alpha^2}} \arctan \left( \frac{\tan \frac{t}{2} - \alpha}{\sqrt{1-\alpha^2}} \right) + C_1 \]
\[ \int \frac{dt}{1+\alpha \sin t} \]
\[
\frac{2}{\sqrt{1-\alpha^2}} \arctg \left( \frac{\frac{t}{2} + \alpha}{\sqrt{1-\alpha^2}} \right) + C_2.
\]

从而

\[
I'(\alpha) = 2 \alpha \int_0^\frac{\pi}{2} \left( \frac{1}{1-\alpha \sin t} \right. \\
+ \frac{1}{1+\alpha \sin t} \left. \right) dt
\]

\[
= -\frac{2\alpha}{\sqrt{1-\alpha^2}} \left[ \arctg \left( \frac{\frac{t}{2} - \alpha}{\sqrt{1-\alpha^2}} \right) \\
+ \arctg \left( \frac{\frac{t}{2} + \alpha}{\sqrt{1-\alpha^2}} \right) \right]_0^{\frac{\pi}{2}}
\]

\[
= -\frac{\pi \alpha}{\sqrt{1-\alpha^2}} \quad (|\alpha| < 1).
\]

两端积分，得

\[
I(\alpha) = -\pi \int \frac{\alpha \, da}{\sqrt{1-\alpha^2}}
\]

\[
= \pi \sqrt{1-\alpha^2} + C \quad (|\alpha| < 1),
\]

其中 C 是待定常数。令 \(\alpha = 0\)，得

\[I(0) = 0 = \pi + C,\]

故 \(C = -\pi\)，从而

\[I(\alpha) = -\pi \left( 1 - \sqrt{1-\alpha^2} \right) \quad (|\alpha| < 1).\]

在此式两端令 \(\alpha \to 1 - 0\) 及 \(\alpha \to -1 + 0\) 取极限，并注意到 \(I(\alpha)\) 在 \(-1 < \alpha < 1\) 上的连续性，即得

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\[ I(1) = I(-1) = -\pi. \]

于是，当 \(|\alpha| \leq 1\) 时，

\[
\int_0^1 \frac{\ln \left(1 - \alpha^2 x^2\right)}{x^2 \sqrt{1 - x^2}} \, dx = -\pi \left(1 - \sqrt{1 - \alpha^2}\right).
\]

3798. \[ \int_0^1 \frac{\ln \left(1 - \alpha^2 x^2\right)}{\sqrt{1 - x^2}} \, dx \quad (|\alpha| \leq 1). \]

解 同3797题，我们可以证明，

\[ I(\alpha) = \int_0^1 \frac{\ln \left(1 - \alpha^2 x^2\right)}{\sqrt{1 - x^2}} \, dx \]

当 \(-1 \leq \alpha \leq 1\) 时连续，且当 \(|\alpha| \leq \alpha_0 < 1\) 时可应用 莱布尼兹法则。于是，

\[ I'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha} \left[ \frac{\ln \left(1 - \alpha^2 x^2\right)}{\sqrt{1 - x^2}} \right] \, dx \]

\[ = \int_0^1 \frac{-2 \alpha x^2}{(1 - \alpha^2 x^2) \sqrt{1 - x^2}} \, dx \]

\[ = \frac{2}{\alpha} \int_0^1 \frac{(1 - \alpha^2 x^2) - 1}{(1 - \alpha^2 x^2) \sqrt{1 - x^2}} \, dx \]

\[ = \frac{2}{\alpha} \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \]

\[ = \frac{2}{\alpha} \left( -\int_0^1 \frac{dx}{\sqrt{1 - x^2}} \right) \]

\[ = \frac{\pi}{\alpha} - \frac{\pi}{\alpha} \frac{\pi}{2 \sqrt{1 - \alpha^2}} \quad (|\alpha| \leq \alpha_0, \alpha \neq 0). \]
由 $\alpha_0 < 1$ 的任意性知，上式对一切 $0 < \alpha < 1$ 均成立。积分得

$$I(\alpha) = \int \left( \frac{\pi}{\alpha} - \frac{\pi}{\alpha\sqrt{1-\alpha^2}} \right) d\alpha$$

$$= \pi \ln |\alpha| + \pi \ln \left| \frac{1 + \sqrt{1-\alpha^2}}{\alpha} \right| + C$$

$$= \pi \ln (1 + \sqrt{1-\alpha^2}) + C,$$

其中 $|\alpha| < 1, \alpha \neq 0, C$ 为待定常数。令 $\alpha \to 0$，并注意到 $I(\alpha)$ 在 $\alpha = 0$ 的连续性，即得

$$I(0) = 0 = \pi \ln 2 + C,$$

故 $C = -\pi \ln 2$，从而得

$$I(\alpha) = \pi \ln \frac{1 + \sqrt{1-\alpha^2}}{2} \quad (|\alpha| < 1).$$

在上式中令 $\alpha \to 1 - 0$ 及 $\alpha \to 1 + 0$，并注意到 $I(\alpha)$ 在 $-1 < \alpha < 1$ 上的连续性，即知上式当 $\alpha = \pm 1$ 时也成立，即

$$\int_0^1 \ln \frac{(1-\alpha^2) \frac{1}{x^2}}{\sqrt{1-x^2}} dx$$

$$= \pi \ln \frac{1 + \sqrt{1-\alpha^2}}{2} \quad (|\alpha| < 1).$$

3799. $\int_1^{+\infty} \frac{\arctg \alpha x}{x^2 \sqrt{x^2 - 1}} dx.$

解 设 $I(\alpha) = \int_1^{+\infty} \frac{\arctg \alpha x}{x^2 \sqrt{x^2 - 1}} dx$，显然有 $I(0) = 0$. 

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当 $\alpha > 0$ 时，由于 $\lim_{x \to \infty} \frac{\operatorname{arc\,tg} \alpha x}{x^2 \sqrt{x^2 - 1}} = \frac{\pi}{2}$，故

$I(\alpha)$ 收敛。其次，易知积分

$$
\int_1^{+\infty} \frac{d}{dx} \left( \frac{\operatorname{arc\,tg} \alpha x}{x^2 \sqrt{x^2 - 1}} \right) dx
$$

$$
= \int_1^{+\infty} \frac{d x}{x \left(1 + \alpha^2 x^2\right) \sqrt{x^2 - 1}}
$$

$$
= \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^2 (t^2 + \alpha^2)}}
$$

对 $\alpha \geq 0$ 一致收敛。事实上，当 $\alpha \geq 0$，$0 \leq t \leq 1$ 时，有

$$
\left| \frac{t^2}{\sqrt{1 - t^2 (t^2 + \alpha^2)}} \right| \leq \frac{1}{\sqrt{1 - t^2}}
$$

且 $\int_0^1 \frac{dt}{\sqrt{1 - t^2}}$ 收敛。于是，可应用莱布尼兹法则，得

$$
I'(\alpha) = \int_1^{+\infty} \frac{d}{dx} \left( \frac{\operatorname{arc\,tg} \alpha x}{x^2 \sqrt{x^2 - 1}} \right) dx
$$

$$
= \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^2 (t^2 + \alpha^2)}}
$$

$$
= \int_0^1 \frac{(t^2 + \alpha^2) - \alpha^2}{\sqrt{1 - t^2 (t^2 + \alpha^2)}} dt
$$

$$
= \int_0^1 \frac{dt}{\sqrt{1 - t^2}}
$$

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\[-\alpha^2 \int_0^1 \frac{dt}{\sqrt{1 - t^2(t^2 + \alpha^2)}} \]

\[= \frac{\pi}{2} - \alpha^2 \cdot \frac{\pi}{2\alpha\sqrt{\alpha^2 + 1}} \]

\[= \frac{\pi}{2} - \frac{\alpha}{2} \frac{\pi}{\sqrt{1 + \alpha^2}} \quad (\alpha \geq 0), \]

从而有

\[I(\alpha) = \frac{\pi}{2} \alpha - \frac{\pi}{2} \int \frac{\alpha \, d\alpha}{\sqrt{1 + \alpha^2}} \]

\[= \frac{\pi}{2} \alpha - \frac{\pi}{2} \sqrt{1 + \alpha^2} + C \quad (\alpha \geq 0), \]

其中 C 为待定常数。令 \(\alpha = 0\)，得

\[I(0) = 0 = -\frac{\pi}{2} + C,\]

故 \(C = \frac{\pi}{2}\)。于是，当 \(\alpha \geq 0\) 时，

\[\int_1^{+\infty} \frac{\text{arc tg} \, \alpha x}{x^2 \sqrt{x^2 - 1}} \, dx = \frac{\pi}{2} \left(1 + \alpha - \sqrt{1 + \alpha^2}\right).\]

当 \(\alpha < 0\) 时，

\[\int_1^{+\infty} \frac{\text{arc tg} \, \alpha x}{x^2 \sqrt{x^2 - 1}} \, dx \]

\[= -\int_1^{+\infty} \frac{\text{arc tg} \, (-\alpha) x}{x^2 \sqrt{x^2 - 1}} \, dx \]

\[= -\frac{\pi}{2} \left(1 + \alpha - \sqrt{1 + \alpha^2}\right).\]

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于是，当 $-\infty \leq a \leq +\infty$ 时，

$$
\int_{1}^{+\infty} \frac{\text{arc tan } ax}{x^{2} \sqrt{x^{2} - 1}} \, dx
= \frac{\pi}{2} \left( 1 + |a| - \sqrt{1 + a^{2}} \right) \text{sgn } a.
$$

3800. \quad \int_{0}^{+\infty} \frac{\ln (a^{2} + x^{2})}{\beta^{2} + x^{2}} \, dx.

解

我们首先计算积分

$$
I_{f}(\alpha) = \int_{0}^{+\infty} \frac{\ln (1 + \alpha^{2} x^{2})}{\beta^{2} + x^{2}} \, dx
$$

（$\alpha \geq 0$ 是参数，$\beta > 0$ 固定）。

首先注意，此积分当 $0 \leq a \leq a_{1}$ ($a_{1} > 0$ 为任何有限数) 时一致收敛。事实上，当 $0 \leq a \leq a_{1}$ 时，

$$
0 \leq \frac{\ln (1 + \alpha^{2} x^{2})}{\beta^{2} + x^{2}}
\leq \frac{\ln (1 + a_{1}^{2} x^{2})}{\beta^{2} + x^{2}} \quad (0 \leq x \leq +\infty),
$$

而积分 $\int_{0}^{+\infty} \frac{\ln (1 + a_{1}^{2} x^{2})}{\beta^{2} + x^{2}} \, dx$ 收敛（因为易知

$$
\lim_{x \to +\infty} x^{\frac{3}{2}} \frac{\ln (1 + a_{1}^{2} x^{2})}{\beta^{2} + x^{2}} = 0.
$$

于是，$I_{f}(\alpha)$ 是 $0 \leq a \leq a_{1}$ 上的连续函数。由 $a_{1} > 0$ 的任意性知，$I_{f}(\alpha)$ 当 $0 \leq a \leq +\infty$ 时连续。

其次，易证积分
\[
\int_0^{+\infty} \frac{\partial}{\partial \alpha} \left[ \frac{\ln(1+\alpha^2x^2)}{\beta^2+x^2} \right] dx
\]

\[
= \int_0^{+\infty} \frac{2\alpha x^2}{(\beta^2+x^2)(1+\alpha^2x^2)} dx = \frac{\pi}{a\beta+1}
\]

当 \(0 \leq \alpha_0 \leq \alpha \leq \alpha_1\) 时是一致收敛的。事实上，此时

\[
0 \leq \frac{2\alpha x^2}{(\beta^2+x^2)(1+\alpha^2x^2)} \leq \frac{2\alpha_1 x^2}{(\beta^2+x^2)(1+\alpha_0^2x^2)} \quad (0 \leq x < +\infty),
\]

积分 \(\int_0^{+\infty} \frac{2\alpha_1 x^2}{(\beta^2+x^2)(1+\alpha_0^2x^2)} dx\) 收敛。于是，根据莱布尼兹法则，当 \(0 \leq \alpha_0 \leq \alpha \leq \alpha_1\) 时，可在积分号下求导数，得

\[
I_\beta'(\alpha) = \frac{\pi}{a\beta+1}.
\]

由 \(\alpha_1\) 与 \(\alpha_0\) 的任意性知，式子对一切 \(0 \leq \alpha \leq +\infty\) 均成立。两端积分，得

\[
I_\beta(\alpha) = \frac{\pi}{\beta} \ln(1+a\beta) + C \quad (0 \leq \alpha \leq +\infty),
\]

其中 \(C\) 是某常数。在此式中令 \(a \to +0\) 取极限，并注意到 \(I_\beta(\alpha)\) 在 \(0 \leq \alpha \leq +\infty\) 上连续，得

\[
0 = I_\beta(0) = 0 + C,
\]

故 \(C = 0\)。因此

\[
I_\beta(\alpha) = \frac{\pi}{\beta} \ln(1+a\beta) \quad (0 \leq \alpha \leq +\infty).
\]

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对于所求积分，只要作适当变形即得。当 $\alpha \to 0$, $\beta \to 0$ 时，有

$$\int_0^{+\infty} \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx$$

$$= \int_0^{+\infty} \frac{2 \ln \alpha + \ln \left(1 + \frac{1}{\alpha^2} x^2\right)}{\beta^2 + x^2} \, dx$$

$$= 2 \ln \alpha \int_0^{+\infty} \frac{dx}{\beta^2 + x^2}$$

$$+ \int_0^{+\infty} \frac{\ln \left(1 + \frac{1}{\alpha^2} x^2\right)}{\beta^2 + x^2} \, dx$$

$$= \frac{\pi \ln \alpha}{\beta} + \frac{\pi}{\beta} \ln \left(1 + \frac{\beta}{\alpha}\right) = \frac{\pi}{\beta} \ln(\alpha + \beta).$$

此式当 $\alpha = 0$ 时也成立，只要在两端令 $\alpha \to +0$ 取极限即可。这是因为积分 $I(\alpha) = \int_0^{+\infty} \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx$ $(\beta \to 0$ 固定) 当 $0 < \alpha \leq \frac{1}{2}$ 一致收敛 (易知

$$\int_0^{\frac{1}{2}} \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx \quad \text{与} \quad \int_{\frac{1}{2}}^{+\infty} \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx$$

当 $0 \leq \alpha \leq \frac{1}{2}$ 时都一致收敛，事实上，

$$\left| \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \right|$$

$$\leq \frac{2 \ln x}{\beta^2 + x^2} \left( 0 < x \leq \frac{1}{2}, \quad 0 \leq \alpha \leq \frac{1}{2} \right),$$

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而 \( \int_{\frac{1}{2}}^{\frac{1}{\alpha}} \frac{\ln x}{\beta^2 + x^2} \, dx \) 收敛，

\[
0 \leq \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \leq \frac{\ln\left(\frac{1}{4} + x^2\right)}{\beta^2 + x^2} \quad \left(\frac{1}{2} \leq x \leq +\infty, \quad 0 \leq \alpha \leq \frac{1}{2}\right),
\]

而 \( \int_{\frac{1}{2}}^{\infty} \frac{\ln\left(\frac{1}{4} + x^2\right)}{\beta^2 + x^2} \, dx \) 收敛，故 \( J(\alpha) \) 在点 \( \alpha = 0 \) (右) 连续。

对于任意的 \( \alpha \) 与 \( \beta \) (\( \beta \neq 0 \))，有

\[
\int_{0}^{\infty} \frac{\ln(\alpha^2 + x^2)}{\beta^2 + x^2} \, dx = \int_{0}^{\infty} \frac{\ln(|\alpha|^2 + x^2)}{|\beta|^2 + x^2} \, dx = \frac{\pi}{|\beta|} \ln(|\alpha| + |\beta|).
\]

注意，当 \( \beta = 0 \) 时上式不成立，右端无意义，左端的积分 \( \int_{0}^{\infty} \frac{\ln(\alpha^2 + x^2)}{x^2} \, dx \) 易知是发散的。

3801. \( \int_{0}^{\infty} \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \, dx \).

解 先设 \( \alpha \geq 0, \beta \geq 0 \)。显然 \( x = 0 \) 不是极点，因为

\[
\lim_{x \to 0} \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} = \alpha \beta.
\]

由于当 \( \alpha \geq 0, \beta \geq 0 \) 时，
\[ \left| \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \right| \leq \frac{\pi^2}{4} \cdot \frac{1}{x^2} \quad (1 \leq x < +\infty), \]

而积分 \( \int_1^{+\infty} \frac{dx}{x^2} \) 收敛，故积分

\[ \int_1^{+\infty} \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \, dx \quad \text{在} \ a \geq 0, \ \beta \geq 0 \ 时一一致收敛，从而积分} \int_0^{+\infty} \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \, dx \quad \text{也}

在 \ a \geq 0, \ \beta \geq 0 \ 时一致收敛。因此，函数

\[ I(\alpha, \beta) = \int_0^{+\infty} \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \, dx \]

是 \ a \geq 0, \ \beta \geq 0 \ 上的二元连续函数。再考察两个积分

\[ J(\alpha, \beta) = \int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{\arctg \alpha x \cdot \arctg \beta x}{x^2} \right) \, dx \]

\[ = \int_0^{+\infty} \frac{\arctg \beta x}{x(1+\alpha^2 x^2)} \, dx, \]

\[ K(\alpha, \beta) = \int_0^{+\infty} \frac{\partial}{\partial \beta} \left( \frac{\arctg \beta x}{x(1+\alpha^2 x^2)} \right) \, dx \]

\[ = \int_0^{+\infty} \frac{dx}{(1+\alpha^2 x^2)(1+\beta^2 x^2)}. \]

由于当 \ a \geq a_0 > 0, \ \beta \geq 0 \ 时 \left| \frac{\arctg \beta x}{x(1+\alpha^2 x^2)} \right| \leq \frac{\pi}{2} \]

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\[
\frac{1}{x(1+\alpha^2x^2)} \quad (1 \leq x \leq +\infty), \quad \text{而积分}
\]

\[
\int_1^{+\infty} \frac{dx}{x(1+\alpha^2x^2)} \quad \text{收敛，故积分} \quad \int_1^{+\infty} \frac{\arctan \beta x}{x(1+\alpha^2x^2)} \, dx
\]

当 \(\alpha \geq \alpha_0, \beta \geq 0\) 时一致收敛，从而积分

\[
\int_0^{+\infty} \frac{\arctan \beta x}{x(1+\alpha^2x^2)} \, dx \quad \text{当} \quad \alpha \geq \alpha_0, \beta \geq 0 \quad \text{时也一致收敛}
\]

(因为 \(\lim_{x \to +0} \frac{\arctan \beta x}{x(1+\alpha^2x^2)} = \beta\)，故 \(x = 0\) 不是瑕点)。

因此，\( J(\alpha, \beta) \) 当 \(\alpha \geq \alpha_0, \beta \geq 0\) 时连续，并且此时 \( I(\alpha, \beta) \) 可在积分号下对 \(\alpha\) 求导数，得

\[
I'_\alpha(\alpha, \beta) = \int_0^{+\infty} \frac{\arctan \beta x}{x(1+\alpha^2x^2)} \, dx = J(\alpha, \beta). \quad (1)
\]

由 \(\alpha_0 \geq 0\) 的任意性知，(1) 式对一切 \(\alpha \geq 0, \beta \geq 0\) 成立，并且 \(J(\alpha, \beta)\) 是 \(\alpha \geq 0, \beta \geq 0\) 上的二元连续函数。

其次，由于当 \(\beta \geq \beta_0 > 0, \alpha > 0\) 时，

\[
0 \leq \frac{1}{(1+\alpha^2x^2)(1+\beta^2x^2)}
\]

\[
\leq \frac{1}{1+\beta_0^2x^2} \quad (0 \leq x \leq +\infty).
\]

而积分 \(\int_0^{+\infty} \frac{dx}{1+\beta_0^2x^2} \) 收敛，故积分

\[
\int_0^{+\infty} \frac{dx}{(1+\alpha^2x^2)(1+\beta^2x^2)}
\]
当 $\beta \geqslant \beta_0$, $\alpha \gg 0$ 时一致收敛。因此，$K(\alpha, \beta)$ 是 $\alpha \gg 0$, $\beta \geqslant \beta_0$ 上的连续函数，并且(1) 式中的积分 当 $\beta \geqslant \beta_0$ ($\alpha \gg 0$) 时可在积分号下对 $\beta$ 求导数，得

$$I''_{\alpha\beta} (\alpha, \beta) = I'_\beta (\alpha, \beta)$$

$$= \int_0^{+\infty} \frac{dx}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)}$$

$$= \frac{\alpha^2}{\alpha^2 - \beta^2} \int_0^{+\infty} \frac{dx}{1 + \alpha^2 x^2}$$

$$- \frac{\beta^2}{\alpha^2 - \beta^2} \int_0^{+\infty} \frac{dx}{1 + \beta^2 x^2}$$

$$= \frac{\alpha \pi}{2(\alpha^2 - \beta^2)} - \frac{\beta \pi}{2(\alpha^2 - \beta^2)}$$

$$= \frac{\pi}{2(\alpha + \beta)}.$$  

由 $\beta_0 \gg 0$ 的任意性知，对任何 $\alpha \gg 0$, $\beta \gg 0$ 均有

$$I''_{\alpha\beta} (\alpha, \beta) = I'_\beta (\alpha, \beta) = \frac{\pi}{2(\alpha + \beta)}. \tag{2}$$

(注意，在推导此式时应设 $\alpha \neq \beta$，因为推导过程中分母内有 $\alpha^2 - \beta^2$，但由于 $K(\alpha, \beta)$ 是 $\alpha \gg 0$, $\beta \geqslant 0$ 上的连续函数，故通过取极限即知(2) 式当 $\alpha = \beta$ 时也成立)。在 (2) 式中固定 $\alpha \gg 0$, 对 $\beta$ 积分，得

$$I'_\alpha (\alpha, \beta) = I(\alpha, \beta)$$

$$= \frac{\pi}{2} \ln(\alpha + \beta) + C(\alpha) \quad (0 < \beta < +\infty),$$

其中 $C(\alpha)$ 是依赖于 $\alpha$ 的常数。在此式中令 $\beta \rightarrow +0$, 并注意到 $I(\alpha, \beta)$ 在 $\alpha \gg 0$, $\beta \geqslant 0$ 上连续，得
$$0 = I(\alpha, 0) = \lim_{\beta \to 0} I(\alpha, \beta) = \frac{\pi}{2} \ln \alpha + C(\alpha),$$

故

$$C(\alpha) = -\frac{\pi}{2} \ln \alpha.$$ 

因此，

$$I_\varepsilon(\alpha, \beta) = \frac{\pi}{2} \ln \frac{\alpha + \beta}{\alpha} \quad (\alpha \geq 0, \beta \geq 0).$$

再固定 $\beta \geq 0$，对 $\alpha$ 积分（右端利用分部积分法），得

$$I(\alpha, \beta) = \frac{\pi}{2} \alpha \ln \frac{\alpha + \beta}{\alpha}$$

$$+ \frac{\pi}{2} \beta \ln(\alpha + \beta) + C^*(\beta),$$

其中 $C^*(\beta)$ 是依赖于 $\beta$ 的常数。在此式中令 $\alpha \to +0$，并注意到 $I(\alpha, \beta)$ 在 $\alpha \geq 0, \beta \geq 0$ 上连续，得

$$0 = I(0, \beta) = \lim_{\alpha \to +0} I(\alpha, \beta)$$

$$= \frac{\pi}{2} \beta \ln \beta + C^*(\beta),$$

故

$$C^*(\beta) = -\frac{\pi}{2} \beta \ln \beta,$$

于是，

$$I(\alpha, \beta) = \frac{\pi}{2} \ln \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta} \quad (\alpha \geq 0, \beta \geq 0).$$
显然，对于任何 \( \alpha \) 与 \( \beta \)，有

\[
\int_{0}^{+\infty} \frac{\arctan \alpha x \cdot \arctan \beta x}{x^2} \, dx
\]

\[
= \begin{cases} 
\frac{\text{sgn}(\alpha \beta) \cdot \pi}{2} \ln \frac{(|\alpha| + |\beta|)^{1+\beta_1}}{|\alpha|^{1+\alpha_1} \cdot |\beta|^{1+\beta_1}}, & \text{当} \alpha \beta \neq 0 \text{ 时}; \\
0, & \text{当} \alpha \beta = 0 \text{ 时}.
\end{cases}
\]

3802. \( \int_{0}^{+\infty} \frac{\ln(1+\alpha^2 x^2) \ln(1+\beta^2 x^2)}{x^4} \, dx \)

解：先设 \( \alpha \geq 0 \)，\( \beta \geq 0 \)。首先，注意，\( x = 0 \) 不是瑕点，因为

\[
\lim_{x \to 0} \frac{\ln(1+\alpha^2 x^2) \ln(1+\beta^2 x^2)}{x^4} = \alpha^2 \beta^2.
\]

由于当 \( 0 \leq \alpha \leq \alpha_1 \)，\( 0 \leq \beta \leq \beta_1 \) 时，恒有

\[
0 \leq \frac{\ln(1+\alpha^2 x^2) \ln(1+\beta^2 x^2)}{x^4} \leq \frac{\ln(1+\alpha_1^2 x^2) \ln(1+\beta_1^2 x^2)}{x^4},
\]

而 \( \int_{0}^{+\infty} \frac{\ln(1+\alpha^2 x^2) \ln(1+\beta^2 x^2)}{x^4} \, dx \) 收敛（因为

\[
\lim_{x \to +\infty} x^2 \cdot \frac{\ln(1+\alpha_1^2 x^2) \ln(1+\beta_1^2 x^2)}{x^4} = 0
\]

故积分 \( \int_{0}^{+\infty} \frac{\ln(1+\alpha^2 x^2) \ln(1+\beta^2 x^2)}{x^4} \, dx \) 当 \( 0 \leq \alpha \leq \alpha_1 \)，\( 0 \leq \beta \leq \beta_1 \) 时一致收敛。因此，函数
\[ l(\alpha, \beta) = \int_0^{+\infty} \frac{\ln(1 + \alpha^2 x^2) \ln(1 + \beta^2 x^2)}{x^4} \, dx \quad (1) \]

是 \( 0 \leq \alpha \leq \alpha_1, \ 0 \leq \beta \leq \beta_1 \) 上的二元连续函数。由 \( \alpha_1 \to 0, \ \beta_1 \to 0 \) 的任意性知，\( l(\alpha, \beta) \) 是 \( \alpha \geq 0, \ \beta \geq 0 \) 上的二元连续函数。再考察两个积分

\[ J(\alpha, \beta) = \int_0^{+\infty} \frac{\partial}{\partial \alpha} \left[ \frac{\ln(1 + \alpha^2 x^2) \ln(1 + \beta^2 x^2)}{x^4} \right] \, dx \]

\[ = \int_0^{+\infty} \frac{2 \alpha \ln(1 + \beta^2 x^2)}{x^2(1 + \alpha^2 x^2)} \, dx, \quad (2) \]

\[ K(\alpha, \beta) = \int_0^{+\infty} \frac{\partial}{\partial \beta} \left[ \frac{2 \alpha \ln(1 + \beta^2 x^2)}{x^2(1 + \alpha^2 x^2)} \right] \, dx \]

\[ = \int_0^{+\infty} \frac{2 \alpha \beta}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)} \, dx \]

\[ = \frac{2 \pi \alpha \beta}{\alpha + \beta} \quad (\alpha \geq 0, \ \beta \geq 0). \quad (3) \]

由于当 \( 0 \leq \alpha \leq \alpha_1, \ 0 \leq \beta \leq \beta_1 \) 时，恒有

\[ 0 \leq \frac{2 \alpha \ln(1 + \beta^2 x^2)}{x^2(1 + \alpha^2 x^2)} \leq \frac{2 \alpha_1 \ln(1 + \beta_1^2 x^2)}{x^2(1 + \alpha_1^2 x^2)} \quad (0 \leq x \leq +\infty), \]

而易知积分 \( \int_0^{+\infty} \frac{2 \alpha_1 \ln(1 + \beta_1^2 x^2)}{x^2(1 + \alpha_1^2 x^2)} \, dx \) 收敛，故 (2) 式中的积分在 \( 0 \leq \alpha \leq \alpha_1, \ 0 \leq \beta \leq \beta_1 \) 上一致收敛。由此可知 \( J(\alpha, \beta) \) 是 \( \alpha \leq \alpha \leq \alpha_1, \ 0 \leq \beta \leq \beta_1 \) 上的连续函数，并且在其上 (1) 中的积分可在积分号
下对 $\alpha$ 求导数，得

$$I^*_1(\alpha, \beta) = \int_0^{+\infty} \frac{2\alpha \ln(1 + \beta^2 x^2)}{x^2(1 + \alpha^2 x^2)} \, dx$$

$$= J(\alpha, \beta). \quad (4)$$

由 $\alpha_1 > \alpha_2 > 0$ 及 $\beta_1 > 0$ 的任意性知，$J(\alpha, \beta)$ 是 $\alpha > 0$, $\beta > 0$ 上的连续函数，并且 (4) 式对一切 $\alpha > 0$, $\beta > 0$ 都成立.

其次，当 $0 < \alpha \leq \alpha_1$, $0 < \beta_0 \leq \beta \leq \beta_1$ 时，恒有

$$0 \leq \frac{4 \alpha \beta}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)}$$

$$\leq \frac{4 \alpha_1 \beta_1}{1 + \beta_0^2 x^2} \quad (0 \leq x \leq +\infty),$$

而积分 $\int_0^{+\infty} \frac{4 \alpha_1 \beta_1}{1 + \beta_0^2 x^2} \, dx$ 收敛，故 (3) 式中的积分在 $0 < \alpha \leq \alpha_1$, $0 < \beta_0 \leq \beta \leq \beta_1$ 上一致收敛。于是，在其上 (2) 式中的积分可在积分号下对 $\beta$ 求导数，得

$$I^*_2(\alpha, \beta) = J^*_1(\alpha, \beta)$$

$$= \int_0^{+\infty} \frac{4 \alpha \beta}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)} \, dx$$

$$= \frac{2 \pi \alpha \beta}{\alpha + \beta}. \quad (5)$$

由 $\alpha_1 > 0$, $\beta_1 > \beta_0 > 0$ 的任意性知，(5) 式对一切 $\alpha > 0$, $\beta > 0$ 都成立。 (5) 式两端对 $\beta$ 积分之（$\alpha > 0$ 固定），得
\[ I_1(a, \beta) = I(a, \beta) = 2\pi a \beta - 2\pi a^2 \ln(a + \beta) + C(a) \]

（0 ≤ β ≤ +∞）

其中 C(a) 是依赖于 α 的常数。在此式中令 β → +0，取极限，并注意到 I(a, β) 在 α ≥ 0, β ≥ 0 上连续，得

\[ 0 = I(a, 0) = \lim_{\beta \to +0} I(a, \beta) = -2\pi a^2 \ln a + C(a), \]

故

\[ C(a) = 2\pi a^2 \ln a. \]

因此，

\[ I_\alpha(a, \beta) = 2\pi a \beta - 2\pi a^2 \ln(a + \beta) + 2\pi a^2 \ln a \]

（α ≥ 0, β ≥ 0）

两端再对 α 积分 (β ≥ 0 固定)，得

\[ I(a, \beta) = \pi a^2 \beta - \frac{2}{3} \pi a^3 \ln(a + \beta) \]

\[ + \frac{2}{9} \pi(a + \beta)^3 - \frac{2}{3} \pi a^2 \beta \]

\[ - \frac{2}{3} \pi \beta^3 \ln(a + \beta) + \frac{2}{3} \pi a^3 \ln a \]

\[ - \frac{2}{9} a^3 + C^*(\beta) \quad (0 ≤ a ≤ +∞), \]

其中 C^*(β) 是依赖于 β 的常数。在此式两端令 α → +0 取极限，并注意到 I(a, β) 在 α ≥ 0, β ≥ 0 上连续，得
\[ 0 = I(0, \beta) = \lim_{\alpha \to 0} I(\alpha, \beta) \]
\[ = \frac{2\pi}{9} \beta^3 - \frac{2}{3}\pi \beta^3 \ln \beta + C^*(\beta), \]

故

\[ C^*(\beta) = -\frac{2\pi}{9} \beta^3 + \frac{2}{3}\pi \beta^3 \ln \beta. \]

于是

\[ I(\alpha, \beta) = -\frac{2}{3}\pi(\alpha^3 + \beta^3) \ln(\alpha + \beta) \]
\[ + \frac{2\pi}{9} (\alpha + \beta)^3 - \frac{2\pi}{9} \alpha^3 \]
\[ - \frac{2}{9}\pi \beta^3 + \frac{2}{3}\pi(\alpha^3 \ln \alpha + \beta^3 \ln \beta) \]
\[ = \frac{2\pi}{3} \left[ \alpha \beta(\alpha + \beta) + \alpha^3 \ln \alpha + \beta^3 \ln \beta \right. \]
\[ \left. - (\alpha^3 + \beta^3) \ln(\alpha + \beta) \right] (\alpha > 0, \beta > 0). \]

因此，对任意的 \( \alpha, \beta \) 有

\[ \int_{0}^{\infty} \frac{\ln(1 + \alpha^2 x^2) \ln(1 + \beta^2 x^2)}{x^4} \, dx \]
\[ = \begin{cases} \frac{2\pi}{3} \left[ |\alpha \beta|(|\alpha| + |\beta|) + |\alpha|^3 \ln |\alpha| \right. \\
+ |\beta|^3 \ln |\beta| - (|\alpha|^3 + |\beta|^3) \ln |\alpha| \]
\[ \left. + |\beta| \right], \quad \text{当} \alpha \beta \neq 0 \text{ 时} \]
\[ 0, \quad \text{当} \alpha \beta = 0 \text{ 时}. \]
从公式

$$I^2 = \int_0^{+\infty} e^{-x^2} dx \int_0^{+\infty} x e^{-xy^2} dy$$

出发，计算尤拉—普阿松积分

$$I = \int_0^{+\infty} e^{-x^2} dx$$

解 在积分

$$I = \int_0^{+\infty} e^{-x^2} dx$$

中令 $x = ut$，其中 $u$ 为任意正数，即得

$$I = u \int_0^{+\infty} e^{-u^2 t^2} dt.$$  

在上式两端乘以 $e^{-u^2} du$，再对 $u$ 从 0 到 $+\infty$ 积分，得

$$I^2 = \int_0^{+\infty} e^{-u^2} du \int_0^{+\infty} u e^{-u^2 t^2} dt.$$  \hspace{1cm} (1)

由于被积函数 $u e^{-(1+t^2)u^2}$ 是非负的连续函数，并且积分

$$\int_0^{+\infty} e^{-(1+t^2)u^2} u^2 du = \frac{1}{2(1+t^2)}$$

及

$$\int_0^{+\infty} e^{-(1+t^2)u^2} u dt = e^{-u^2}.I$$

分别对于 $t$ 及 $u$ 是连续的，积分互换后的逐次积分显然存在。于是，(1) 式中的积分顺序可以互换 *)，并且
有

\[ I^2 = \int_0^\infty dt \int_0^\infty e^{-(1+t^2)u^2} u \, du. \]

\[ = \frac{1}{2} \int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{4}. \]

由于 \( I \approx 0 \)，故

\[ I = \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}. \]

*）参看菲赫金哥尔茨著《微积分学教程》第二卷

483目定理 V 的推理。

利用尤拉-普阿桑积分，求下列积分之值:

3804. \( \int_{-\infty}^{+\infty} e^{-(ax^2 + 2bx + c)} \, dx \) \( (a \gg 0, \, ac - b^2 \gg 0) \)

解

\[ \int_{-\infty}^{+\infty} e^{-(ax^2 + 2bx + c)} \, dx \]

\[ = \int_{-\infty}^{+\infty} e^{-a[(ax+b)^2 + ax-b^2]} \, dx \]

\[ = e^{\frac{b^2-ac}{a}} \int_{-\infty}^{+\infty} e^{-\frac{1}{a}(ax+b)^2} \, dx \]

\[ = e^{\frac{b^2-ac}{a}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{a}} e^{-t^2} \, dt \]

\[ = \frac{2}{\sqrt{a}} e^{\frac{b^2-ac}{a}} \int_0^{+\infty} e^{-t^2} \, dt \]

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\[
\int_{-\infty}^{+\infty} \left( a_1 x^2 + 2b_1 x + c_1 \right) e^{-\left( a x^2 + 2b x + c \right)} \, dx \\
(a \gg 0, \ ac-b^2 \gg 0) \text{ *)).
\]

设有 \( \frac{1}{\sqrt{a}}(ax+b)=t \)，则 \( x = \frac{\sqrt{a}}{a} \cdot \frac{t-b}{a} \)。代入得

\[
\int_{-\infty}^{+\infty} \left( a_1 x^2 + 2b_1 x + c_1 \right) e^{-\left( a x^2 + 2b x + c \right)} \, dx \\
= \frac{1}{\sqrt{a}} e^{-\frac{b^2-ac}{a}} \int_{-\infty}^{+\infty} \left( \frac{a_1}{a} t^2 + \frac{2(ab_1-a_1b)}{a\sqrt{a}} t \\
+ \frac{a_1b^2-2ab_1}{a^2} + c_1 \right) e^{-t^2} \, dt.
\]

由于

\[
\int_{-\infty}^{+\infty} t^2 e^{-t^2} \, dt = -\frac{1}{2} \int_{-\infty}^{+\infty} t \, d(e^{-t^2}) \\
= -\frac{1}{2} t e^{-t^2} \bigg|_{-\infty}^{+\infty} + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2},
\]

\[
\int_{-\infty}^{+\infty} t \, e^{-t^2} \, dt = 0
\]
及
\[ \int_{-\infty}^{+\infty} e^{-t^2} \, dt = 2 \int_{0}^{+\infty} e^{-t^2} \, dt = \sqrt{\pi}, \]
故得
\[ \int_{-\infty}^{+\infty} (a_1 x^2 + 2b_1 x + c_1) e^{-(ax^2 + 2bx + c)} \, dx \]
\[= \frac{1}{\sqrt{a}} e^{\frac{b_1}{a}} \left[ \frac{a_1}{a} \int_{0}^{\infty} e^{-t^2} \, dt \right] \]
\[+ \left( \frac{a_1 b^2 - 2ab b_1 + c_1}{a^2} \right) \sqrt{\pi} \]
\[= \frac{(a + 2b^2) a_1 - 4abb_1 + 2a^2 c_1}{2a^2} \cdot \sqrt{\pi} e^{\frac{b_1}{a}}. \]

*）只要假定 \(a > 0\)，条件 \(ac - b^2 \geq 0\) 可去掉。

3806. \( \int_{-\infty}^{+\infty} e^{-ax^2} \cosh bx \, dx \quad (a > 0). \)

解
\[ \int_{-\infty}^{+\infty} e^{-ax^2} \cosh bx \, dx \]
\[= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-ax^2} (e^{bx} + e^{-bx}) \, dx \]
\[= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-(ax^2 - bx)} \, dx + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-(ax^2 + bx)} \, dx \]
\[= \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} + \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \]

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3807. \( \int_{0}^{+\infty} e^{-\left( x^2 + \frac{a^2}{x^2} \right)} \, dx \quad (a > 0) \).

解 由于积分
\[
\int_{0}^{+\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2},
\]
故利用2355题的结果，即得
\[
\int_{0}^{+\infty} e^{-\left( x^2 + \frac{a^2}{x^2} \right)} \, dx
= e^{2a} \int_{0}^{+\infty} e^{-\left( x + \frac{a}{x} \right)^2} \, dx
= e^{2a} \int_{0}^{+\infty} e^{-\left( x^2 + 4a \right)} \, dx
= e^{-2a} \int_{0}^{+\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} e^{-2a}.
\]

3808. \( \int_{0}^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} \, dx \quad (\alpha > 0, \beta > 0) \).

解 由分部积分法知
\[
\int_{c}^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} \, dx
= - \int_{0}^{+\infty} (e^{-\alpha x^2} - e^{-\beta x^2}) \, d\left( \frac{1}{x} \right)
\]
\[\int_0^{+\infty} e^{-ax^2} \cos bx \, dx \quad (a > 0).\]

解 令 \(I(b) = \int_0^{+\infty} e^{-ax^2} \cos bx \, dx\)。由于 \(e^{-ax^2} \cos bx\) 与 \(\frac{d}{db}(e^{-ax^2} \cos bx) = -x e^{-ax^2} \sin bx\) 都是 \(x \geq 0\)，

\(-\infty < b < +\infty\) 上的连续函数，并且此时

\[|e^{-ax^2} \cos bx| \leq e^{-ax^2},\]

\[|x e^{-ax^2} \sin bx| \leq x e^{-ax^2},\]

而积分 \(\int_0^{+\infty} e^{-ax^2} \, dx\) 与 \(\int_0^{+\infty} x e^{-ax^2} \, dx\) 都收敛，故积分

\(\int_0^{+\infty} e^{-ax^2} \cos bx \, dx\) 与 \(\int_0^{+\infty} x e^{-ax^2} \sin bx \, dx\) 都在

\(-\infty < b < +\infty\) 上一致收敛，从而可在积分号下求导
数，得

\[ I'(b) = -\int_0^+ x e^{-ax^2} \sin bx \, dx \]

\[ (-\infty < b < +\infty) \]

利用分部积分法，得

\[ \int_0^+ x e^{-ax^2} \sin bx \, dx \]

\[ = -\frac{1}{2a} e^{-ax^2} \sin bx \bigg|_0^+ \]

\[ + \frac{b}{2a} \int_0^+ e^{-ax^2} \cos bx \, dx \]

\[ = \frac{b}{2a} I(b), \]

故

\[ I''(b) = -\frac{b}{2a} I(b) \quad (-\infty < b < +\infty) \]

于是

\[ \int \frac{I'(b)}{I(b)} \, db = -\frac{1}{2a} \int b \, db, \]

即

\[ \ln I(b) = -\frac{b^2}{4a} + C \quad (-\infty < b < +\infty) \]

其中 C 是待定常数，也即

\[ I(b) = C_1 e^{-\frac{b^2}{4a}} \quad (-\infty < b < +\infty) \]

其中 C_1 也是待定常数。但
\[ I(0) = \int_0^{+\infty} e^{-ax^2} \, dx \]
\[ = \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-t^2} \, dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} \]

代入，得 \( C_1 = \frac{1}{2} \sqrt{\frac{\pi}{a}} \)。于是，最后得

\[ \int_0^{+\infty} e^{-ax^2} \cos bx \, dx \]
\[ = I(b) = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \quad (-\infty < b < +\infty) \]

3810. \( \int_0^{+\infty} xe^{-ax^2} \sin bx \, dx \quad (a \gg 0) \)

解 \( \int_0^{+\infty} xe^{-ax^2} \sin bx \, dx \)

\[ = \int_0^{+\infty} \sin bx \, d(xe^{-ax^2}) \]
\[ = \frac{1}{2a} \left. e^{-ax^2} \sin bx \right|_0^{+\infty} \]

\[ + \frac{b}{2a} \int_0^{+\infty} e^{-ax^2} \cos bx \, dx \]
\[ = \frac{b}{2a} \int_0^{+\infty} e^{-ax^2} \cos bx \, dx \]
\[ = \frac{b}{4a} \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}} \]

*) 利用3809题的结果。
解 由3809题得

\[
\int_0^{+\infty} e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.
\] (1)

积分

\[
\int_0^{+\infty} \frac{d^n}{db^n} (e^{-x^2} \cos 2bx) \, dx
\]

\[
= 2^n \int_0^{+\infty} x^n e^{-x^2} \cos \left(2bx + \frac{k\pi}{2}\right) \, dx,
\] (2)

而

\[
| x^n e^{-x^2} \cos \left(2bx + \frac{k\pi}{2}\right) | \leqslant x^n e^{-x^2} \quad (x \geqslant 0).
\]

但是积分 \(\int_0^{+\infty} x^n e^{-x^2} \, dx\) 对于任意的自然数 \(k\) 均收敛，故积分 (2) 当 \(-\infty \leq b \leq +\infty\) 时一致收敛。因此，(1) 式的左端可在积分号下求任意次导数，从而可得

\[
\int_0^{+\infty} \frac{d^{2n}}{db^{2n}} (e^{-x^2} \cos 2bx) \, dx
\]

\[
= \int_0^{+\infty} 2^{2n} x^{2n} e^{-x^2} \cos (2bx + n\pi) \, dx
\]

\[
= 2^{2n} (-1)^n \int_0^{+\infty} x^{2n} e^{-x^2} \cos 2bx \, dx
\]

\[
= \frac{\sqrt{\pi}}{2} \frac{d^{2n}}{db^{2n}} (e^{-b^2}),
\]

即

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\[
\int_0^{+\infty} x^{2n} e^{-x^2} \cos 2b\alpha \, dx
\]
\[-= (-1)^n \cdot \frac{\sqrt{\pi}}{2^{2n+1}} \frac{d^{2n}}{db^{2n}} (e^{-b^2}).\]

3812. 从积分

\[I(\alpha) = \int_0^{+\infty} e^{-ax} \frac{\sin \beta x}{x} \, dx\]

出发，计算迪里克里积分

\[D(\beta) = \int_0^{+\infty} \frac{\sin \beta x}{x} \, dx.\]

解 先设 \(\beta \to 0\)。将 \(\beta\) 固定，\(\alpha\) 视为变量。仿 3760 题的证法，可知积分 \(\int_0^{+\infty} e^{-ax} \frac{\sin \beta x}{x} \, dx\) 当 \(\alpha \geq 0\) 时一致收敛，从而 \(I(\alpha)\) 是 \(\alpha \geq 0\) 上的连续函数（注意，上述积分中 \(x = 0\) 不是瑕点，因为 \(\lim_{x \to 0} e^{-ax} \frac{\sin \beta x}{x} = \beta\)）。由于

\[
\int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( e^{-ax} \frac{\sin \beta x}{x} \right) \, dx
\]

\[-= -\int_0^{+\infty} e^{-ax} \sin \beta x \, dx = -\frac{\beta}{\alpha^2 + \beta^2},\]

易知积分 \(\int_0^{+\infty} e^{-ax} \sin \beta x \, dx\) 当 \(\alpha \geq \alpha_0 \geq 0\) 时一致收敛（因为此时 \(e^{-ax} \sin \beta x \leq e^{-\alpha_0 x}\)，而 \(\int_0^{+\infty} e^{-\alpha_0 x} \, dx\)
收敛），故知当 $\alpha \geqslant \alpha_0$ 时，积分

$$
\int_0^{+\infty} e^{-\alpha x} \frac{\sin \beta x}{x} \, dx
$$

可在积分号下求导数，得

$$
I'(\alpha) = -\frac{\beta}{\alpha^2 + \beta^2}.
$$

由 $\alpha_0 \geqslant 0$ 的任意性知，上式对一切 $0 \leqslant \alpha \leqslant +\infty$ 皆成立。两端对 $\alpha$ 积分，得

$$
I(\alpha) = -\arctg \frac{\alpha}{\beta} + C \quad (0 \leqslant \alpha \leqslant +\infty), \quad (1)
$$

其中 $C$ 是某常数。由 $|\sin u| \leqslant |u|$ 知

$$
|I(\alpha)| \leqslant \beta \int_0^{+\infty} e^{-\alpha x} \, dx = \frac{\beta}{\alpha} \quad (0 \leqslant \alpha \leqslant +\infty),
$$

由此可知 $\lim_{\alpha \to +\infty} I(\alpha) = 0$。在 (1) 式两端令 $\alpha \to +\infty$ 取极限，得 $0 = -\frac{\pi}{2} + C$. 故 $C = \frac{\pi}{2}$, 于是

$$
I(\alpha) = -\arctg \frac{\alpha}{\beta} + \frac{\pi}{2} \quad (0 \leqslant \alpha \leqslant +\infty). \quad (2)
$$

在 (2) 式两端令 $\alpha \to +0$ 取极限，并注意到 $I(\alpha)$ 当 $\alpha \geqslant 0$ 时连续，即得

$$
D(\beta) = I(0) = \lim_{\alpha \to +0} I(\alpha) = \frac{\pi}{2}.
$$

当 $\beta \leqslant 0$ 时，$D(\beta) = -D(-\beta) = -\frac{\pi}{2}$. 又显然有 $D(0) = 0$. 综上所述，有
\[ D(\beta) = \frac{\pi}{2} \text{sgn } \beta. \]

利用迪里黑里和傅茹兰积分，求下列积分之值：

3813. \[ \int_0^\infty e^{-ax^2} - \cos \beta x \quad dx \quad (a > 0). \]

解 令 \( I(\beta) = \int_0^\infty e^{-ax^2} - \cos \beta x \quad dx \). 首先注意到 \( x = 0 \) 不是极点，因为

\[
\lim_{x \to 0} \frac{e^{-ax^2} - \cos \beta x}{x^2} = \lim_{x \to 0} \frac{-2axe^{-ax^2} + \beta \sin \beta x}{2x} = \frac{\beta^2}{2} - a.
\]

由于

\[
\left| \frac{e^{-ax^2} - \cos \beta x}{x^2} \right| \leq \frac{2}{x^2} \quad (x \gg 0),
\]

而 \( \int_1^\infty \frac{dx}{x^2} \) 收敛，故 \( \int_1^\infty e^{-ax^2} - \cos \beta x \quad dx \) 在 \( -\infty \leq \beta < +\infty \) 上一致收敛，从而 \( \int_0^\infty e^{-ax^2} - \cos \beta x \quad dx \) 也在 \( -\infty \leq \beta < +\infty \) 上一致收敛。于是，\( I(\beta) \) 是 \( -\infty \leq \beta < +\infty \) 上的连续函数。设 \( \beta > 0 \)。由于

\[
\int_0^\infty \frac{d}{d\beta} \left( \frac{e^{-ax^2} - \cos \beta x}{x^2} \right) \quad dx
\]

\[
= \int_0^\infty \frac{\sin \beta x}{x} \quad dx = \frac{\pi}{2},
\]
而积分 $\int_0^{+\infty} \frac{\sin \beta x}{x} \, dx$ 在 $\beta \geq \beta_0 \gg 0$ 上一致收敛

（因为当 $x \to +\infty$ 时 $\frac{1}{x}$ 单调递减趋于零，而

$$\left| \int_0^a \sin \beta x \, dx \right| = \left| \frac{1 - \cos \beta A}{\beta} \right| \leq \frac{2}{\beta_0},$$

所以由定理判别法知 $\int_0^{+\infty} \frac{\sin \beta x}{x} \, dx$ 当 $\beta \geq \beta_0$ 时一致收敛。）

于是，当 $\beta \geq \beta_0$ 时，可在积分号下求导数，得

$$I'(\beta) = \int_0^{+\infty} \frac{\sin \beta x}{x} \, dx = \frac{\pi}{2}.$$  (1)

由 $\beta_0 \gg 0$ 的任意性知，式 1）对一切 $\beta \geq 0$ 都成立。于是

$$I(\beta) = \frac{\pi}{2} \beta + C \quad (0 < \beta < +\infty),$$  (2)

其中 $C$ 是某常数。在 2）式两端令 $\beta \to +0$ 取极限，并注意到 $I(\beta)$ 在 $-\infty < \beta < +\infty$ 上的连续性，得

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-\beta x^2}}{x^2} \, dx = I(0) = \lim_{\beta \to +0} I(\beta) = C.$$  (3)

根据3808题结果知

$$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-\beta x^2}}{x^2} \, dx
to$$

$$= \sqrt{\pi} \left( \sqrt{\beta} - \sqrt{\alpha} \right) \quad (a > 0, \ \beta > 0).$$  (4)

令 $I(\beta) = \int_0^{+\infty} \frac{e^{-ax^2} - e^{-\beta x^2}}{x^2} \, dx$ (a > 0)。仿上
面之证，易知 \( \int_0^{+\infty} \frac{e^{-ax^2} - e^{-\beta x^2}}{x^2} \, dx \) 当 \( \beta \geq 0 \) 时一致收敛，故 \( J(\beta) \) 是 \( \beta \geq 0 \) 上的连续函数。于是，在（4）式两端令 \( \beta \to +0 \) 取极限，得

\[
\int_0^{+\infty} \frac{e^{-ax^2} - 1}{x^2} \, dx = J(0) = \lim_{\beta \to +0} J(\beta) = -\sqrt{\pi} \alpha \quad (\alpha > 0),
\]

以此代入（3）式，得 \( C = -\sqrt{\pi} \alpha \)。于是，

\[
I(\beta) = \frac{\pi}{2} \beta - \sqrt{\pi} \alpha \quad (0 \leq \beta < +\infty).
\]

当 \( \beta < 0 \) 时，\( I(\beta) = I(-\beta) = \frac{\pi}{2} (-\beta) - \sqrt{\pi} \alpha \)。

总之，得

\[
\int_0^{+\infty} \frac{e^{-ax^2} - \cos \beta x}{x^2} \, dx = \frac{\pi}{2} |\beta| - \sqrt{\pi} \alpha \quad (\alpha > 0).
\]

*) 利用3812题的结果。

3814. \( \int_0^{+\infty} \frac{\sin \alpha x \sin \beta x}{x} \, dx \).

解 \[
\int_0^{+\infty} \frac{\sin \alpha x \sin \beta x}{x} \, dx = \frac{1}{2} \int_0^{+\infty} \frac{\cos(\alpha - \beta)x - \cos(\alpha + \beta)x}{x} \, dx
\]

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\[ \int_0^{+\infty} \frac{\sin \alpha x \cos \beta x}{x} \, dx. \]

解 \[ \int_0^{+\infty} \frac{\sin \alpha x \cos \beta x}{x} \, dx \]

\[ = \frac{1}{2} \int_0^{+\infty} \frac{\sin (\alpha + \beta)x + \sin (\alpha - \beta)x}{x} \, dx \]

\[ = \frac{1}{2} \int_0^{+\infty} \frac{\sin (\alpha + \beta)x - \sin (\beta - \alpha)x}{x} \, dx \]

\[ = \begin{cases} 
0, & \text{若 } |\alpha| = |\beta| \text{ *)}, \\
\frac{\pi}{4} \text{sgn } \alpha, & \text{若 } |\alpha| = |\beta| \text{ **}, \\
\frac{\pi}{2} \text{sgn } \alpha, & \text{若 } |\alpha| > |\beta| \text{ ***}. 
\end{cases} \]

*) 利用3791题的结果。

**) 及 ***) 利用3812题的结果。

3815. \[ \int_0^{+\infty} \frac{\sin^3 \alpha x}{x} \, dx \]

解 由于 \( \sin 3\alpha x = 3 \sin \alpha x - 4 \sin^3 \alpha x \)，故

\[ \int_0^{+\infty} \frac{\sin^3 \alpha x}{x} \, dx = \int_0^{+\infty} \frac{3 \sin \alpha x - \sin 3\alpha x}{4x} \, dx \]

\[ = \frac{\pi}{2} \text{sgn } \alpha \cdot \left( \frac{3}{4} - \frac{1}{4} \right) \text{ *)} = \frac{\pi}{4} \text{sgn } \alpha. \]
3817. \[ \int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^2 \, dx. \]

解 令 \( I(\alpha) = \int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^2 \, dx. \) 先设 \( \alpha \geq 0. \)

显然 \( x = 0 \) 不是瑕点，因为 \( \lim_{x \to 0} \left( \frac{\sin \alpha x}{x} \right)^2 = \alpha^2. \)

而由于 \( \left( \frac{\sin \alpha x}{x} \right)^2 \leq \frac{1}{x^2} \)，又 \( \int_1^{+\infty} \frac{dx}{x^2} \) 收敛，故

\[ \int_1^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^2 \, dx \text{ 在 } \alpha \geq 0 \text{ 上一致收敛。从而} \]

\[ \int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^2 \, dx \text{ 在 } \alpha \geq 0 \text{ 时一致收敛。因此, } I(\alpha) \]

是 \( \alpha \geq 0 \) 上的连续函数。

又因

\[ \int_0^{+\infty} \frac{\partial}{\partial \alpha} \left( \frac{\sin \alpha x}{x} \right)^2 \, dx \]

\[ = \int_0^{+\infty} \frac{\sin 2\alpha x}{x} \, dx = \frac{\pi}{2}, \]

而积分 \( \int_0^{+\infty} \frac{\sin 2\alpha x}{x} \, dx \) 当 \( \alpha \geq \alpha_0 \geq 0 \) 时一致收敛

（参看3813题的解题过程），故当 \( \alpha \geq \alpha_0 \) 时可在积分号下求导数，得

\[ I'(\alpha) = \int_0^{+\infty} \frac{\sin 2\alpha x}{x} \, dx = \frac{\pi}{2}, \quad (1) \]
由 $\alpha_0 \geq 0$ 的任意性知，(1) 式对一切 $\alpha \geq 0$ 均成立。两端积分，得

$$I(\alpha) = \frac{\pi}{2} \alpha - C \quad (0 \leq \alpha \leq +\infty),$$

其中 $C$ 是某常数。在上式两端令 $\alpha \to +0$ 取极限，并注意到 $I(\alpha)$ 在 $\alpha \geq 0$ 时的连续性知

$$0 = I(0) = \lim_{\alpha \to 0} I(\alpha) = C.$$

于是 $I(\alpha) = \frac{\pi}{2} \alpha \quad (0 \leq \alpha \leq +\infty)$. 当 $\alpha \leq 0$ 时，显然，$I(\alpha) = I(-\alpha) = \frac{\pi}{2}(-\alpha)$，故对于任何 $\alpha$，有

$$\int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^2 dx = I(\alpha) = \frac{\pi}{2} |\alpha|.$$ 3818. $\int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^3 dx$.  

解 $\int_0^{+\infty} \left( \frac{\sin \alpha x}{x} \right)^3 dx$

$$= -\frac{1}{2} \int_0^{+\infty} \sin^3 \alpha x \ d\left(\frac{1}{x^2}\right)$$

$$= -\frac{1}{2} \left. x^2 \sin^3 \alpha x \right|_0^{+\infty}$$

$$+ \frac{1}{2} \int_0^{+\infty} \frac{3 \alpha \sin^2 \alpha x \cos \alpha x}{x^2} dx$$

$$= \frac{3 \alpha}{2} \int_0^{+\infty} \frac{\sin^2 \alpha x \cos \alpha x}{x^2} dx.$$ 673
$$= -\frac{3\alpha}{2} \int_0^{+\infty} \sin^2\alpha x \cos\alpha x \; d\left(\frac{1}{x}\right)$$

$$= -\frac{3\alpha}{2} \int_0^{+\infty} \sin^2\alpha x \cos\alpha x \; dx$$

$$+ \frac{3\alpha}{2} \int_0^{+\infty} \frac{2\alpha \sin\alpha x \cos^2\alpha x - \alpha \sin^2\alpha x}{x} \; dx$$

$$= \frac{3\alpha}{2} \int_0^{+\infty} \frac{2\alpha \sin\alpha x}{x} \; dx$$

$$- \frac{3\alpha}{2} \int_0^{+\infty} \frac{3\alpha \sin^2\alpha x}{x} \; dx$$

$$= 3\alpha^2 \cdot \frac{\pi}{2} \; \text{sgn} \alpha - \frac{9}{2} \alpha^2 \cdot \frac{\pi}{4} \; \text{sgn} \alpha$$

$$= \frac{3\pi}{8} \alpha^2 \; \text{sgn} \alpha = \frac{3\pi}{8} |\alpha| \cdot \text{sgn} \alpha .$$

*) 利用3816题的结果。

3819. \( \int_0^{+\infty} \frac{\sin^4 x}{x^2} \; dx \).

解 \( \int_0^{+\infty} \frac{\sin^4 x}{x^2} \; dx \)

$$= -\frac{1}{x} \left. \sin^4 x \right|_0^{+\infty} + \int_0^{+\infty} \frac{4 \sin^3 x \cos x}{x} \; dx$$

$$= \int_0^{+\infty} \frac{(3 \sin x - \sin 3x) \cos x}{x} \; dx$$

$$= \frac{3}{2} \int_0^{+\infty} \frac{\sin 2x}{x} \; dx - \frac{1}{2} \int_0^{+\infty} \frac{\sin 4x}{x} \; dx$$

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3820. \[ \int_0^{+\infty} \frac{\sin^4 \alpha x - \sin^4 \beta x}{x} \, dx. \]

解 由于 \( \sin^4 x = \frac{1}{8} (\cos 4x - 4 \cos 2x + 3) \)，故

\[ \int_0^{+\infty} \frac{\sin^4 \alpha x - \sin^4 \beta x}{x} \, dx = \frac{1}{8} \int_0^{+\infty} \frac{\cos 4 \alpha x - \cos 4 \beta x}{x} \, dx - \frac{1}{2} \int_0^{+\infty} \frac{\cos 2 \alpha x - \cos 2 \beta x}{x} \, dx \]

\[ = \frac{1}{8} \ln \left| \frac{\beta}{\alpha} \right| - \frac{1}{2} \ln \left| \frac{\beta}{\alpha} \right| = \frac{3}{8} \ln \left| \frac{\alpha}{\beta} \right| (\alpha \neq 0, \beta \neq 0), \]

注 若 \( \alpha = \beta = 0 \)，显然积分为零；若 \( \alpha = 0 (\beta \neq 0) \)
或 \( \beta = 0 (\alpha \neq 0) \)，易知积分发散。

3821. \[ \int_0^{+\infty} \frac{\sin (x^2)}{x} \, dx. \]

解 作代换 \( x = \sqrt{t} \)，则有

\[ \int_0^{+\infty} \frac{\sin (x^2)}{x} \, dx = \frac{1}{2} \int_0^{+\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{4}. \]
3822. \[ \int_0^{+\infty} e^{-ix} \frac{\sin ax \sin \beta x}{x^2} \, dx \quad (k \geq 0, \quad a > 0, \quad \beta > 0). \]

解

\[ \int_0^{+\infty} e^{-ix} \frac{\sin ax \sin \beta x}{x^2} \, dx \]

\[ = -\frac{1}{x} e^{-ix} \sin ax \sin \beta x \bigg|_0^{+\infty} \]

\[ + \int_0^{+\infty} \frac{1}{x} \left\{ -ke^{-ix} \sin ax \sin \beta x \right\} \, dx \]

\[ + e^{-ix} (ax \sin \beta x \cos ax + \beta \sin ax \cos \beta x) \, dx \]

\[ = \int_0^{+\infty} e^{-ix} \frac{ax \sin \beta x \cos ax + \beta \sin ax \cos \beta x}{x} \, dx \]

\[ - k \int_0^{+\infty} e^{-ix} \frac{\sin ax \sin \beta x}{x} \, dx. \]

由于

\[ \int_0^{+\infty} e^{-ix} \frac{ax \sin \beta x \cos ax}{x} \, dx \]

\[ = \frac{a}{2} \int_0^{+\infty} e^{-ix} \frac{\sin(a+\beta)x - \sin(a-\beta)x}{x} \, dx \]

\[ = \frac{a}{2} \left( \arctg \frac{a+\beta}{k} - \arctg \frac{a-\beta}{k} \right) \quad *) \]

\[ \int_0^{+\infty} e^{-ix} \frac{\beta \sin ax \cos \beta x}{x} \, dx \]

\[ = \frac{\beta}{2} \left( \arctg \frac{a+\beta}{k} + \arctg \frac{a-\beta}{k} \right), \]

且

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\[
\int_0^{+\infty} e^{-ix} \frac{\sin ax \sin \beta x}{x} \, dx
\]

\[
= \int_0^{+\infty} \left[ (e^{-ix}-1)+1 \right] \frac{\cos(\alpha-\beta)x-\cos(\alpha+\beta)x}{2x} \, dx
\]

\[
- \frac{1}{2} \int_0^{+\infty} (e^{-ix}-1) \frac{\cos(\alpha-\beta)x}{x} \, dx
\]

\[
- \frac{1}{2} \int_0^{+\infty} (e^{-ix}-1) \frac{\cos(\alpha+\beta)x}{x} \, dx
\]

\[
+ \frac{1}{2} \int_0^{+\infty} \frac{\cos(\alpha-\beta)x-\cos(\alpha+\beta)x}{x} \, dx
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} \ln \frac{(\alpha-\beta)^2}{(\alpha-\beta)^2+k^2}
\]

\[
- \frac{1}{2} \cdot \frac{1}{2} \ln \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2+k^2} \quad \text{**})
\]

\[
+ \frac{1}{2} \ln \left| \frac{\alpha+\beta}{\alpha-\beta} \right|
\]

\[
= \frac{1}{4} \ln \frac{(\alpha+\beta)^2+k^2}{(\alpha-\beta)^2+k^2},
\]

故

\[
\int_0^{+\infty} e^{-ix} \frac{\sin ax \sin \beta x}{x^2} \, dx
\]

\[
= \frac{\alpha+\beta}{2} \arctg \frac{\alpha+\beta}{k} - \frac{\alpha-\beta}{2} \arctg \frac{\alpha-\beta}{k}
\]

\[
+ \frac{k}{4} \ln \frac{(\alpha-\beta)^2+k^2}{(\alpha+\beta)^2+k^2}.
\]

*) 利用3812题的结果。

**) 易知3796题的结果当 \( \alpha \to 0, \beta = 0 \) 时也成立。
3823. 于不同的 $x$ 值，求得里黑里间断乘数

$$D(x) = 2 \pi \int_{0}^{+\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} \frac{d\lambda}{\lambda},$$

作出函数 $y = D(x)$ 的图形。

解

$$D(x) = \frac{1}{\pi} \int_{0}^{+\infty} \frac{\sin(1+x)\lambda + \sin(1-x)\lambda}{\lambda} d\lambda.$$  

当 $|x| < 1$ 时，$1+x > 0$ 及 $1-x > 0$，利用3812题的结果，即得

$$D(x) = \frac{1}{\pi} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 1;$$

当 $|x| = 1$ 时，$1+x$ 及 $1-x$ 中总有一个为零，一个为正值，即得

$$D(x) = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2};$$

当 $|x| > 1$ 时，$(1+x)(1-x) < 0$，即得 $D(x) = 0$.  

如图7.3所示。

![图7.3](image)

3824. 计算积分:

(a) $V \cdot P. \int_{-\infty}^{+\infty} \frac{\sin ax}{x+b} dx;$

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(6) \( \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\cos \alpha x}{x + b} \, dx \).

解 (a) \( \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\sin \alpha x}{x + b} \, dx \)

\[ = \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\sin \alpha (t - b)}{t} \, dt \]

\[ = \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\sin \alpha t \cos \beta b}{t} \, dt \]

\[ - \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\cos \alpha t \sin \beta b}{t} \, dt \]

\[ = 2 \int_{0}^{+\infty} \frac{\sin \alpha t \cos \beta b}{t} \, dt = \pi \sgn \alpha \cos \beta b. \]

类似地，可求得

(6) \( \text{V}. \ P. \ \int_{-\infty}^{+\infty} \frac{\cos \alpha x}{x + b} \, dx = \pi \sgn \alpha \sin \beta b. \)

3825. 利用公式

\[ \frac{1}{1 + x^2} = \int_{0}^{+\infty} e^{-y(1 + x^2)} \, dy, \]

计算拉普拉斯积分

\[ L = \int_{0}^{+\infty} \frac{\cos \alpha x}{1 + x^2} \, dx. \]

解 \( L = \int_{0}^{+\infty} \cos \alpha x \, dx \int_{0}^{+\infty} e^{-y(1 + x^2)} \, dy \)。由于被积函数 \( \cos \alpha x \ e^{-y(1 + x^2)} \) 是 \( 0 \leq x \leq +\infty, \ 0 \leq y \leq +\infty \) 上的连续函数，并且绝对值的积分
\[ \int_{0}^{\infty} d y \int_{0}^{\infty} e^{-y(1+x^2)} \cos \alpha x \, d x \]

\[ \leq \int_{0}^{\infty} e^{-y} d y \int_{0}^{\infty} e^{-y x^2} d x \]

\[ = \frac{\sqrt{\pi}}{2} - \int_{0}^{\infty} \frac{e^{-y}}{\sqrt{y}} d y = \sqrt{\pi} \int_{0}^{\infty} e^{-t^2} d t \]

\[ = \frac{\pi}{2} e^{-\infty}, \]

故原逐次积分可交换积分顺序，得

\[ L = \int_{0}^{\infty} e^{-y} d y \int_{0}^{\infty} e^{-y x^2} \cos \alpha x \, d x \]

\[ = \int_{0}^{\infty} e^{-y} \cdot \frac{1}{2} \sqrt{\frac{\pi}{y}} e^{-\frac{a^2}{4y}} d y \]

\[ = \int_{0}^{\infty} \sqrt{\pi} e^{-\left[ t^2 + \frac{1}{t^2} \left( \frac{|a|}{2} \right)^2 \right]} d t \]

\[ = \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2} e^{-2 \cdot \frac{|a|}{2}} \]

\[ = \frac{\pi}{2} e^{-1 |a|}. \]

*) 利用3809题的结果。

**) 利用3807题的结果。

3826. 计算积分

\[ L_1 = \int_{0}^{\infty} \frac{x \sin \alpha x}{1 + x^2} \, d x. \]

解 由于 \[ \frac{\partial}{\partial \alpha} \left( \frac{\cos \alpha x}{1 + x^2} \right) = -\frac{x \sin \alpha x}{1 + x^2}, \] 故我们
考虑积分 \( L = \int_0^{+\infty} \frac{\cos a x}{1 + x^2} \, dx \)。由于 \( \left| \frac{\cos a x}{1 + x^2} \right| \leq \frac{1}{1 + x^2} \)。其中，而 \( \int_0^{+\infty} \frac{dx}{1 + x^2} \) 收敛，故 \( \int_0^{+\infty} \frac{\cos a x}{1 + x^2} \, dx \) 当 \(-\infty < a < +\infty\) 时一致收敛。又由于当 \( a \geq a_0 > 0 \) 时，

\[ \left| \int_0^{A} \sin a x \, dx \right| = \left| \frac{1 - \cos a A}{a} \right| \leq \frac{2}{a_0}, \]

而 \( \frac{x}{1 + x^2} \) 当 \( x \to 1 \) 时递减，且当 \( x \to +\infty \) 时趋于零；于是，由迪里黑里判别法知积分 \( \int_0^{+\infty} \frac{x \sin a x}{1 + x^2} \, dx \) 当 \( a \geq a_0 \) 时一致收敛。因此，当 \( a \geq a_0 \) 时可在积分号下求导数，得

\[ \frac{dL}{da} = -L_1. \]  

(1)

由 \( a_0 > 0 \) 的任意性知，(1) 式对一切 \( a > 0 \) 成立。由3825题知 当 \( a > 0 \) 时 \( L = \frac{\pi}{2} e^{-a} \)。于是，由(1) 式知

\[ L_1 = -\frac{dL}{da} = \frac{\pi}{2} e^{-a} \quad (a > 0). \]

显然，当 \( a < 0 \) 时，

\[ L_1 = \int_0^{+\infty} \frac{x \sin(-a) x}{1 + x^2} \, dx = \frac{\pi}{2} e^a; \]

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而当 $\alpha = 0$ 时，$L_1 = 0$。综上所述，有

$$L_1 = \frac{\pi}{2} \text{sgn} \alpha \cdot e^{-\alpha^2}.$$ 

计算积分：

3827. $\int_0^{+\infty} \frac{\sin^2 x}{1 + x^2} dx$.

解

$$\int_0^{+\infty} \frac{\sin^2 x}{1 + x^2} dx = \frac{1}{2} \int_0^{+\infty} \frac{dx}{1 + x^2} - \frac{1}{2} \int_0^{+\infty} \frac{\cos 2x}{1 + x^2} dx$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} e^{-2} = \frac{\pi}{4} (1 - e^{-2}).$$

*) 利用3825题的结果。

3828. $\int_0^{+\infty} \frac{\cos ax}{(1 + x^2)^2} dx$.

解

$$\int_0^{+\infty} \frac{\cos ax}{(1 + x^2)^2} dx = \int_0^{+\infty} \frac{\cos ax}{1 + x^2} dx - \int_0^{+\infty} \frac{x \cos ax}{(1 + x^2)^2} dx$$

$$= \frac{\pi}{2} e^{-\alpha^2} + \frac{1}{2} \int_0^{+\infty} x \cos ax d \left( \frac{1}{1 + x^2} \right)$$

$$= \frac{\pi}{2} e^{-\alpha^2} + \frac{1}{2} \cdot \frac{x \cos ax}{1 + x^2} \bigg|_0^{+\infty}$$

$$= -\frac{1}{2} \int_0^{+\infty} \frac{\cos ax - ax \sin ax}{1 + x^2} dx.$$
\[\begin{align*}
&= \frac{x}{2} e^{-it} - \frac{\pi}{2} e^{-i\alpha t} + \frac{\alpha}{2} \cdot \frac{\pi}{2} \text{sgn} \cdot e^{-i\alpha t} \\
&= \frac{\pi}{4} (1 + |a|) e^{-i\alpha t} .
\end{align*}\]

*) 利用3825题与3826题的结果。

3828. \( \int_{-\infty}^{+\infty} \frac{\cos ax}{ax^2 + 2bx + c} dx \) (\( a > 0 , \ ac - b^2 > 0 \)).

解 \( ax^2 + 2bx + c = a \left[ \left( x + \frac{b}{a} \right)^2 + \frac{ac - b^2}{a^2} \right] \). 令

\[m = \frac{\sqrt{ac - b^2}}{a} , \quad t = \frac{1}{m} \left( x + \frac{b}{a} \right) (m > 0) , \]

则 \( ax^2 + 2bx + c = am^2 (t^2 + 1) , \)

\[\cos ax = \cos a \left( mt - \frac{b}{a} \right) = \cos a \ mt \cos \frac{b \alpha}{a} + \sin a \ mt \sin \frac{b \alpha}{a} . \]

于是,

\[\begin{align*}
&\int_{-\infty}^{+\infty} \frac{\cos ax}{ax^2 + 2bx + c} dx \\
&= \frac{1}{am} \int_{-\infty}^{+\infty} \frac{\cos a \ mt \ cos \frac{b \alpha}{a}}{1 + t^2} dt
\end{align*}\]
\[ + \frac{1}{am} \int_{-\infty}^{\infty} \frac{\sin\alpha mt \sin \frac{ba}{a}}{1 + t^2} dt. \]

由于 \( \frac{\cos \alpha mt}{1 + t^2} \) \( \ll \frac{1}{1 + t^2} \)，而 \( \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} = \pi \) 收敛，故积分 \( \int_{-\infty}^{\infty} \frac{\cos \alpha mt}{1 + t^2} dt \) 收敛。同理，积分 \( \int_{-\infty}^{\infty} \frac{\sin \alpha mt}{1 + t^2} dt \) 收敛。又由于 \( \frac{\cos \alpha mt}{1 + t^2} \) 为偶函数，\( \frac{\sin \alpha mt}{1 + t^2} \) 为奇函数，故

\[ \int_{-\infty}^{\infty} \frac{\cos \alpha mt}{1 + t^2} dt = 2 \int_{0}^{\infty} \frac{\cos \alpha mt}{1 + t^2} dt = \pi e^{-\frac{b^2}{a}} \text{ (1)} \]

\[ \int_{-\infty}^{\infty} \frac{\sin \alpha mt}{1 + t^2} dt = 0. \]

从而得

\[ \int_{-\infty}^{\infty} \frac{\cos \alpha x}{ax^2 + 2bx + c} dx = \frac{1}{am} \cos \frac{ba}{a} \cdot \pi e^{-\frac{b^2}{a}} \]

\[ = \frac{\pi}{\sqrt{ac - b^2}} \cos \frac{ba}{a} e^{-\frac{\sqrt{ac - b^2}}{a}}. \]

\[ * \) 利用3825题的结果。

3830. 利用公式

\[ \frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x y^2} dy \quad (x > 0), \]

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计算傅里叶积分

\[
\int_0^{+\infty} \sin(x^2) \, dx = \frac{1}{2} \int_0^{+\infty} \frac{\sin x}{\sqrt{x}} \, dx
\]

及

\[
\int_0^{+\infty} \cos(x^2) \, dx = \frac{1}{2} \int_0^{+\infty} \frac{\cos x}{\sqrt{x}} \, dx.
\]

解  在积分

\[
\frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-xy^2} \, dy
\]

的两端乘以 \(\sin x\)，再在 \(0 < x_0 < x \leq x_1\) 上积分，则得

\[
\int_{x_0}^{x_1} \frac{\sin x}{\sqrt{x}} \, dx
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{x_0}^{x_1} dx \int_0^{+\infty} \sin x \cdot e^{-xy^2} \, dy.
\]

由于 \(|\sin x \cdot e^{-xy^2}| \leq e^{-x_0y^2}\)，而 \(\int_0^{+\infty} e^{-x_0y^2} \, dy\) 收敛，故积分 \(\int_0^{+\infty} \sin x \cdot e^{-xy^2} \, dy\) 对 \(x_0 < x \leq x_1\) 一致收敛，从而可进行积分顺序的互换，得

\[
\int_{x_0}^{x_1} \frac{\sin x}{\sqrt{x}} \, dx
\]

\[
= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dy \int_{x_0}^{x_1} \sin x \cdot e^{-xy^2} \, dx
\]

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\[= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \left[ -\frac{e^{-x_0^2} (y^2 \sin x + \cos x)}{1 + y^4} \right] e^{x_1 y^2} \, dy \bigg|_{x_0 = 0}^{x_1} \]

\[= \frac{2}{\sqrt{\pi}} \sin x_0 \int_0^{+\infty} \frac{y^2 e^{-x_0 y^2}}{1 + y^4} \, dy \]

\[+ \frac{2}{\sqrt{\pi}} \cos x_0 \int_0^{+\infty} \frac{e^{-x_0 y^2}}{1 + y^4} \, dy \]

\[- \frac{2}{\sqrt{\pi}} \sin x_1 \int_0^{+\infty} \frac{y^2 e^{-x_1 y^2}}{1 + y^4} \, dy \]

\[+ \frac{2}{\sqrt{\pi}} \cos x_1 \int_0^{+\infty} \frac{e^{-x_1 y^2}}{1 + y^4} \, dy. \]

上述等式右端的诸积分分别对 \(0 \leq x_0 \leq +\infty\), \(0 \leq x_1 \leq +\infty\) 都是一致收敛的（事实上，\(e^{-x_0 y^2} \leq 1\), \(e^{-x_1 y^2} \leq 1\)，且积分 \(\int_0^{+\infty} \frac{y^2}{1 + y^4} \, dy\) 及 \(\int_0^{+\infty} \frac{dy}{1 + y^4}\) 均收敛）。于是，它们分别都是 \(x_0\), \(x_1\) （\(0 \leq x_n \leq +\infty\)。
由于上式右端的后两个积分均不超过积分

\[ \int_0^{+\infty} e^{-x_1^2} \, dx_1 = \frac{1}{2} \sqrt{\frac{\pi}{x_1}}, \]

且 \( \lim_{x_1 \to +\infty} \sqrt{\frac{\pi}{x_1}} = 0 \)，故令 \( x_1 \to +\infty \)，即得

\[ \int_0^{+\infty} \frac{\sin x}{\sqrt{x}} \, dx = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{dy}{1+y^4} = \frac{2}{\sqrt{\pi}} \cdot \frac{\pi}{2 \sqrt{2}} = \sqrt{\frac{\pi}{2}}. \]

最后得

\[ \int_0^{+\infty} \sin(x^2) \, dx = \frac{1}{2} \int_0^{+\infty} \frac{\sin x}{\sqrt{x}} \, dx = -\frac{\sqrt{\pi}}{2 \sqrt{2}}. \]

同法可得

\[ \int_0^{+\infty} \cos(x^2) \, dx = \frac{\sqrt{\pi}}{2 \sqrt{2}}. \]

求下列积分之值：

3831. \( \int_{-\infty}^{+\infty} \sin(ax^2 + 2bx + c) \, dx \quad (a \neq 0) \).

解

\[ \int_{-\infty}^{+\infty} \sin(ax^2 + 2bx + c) \, dx = \int_{-\infty}^{+\infty} \sin a \left[ \left( x + \frac{b}{a} \right)^2 + \frac{ac-b^2}{a^2} \right] \, dx = \int_{-\infty}^{+\infty} \sin \left( at^2 + \frac{ac-b^2}{a} \right) \, dt \]

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$$= \cos \frac{ac - b^2}{a} \int_{-\infty}^{+\infty} \sin at^2 dt$$

$$+ \sin \frac{ac - b^2}{a} \int_{-\infty}^{+\infty} \cos at^2 dt$$

$$= \text{sgn } a \cdot \cos \frac{ac - b^2}{a} \cdot \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} \sin y^2 dy$$

$$+ \sin \frac{ac - b^2}{a} \cdot \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} \cos y^2 dy$$

$$= \sqrt{\frac{\pi}{2|a|}} \left( \text{sgn } a \cdot \cos \frac{ac - b^2}{a} \right. $$

$$\left. + \sin \frac{ac - b^2}{a} \right) ^*$$

$$= \sqrt{\frac{\pi}{|a|}} \sin \left( \frac{ac - b^2}{a} + \frac{\pi}{4} \text{sgn } a \right).$$

$$^*$$ 利用3830题的结果。

3832. $\int_{-\infty}^{+\infty} \sin x^2 \cdot \cos 2ax \, dx$.

解

$$\int_{-\infty}^{+\infty} \sin x^2 \cdot \cos 2ax \, dx$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \left[ \sin(x^2 + 2ax) + \sin(x^2 - 2ax) \right] \, dx$$

$$= \frac{1}{2} \left[ \sqrt{\pi} \sin \left( \frac{\pi}{4} - a^2 \right) + \sqrt{\pi} \sin \left( \frac{\pi}{4} - a^2 \right) \right] ^*$$

$$= \sqrt{\pi} \sin \left( \frac{\pi}{4} - a^2 \right) = \sqrt{\pi} \cos \left( \frac{\pi}{4} + a^2 \right).$$
3833. \[ \int_{-\infty}^{\infty} \cos x^2 \cdot \cos 2ax \, dx. \]

解

\[ \int_{-\infty}^{\infty} \cos x^2 \cdot \cos 2ax \, dx \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \left[ \cos (x^2 + 2ax) + \cos (x^2 - 2ax) \right] \, dx \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \left[ \sin (x^2 + 2ax + \frac{\pi}{2}) + \sin (x^2 - 2ax + \frac{\pi}{2}) \right] \, dx \]

\[ = \frac{1}{2} \cdot 2 \sqrt{\pi} \sin \left( \frac{\pi}{2} - a^2 + \frac{\pi}{4} \right) \]

\[ = \sqrt{\pi} \sin \left( \frac{\pi}{4} + a^2 \right). \]

*） 利用3831题的结果。

3834. 证明公式：

1） \[ \int_{0}^{\infty} \frac{\cos ax}{a^2 - x^2} \, dx = -\frac{\pi}{2a} \sin a \alpha \quad (\alpha > 0), \]

2） \[ \int_{0}^{\infty} \frac{x \sin ax}{a^2 - x^2} \, dx = -\frac{\pi}{2} \cos a \alpha \quad (\alpha > 0), \]

这里 \( \alpha \neq 0 \)，积分应了解为在广义意义上的主值。

证 1） \[ \int_{0}^{\infty} \frac{\cos ax}{a^2 - x^2} \, dx \]
\[
\lim_{\eta \to 0} \left[ \int_{0}^{\eta} \frac{\cos \alpha x}{a^2 - x^2} \, dx + \int_{\eta}^{A} \frac{\cos \alpha x}{a^2 - x^2} \, dx \right]
\]
\[
= \frac{1}{2a} \lim_{\eta \to 0} \left[ - \int_{a}^{2a} \frac{\cos \alpha (t-a)}{t} \, dt + \int_{a}^{A} \frac{\cos \alpha (t-a)}{t} \, dt - \int_{A-a}^{A} \frac{\cos \alpha (t+a)}{t} \, dt + \int_{2a+\eta}^{A+a} \frac{\cos \alpha (t-a)}{t} \, dt \right]
\]
\[
= \frac{1}{2a} \lim_{\eta \to 0} \left[ \int_{\eta}^{A-a} \frac{\cos \alpha (t-a)}{t} \, dt + \int_{A-a}^{A+a} \frac{\cos \alpha (t-a)}{t} \, dt + \int_{2a+\eta}^{2a-n} \frac{\cos \alpha (t-a)}{t} \, dt \right]
\]
\[- \int_{\eta}^{A-\eta} \frac{\cos \alpha(t+\eta)}{t} \, dt \]

\[= \frac{1}{2a} \lim_{\eta \to 0} \int_{\eta}^{A-\eta} \frac{\cos \alpha(t-\eta) - \cos \alpha(t+\eta)}{t} \, dt \]

\[+ \int_{A-\eta}^{A+\eta} \frac{\cos \alpha(t-\eta)}{t} \, dt \]

\[= \frac{1}{2a} \lim_{\eta \to 0} \int_{\eta}^{A-\eta} \frac{2 \sin \alpha t \sin \alpha \eta}{t} \, dt \]

\[+ \frac{1}{2a} \lim_{\eta \to +\eta} \int_{A-\eta}^{A+\eta} \frac{\cos \alpha(t-\eta)}{t} \, dt \]

\[= \frac{1}{2a} \lim_{\eta \to +\eta} \int_{2a-\eta}^{2a+\eta} \frac{\cos \alpha(t-\eta)}{t} \, dt \]

\[= \frac{\sin \alpha \xi}{a} \int_{0}^{+\infty} \frac{\sin \alpha t}{t} \, dt = \frac{\pi}{2a} \sin \alpha \eta \]

\[= \frac{\sin \alpha \xi}{a} \int_{0}^{+\infty} \frac{\sin \alpha t}{t} \, dt = \frac{\pi}{2a} \sin \alpha \eta \]

\[2) \int_{0}^{+\infty} \frac{x \sin \alpha x}{a^2 - x^2} \, dx \]

\[= \lim_{\eta \to +\eta} \left[ \int_{0}^{a-\eta} \frac{x \sin \alpha x}{a^2 - x^2} \, dx \right. \]

\[+ \int_{a+\eta}^{A} \frac{x \sin \alpha x}{a^2 - x^2} \, dx \]

\[= -\frac{1}{2} \lim_{\eta \to +\eta} \left[ \int_{0}^{a-\eta} \frac{\sin \alpha x}{x-a} \, dx \right. \]
\begin{align*}
&+ \int_{\sigma}^{\sigma-a} \frac{\sin \alpha x}{x+a} \, dx + \int_{\sigma+a}^{A} \frac{\sin \alpha x}{x-a} \, dx \\
&+ \int_{\sigma+a}^{A} \frac{\sin \alpha x}{x+a} \, dx \\
&= -\frac{1}{2} \lim_{\sigma \to 0} \left[ \int_{\sigma}^{\sigma-a} \frac{\sin \alpha(t-a)}{t} \, dt \\
&+ \int_{\sigma}^{2 \sigma-\eta} \frac{\sin \alpha(t-a)}{t} \, dt \\
&+ \int_{\sigma}^{A-\eta} \frac{\sin \alpha(t+a)}{t} \, dt \\
&+ \int_{\sigma+\eta}^{A+\eta} \frac{\sin \alpha(t-a)}{t} \, dt \right] \\
&= -\frac{1}{2} \lim_{\sigma \to 0} \left[ \int_{\sigma}^{A-a} \frac{\sin \alpha(t-a) + \sin \alpha(t+a)}{t} \, dt \\
&+ \int_{\sigma}^{A+a} \frac{\sin \alpha(t-a)}{t} \, dt \right]
\end{align*}
\[ + \int_{2n+1}^{2n+1} \frac{\sin \alpha(t-a)}{t} \, dt \]

\[ = -\frac{1}{2} \lim_{n \to \infty} \int_{2n+1}^{2n+1} \frac{2 \sin \alpha t \cos \alpha}{} \, dt \]

\[ = -\frac{1}{2} \lim_{n \to \infty} \int_{2n+1}^{2n+1} \frac{\sin \alpha(t-a)}{t} \, dt \]

\[ + \frac{1}{2} \lim_{n \to \infty} \int_{2n+1}^{2n+1} \frac{\sin \alpha(t-a)}{t} \, dt \]

\[ = -\cos \alpha \alpha \int_{0}^{+\infty} \frac{\sin \alpha t}{t} \, dt \]

\[ = -\frac{\pi}{2} \cos \alpha \alpha \, . \]

*）利用3812题的结果。

编者注：原题1）应加上条件 \( \alpha \geq 0 \)。当 \( \alpha < 0 \) 时，有

\[ \int_{0}^{+\infty} \frac{\cos \alpha x}{a^2 - x^2} \, dx \]

\[ = \int_{0}^{+\infty} \frac{\cos (-\alpha) x}{a^2 - x^2} \, dx = \frac{\pi}{2a} \sin \alpha(-\alpha) \]

\[ = -\frac{\pi}{2a} \sin \alpha \alpha . \]

原题2）应加上条件 \( \alpha \geq 0 \)。当 \( \alpha = 0 \) 时等式显然不成立（左端等于 0，右端等于 \(-\frac{\pi}{2}\)）；当 \( \alpha < 0 \) 时，有
\[
\int_{0}^{+\infty} \frac{x \sin ax}{a^2 - x^2} \, dx \\
= - \int_{0}^{+\infty} \frac{x \sin(-ax)}{a^2 - x^2} \, dx \\
= - \left[ -\frac{\pi}{2} \cos a(-a) \right] = \frac{\pi}{2} \cos a. 
\]

3835. 对于函数 \( f(t) \)，求拉普拉斯变换

\[
F(p) = \int_{0}^{+\infty} e^{-pt} f(t) \, dt \quad (p > 0). 
\]

设，

(a) \( f(t) = t^n \) (\( n \) 为自然数)；

(b) \( f(t) = t^m \); 

(c) \( f(t) = e^{-at} \);

(d) \( f(t) = e^{-at} t \);

(e) \( f(t) = \cos at \);

(f) \( f(t) = \frac{1 - e^{-t}}{t} \);

(g) \( f(t) = \sin at \sqrt{t} \).

解 (a) \( F(p) = \int_{0}^{+\infty} e^{-pt} t^n \, dt \)

\[
= -\frac{1}{p} e^{-pt} t^n \bigg|_{0}^{+\infty} + \frac{n}{p} \int_{0}^{+\infty} e^{-pt} t^{n-1} \, dt \\
= \frac{n}{p} \int_{0}^{+\infty} e^{-pt} t^{n-1} \, dt \\
= \cdots = \frac{n!}{p^n} \int_{0}^{+\infty} e^{-pt} \, dt = -\frac{n!}{p^{n+1}}. 
\]

(b) \( F(p) = \int_{0}^{+\infty} e^{-pt} \sqrt{t} \, dt \)
\[ = -\frac{1}{p} e^{-pt} \sqrt{t} \bigg|_0^{+\infty} \]
\[ + \frac{1}{2p} \int_0^{+\infty} e^{-pt} \frac{dt}{\sqrt{t}} \]
\[ = \frac{1}{p} \int_0^{+\infty} e^{-pt} du = \frac{\sqrt{\pi}}{2p\sqrt{p}}. \]

\((\text{R})\) \(F(p) = \int_0^{+\infty} e^{-pt} e^{\tau t} dt = \int_0^{+\infty} e^{\left(1 - \frac{p}{p} - \alpha\right)t} dt.\)

当 \(p > \alpha\) 时，\(F(p) = \frac{1}{p - \alpha}\); 当 \(p \leq \alpha\) 时，积分发散。

\((\text{r})\) \(F(p) = \int_0^{+\infty} e^{-pt} t e^{-p t} dt\)
\[ = \int_0^{+\infty} t e^{-(p + \alpha)t} dt\]
\[ = \frac{1}{(p + \alpha)^2} \quad (p + \alpha \gg 0) \quad (*) \]

*) 利用本题（a）的结果，\(n = 1\)。

\((\text{a})\) \(F(p) = \int_0^{+\infty} e^{-pt} \cos t dt\)
\[ = \frac{-p \cos t + \sin t}{p^2 + 1} e^{-pt} \bigg|_0^{+\infty} \]
\[ = \frac{p}{p^2 + 1}. \]

\((\theta)\) \(F(p) = \int_0^{+\infty} e^{-pt} \frac{1 - e^{-t}}{t} dt.\)

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由于 \( \lim_{t \to 0} \frac{1-e^{-t}}{t} = 1, \lim_{t \to +\infty} \frac{1-e^{-t}}{t} = 0 \)，故函数 \( \frac{1-e^{-t}}{t} \) 有界。

\[
0 \leq \frac{1-e^{-t}}{t} \leq M = \text{常数} \quad (0 \leq t < +\infty).
\]

由此可知，当 \( p \gg 0 \) 时，积分 \( \int_0^{+\infty} e^{-pt} \frac{1-e^{-t}}{t} \, dt \) 收敛，并且

\[
0 \leq F(p) \leq M \int_0^{+\infty} e^{-pt} \, dt = \frac{M}{p} \quad (0 \leq p < +\infty), \quad (1)
\]

再考虑积分

\[
\int_0^{+\infty} \frac{\partial}{\partial p} \left( e^{-pt} \frac{1-e^{-t}}{t} \right) \, dt
= \int_0^{+\infty} e^{-pt}(e^{-t} - 1) \, dt
= \int_0^{+\infty} e^{-(p+1)t} \, dt - \int_0^{+\infty} e^{-pt} \, dt
= \frac{1}{p+1} - \frac{1}{p} \quad (p \gg 0),
\]

它对 \( p \gg p_0 > 0 \) 是一致收敛的。因此，当 \( p \gg p_0 \) 时，可对函数 \( F(p) \) 应用莱布尼兹法则，得

\[
F'(p) = -\frac{1}{p+1} - \frac{1}{p} \quad (\text{当} p \gg p_0 \text{时}).
\]
由 $p_0 > 0$ 的任意性知，上式对一切 $p > 0$ 均成立。两端积分，得

$$F(p) = \ln \frac{p+1}{p} + C \quad (0 < p < +\infty), \quad (2)$$

其中 $C$ 是某常数。由 (1) 式知

$$\lim_{p \to +\infty} F(p) = 0.$$  

于是，在 (2) 式两端令 $p \to +\infty$，取极限，得 $C = 0$。由此可知

$$F(p) = \ln \frac{p+1}{p} = \ln \left(1 + \frac{1}{p}\right).$$

(3) 有

$$F(p) = \int_0^{+\infty} e^{-it} \sin \alpha \sqrt{t} \, dt$$

$$= 2 \int_0^{+\infty} u e^{-t} \sin \alpha u \, du$$

$$= \frac{\alpha \sqrt{\pi}}{2p \sqrt{p}} e^{-\frac{a^2}{4p}}.$$

*）利用3810题的结果。

3836. 证明公式（李普希兹积分）

$$\int_0^{+\infty} e^{-at} J_0(bt) \, dt = \frac{1}{\sqrt{a^2 + b^2}} \quad (a > 0),$$

其中 $J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \varphi) \, d\varphi$ 为足指数是 0 的贝塞尔函数（参阅3726题）。

证

$$\int_0^{+\infty} e^{-at} J_0(bt) \, dt$$

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\[-\frac{1}{\pi} \int_0^{+\infty} e^{-at} dt \int_0^\varphi \cos(bt \sin \varphi) d\varphi. \]

由于积分

\[\int_0^{+\infty} e^{-at} \cos(bt \sin \varphi) dt \text{ 对 } 0 \leq \varphi \leq \pi \] 一致收敛的，故可交换积分顺序，得

\[\int_0^{+\infty} e^{-at} f_0(bt) dt\]

\[= \frac{1}{\pi} \int_0^\pi d\varphi \int_0^{+\infty} e^{-at} \cos(bt \sin \varphi) dt\]

\[= \frac{1}{\pi} \int_0^\pi \left( -\frac{a \cos(bt \cos \varphi) + b \sin \varphi \cdot \sin(bt \sin \varphi)}{a^2 + b^2 \sin^2 \varphi} e^{-at} \right) \bigg|_0^{+\infty} d\varphi\]

\[= \frac{a}{\pi} \int_0^\pi \frac{d\varphi}{a^2 + b^2 \sin^2 \varphi} = \frac{2a}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{a^2 + b^2 \sin^2 \varphi}\]

\[= \frac{2a}{\pi} \int_0^{\frac{\pi}{2}} \frac{d(tg \varphi)}{(a^2 + b^2) tg^2 \varphi + a^2}\]

\[= \frac{2a}{\pi} \int_0^{+\infty} \frac{dt}{(a^2 + b^2) t^2 + a^2}\]

\[= \frac{2a}{\pi} \cdot \frac{1}{a \sqrt{a^2 + b^2}} \arctg \left\{ \frac{\sqrt{a^2 + b^2} t}{a} \right\} \bigg|_0^{+\infty}\]

\[= \frac{1}{\sqrt{a^2 + b^2}}.\]

3837. 求外耳伊特拉斯变换

\[F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2} f(y) dy.\]
设：
\( a) \ f(y) = 1; \quad (6) \ f(y) = y^2; \)
\( b) \ f(y) = e^{2xy}; \quad (10) \ f(y) = \cos ay. \)

解
\( a) \ F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2} dy
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du
= \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1. \)

\( 6) \ F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2} y^2 dy
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} (x+u)^2 du
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} u^2 du
+ \frac{2x}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} u du
+ \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du. \)

由于
\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} u^2 du \]
\[ = \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} u^2 e^{-u^2} du = -\frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} u \ d(e^{-u^2}) \]
\[ = -\frac{1}{\sqrt{\pi}} \left. u e^{-u^2} \right|_{0}^{+\infty} + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-u^2} du \]
\[
\int_{-\infty}^{+\infty} e^{-u^2} du = 0.
\]

故得

\[
F(x) = \frac{1}{2} + \frac{2x^2}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{2} = x^2 + \frac{1}{2},
\]

\(\Box\) \quad F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2} e^{2xy} dy

= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2 + 2xy} dy

= \frac{1}{\sqrt{\pi}} e^{a^2 + 2ax} \cdot \int_{-\infty}^{+\infty} e^{-(y-x-a)^2} dy

= \frac{1}{\sqrt{\pi}} e^{a^2 + 2ax} \cdot 2 \cdot \frac{\sqrt{\pi}}{2}

= e^{a^2 + 2ax}.

\(\Box\) \quad F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(x-y)^2} \cos a y dy

= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} \cos a(x+u) du

= \frac{\cos ax}{\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} e^{-u^2} \cos au du

= -\frac{\sin ax}{\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} e^{-u^2} \sin au du

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\[
H_n(x) = \left( -1 \right)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) \quad (n=0,1,2,\ldots)
\]

而定义，证明

\[
\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx
\]

\[
= \begin{cases} 
0, & \text{若 } m \neq n, \\
2^n n! \sqrt{\pi}, & \text{若 } m = n.
\end{cases}
\]

证 由1231题的结果知，\(H_n(x)\)为一个\(n\)次多项式，且

\[x^n\]的系数为\(2^n\)．不妨设\(m \leq n\)，则

\[
\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx
= \int_{-\infty}^{+\infty} \left( -1 \right)^n H_n(x) \frac{d^n}{dx^n} \left( e^{-x^2} \right) dx
= (-1)^n \int_{-\infty}^{+\infty} H_n(x) d \left[ \frac{d^{n-1}}{dx^{n-1}} \left( e^{-x^2} \right) \right]
= (-1)^{n+1} \int_{-\infty}^{+\infty} H'_n(x) \frac{d^{n-1}}{dx^{n-1}} \left( e^{-x^2} \right) dx
= \cdots = (-1)^{n+n} \int_{-\infty}^{+\infty} H_n^{(n)}(x) \frac{d^{n-n}}{dx^{n-n}} \left( e^{-x^2} \right) dx
\]

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\[= (-1)^n \int_{-\infty}^{+\infty} H_n(x) e^{-x^2} dx.\]

当 \(m = n\) 时，\(H_n(x) = 0\)，故

\[
\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = 0.
\]

当 \(m = n\) 时，\(H_n(x) = 2^n n!\)，故

\[
\int_{-\infty}^{+\infty} H_n(x) H_n(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}.
\]

3839. 计算在概率论中有重要意义的积分

\[
\varphi(x) = \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{2\sigma_1^2}} e^{-\frac{\xi^2}{2\sigma_2^2}} d\xi.
\]

\((\sigma_1 > 0, \sigma_2 > 0)\).

解 注意到

\[
\frac{\xi^2}{2\sigma_1^2} + \frac{(x-\xi)^2}{2\sigma_2^2}
\]

\[
= \frac{1}{2\sigma_1^2 \sigma_2^2} \left[ (\sigma_1^2 + \sigma_2^2) \xi^2 - 2\sigma_1^2 x\xi + \sigma_1^2 x^2 \right],
\]

并令

\[
a = \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2 \sigma_2^2}, \quad b = -\frac{\sigma_1^2 x}{2\sigma_1^2 \sigma_2^2},
\]

\[
c = \frac{\sigma_1^2 x^2}{2\sigma_1^2 \sigma_2^2}.
\]
即得

$$
\varphi(x) = \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{+\infty} e^{-(a \xi^2 + 2b \xi + c)} d\xi
$$

$$
= \frac{1}{2\pi \sigma_1 \sigma_2} \sqrt{\frac{\pi}{a}} e^{\frac{-ac - b^2}{a}}.
$$

将 $a, b, c$ 的表达式代入上式，并令 $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$，化简整理得

$$
\varphi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.
$$

**）利用3804题的结果。

3840. 设函数 $f(x)$ 在区间 $(—\infty, +\infty)$ 内连续且绝对可积分 ** *。证明：积分

$$
u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(\xi - x)^2}{4a^2t}} d\xi
$$

满足热传导方程式

$$
\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}
$$

及初值条件

$$
\lim_{t \to 0} u(x, t) = f(x).
$$

证 当 $t > 0$, $—\infty < x < +\infty$ 时，

$$
\left| f(\xi) e^{-\frac{(\xi - x)^2}{4a^2t}} \right| \leq |f(\xi)|, \text{而 } \int_{-\infty}^{+\infty} |f(\xi)| d\xi
$$

$$
\leq +\infty, \text{故积分 } \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(\xi - x)^2}{4a^2t}} d\xi \text{ 在 } t > 0,
$$

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\(-\infty < x < +\infty\) 上一致收敛，从而 \(u(x,t)\) 是 \(t \to 0\)，
\(-\infty < x < +\infty\) 上的连续函数。考虑积分
\[
\int_{-\infty}^{+\infty} \frac{\partial}{\partial t} \left( f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \right) d\xi
\]

\[
= \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \frac{(\xi-x)^2}{4a^2t^2} d\xi,
\]  \hspace{1cm} (1)

\[
\int_{-\infty}^{+\infty} \frac{\partial}{\partial x} \left( f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \right) d\xi
\]

\[
= \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \frac{(\xi-x)}{2a^2t} d\xi,
\]  \hspace{1cm} (2)

\[
\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial x^2} \left( f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \right) d\xi
\]

\[
= \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \left[ -\frac{1}{2a^2t} \right.
\]

\[
+ \frac{(\xi-x)^2}{4a^4t^2} \left] d\xi, \hspace{1cm} (3)
\]

先考察（1）式中的积分：
由于对 \(|x| \leq x_0\)， \(0 < t_0 \leq t \leq t_1\) （\(x_0, t_0, t_1\) 任意固定），当 \(|\xi| > x_0\) 时，有

\[
|f(\xi) e^{-\frac{(\xi-x)^2}{4a^2t}} \frac{(\xi-x)^2}{4a^2t^2}| \leq |f(\xi)| e^{-\frac{|\xi|-x_0}{4a^2t_1}} \frac{|\xi|+x_0}{4a^2t_0},
\]

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\[ \frac{\partial u}{\partial t} = \frac{1}{4a \sqrt{\pi t}} \left[ - \int_{-\infty}^{\infty} e^{-\frac{(t-x)^2}{4at^2}} \, dx + \frac{1}{2 \sqrt{\pi t}} \right] \]

\[ \frac{\partial u}{\partial x} = \frac{1}{2 \sqrt{\pi t}} \left[ - \int_{-\infty}^{\infty} e^{-\frac{(t-x)^2}{4at^2}} \, dx \right] \]

\[ \frac{\partial u}{\partial x} \bigg|_{x=0} = 0 \]

其中 \( a \) 是常数。当 \( x = 0 \) 时，有

\[ f(\xi) e^{-\frac{(\xi-x)^2}{4at^2}} \]

由外氏积分法，(1) 式中的积分在 \( x \in (-\infty, \infty) \) 一致收敛，根据积分的分布性，(2) 式中的积分在 \( |x| \leq M \) 上一致收敛。于是，用外氏积分法得

\[ \int_{-\infty}^{\infty} e^{-\frac{(t-x)^2}{4at^2}} \, dx = \sqrt{\pi t} \]

\[ \frac{\partial u}{\partial x} \bigg|_{x=0} = 0 \]
由 $x_0$, $t_0$, $t_1$ 的任意性知，(4)、(5)、(6) 三式对一切 $-\infty < x < +\infty$, $t > 0$ 都成立。根据 (4) 式及 (6) 式，即得

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} (-\infty < x < +\infty, t > 0).$$

下面证明

$$\lim_{t \to +0} u(x, t) = f(x) (-\infty < x < +\infty). \quad (7)$$

任意固定 $x$, 易知 ($t > 0$，作变量替换 $u = \frac{\xi - x}{2a \sqrt{t}}$)

$$\int_{-\infty}^{+\infty} e^{-\frac{(\xi - x)^2}{4a^2t}} d\xi$$

$$= 2a \sqrt{\frac{t}{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du = 2a \sqrt{\pi} t,$$

故

$$u(x, t) - f(x)$$

$$= \frac{1}{2a \sqrt{\pi} t} \int_{-\infty}^{+\infty} [f(\xi) - f(x)] e^{-\frac{(\xi - x)^2}{4a^2t}} d\xi.$$ 任给 $\varepsilon > 0$，根据 $f(x)$ 在点 $x$ 的连续性，可取某 $\delta > 0$，使当 $|\xi - x| \leq \delta$ 时，恒有 $|f(\xi) - f(x)| \leq \frac{\varepsilon}{3}$. 有

$$u(x, t) - f(x)$$

$$= \frac{1}{2a \sqrt{\pi} t} \left( \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{+\infty} \right)$$

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\[ + \int_{x+\delta}^{+\infty} \left[ f(\xi) - f(x) \right] e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \]

\[ = I_1 + I_2 + I_3. \]

下面分别估计 \( I_1, I_2 \) 与 \( I_3 \)，我们有

\[ |I_2| = \left| \frac{1}{2a\sqrt{\pi t}} \int_{x-\delta}^{x+\delta} \left[ f(\xi) - f(x) \right] e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \right| \]

\[ \leq \frac{e}{3} \left( \frac{1}{2a\sqrt{\pi t}} \int_{x-\delta}^{x+\delta} e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \right) \]

\[ \leq \frac{e}{3} \left( \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \right) = \frac{e}{3}. \]

又有

\[ |I_3| = \left| \frac{1}{2a\sqrt{\pi t}} \int_{x+\delta}^{+\infty} \left[ f(\xi) - f(x) \right] e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \right| \]

\[ \leq \frac{1}{2a\sqrt{\pi t}} e^{-\frac{\delta^2}{4a^2t}} \int_{x+\delta}^{+\infty} |f(\xi)| \, d\xi \]

\[ + \frac{|f(x)|}{2a\sqrt{\pi t}} \int_{x+\delta}^{+\infty} e^{-\frac{(\xi-x)^2}{4a^2t}} \, d\xi \]

\[ \leq \frac{1}{2a\sqrt{\pi t}} e^{-\frac{\delta^2}{4a^2t}} \int_{-\infty}^{+\infty} |f(\xi)| \, d\xi \]

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由此可知 \( \lim_{t \to 0} I_3 = 0 \). 同理可知 \( \lim_{t \to 0} I_1 = 0 \). 于是，存在 \( \eta > 0 \), 使得当 \( 0 < t < \eta \) 时，恒有

\[ |I_3| \leq \frac{\varepsilon}{3}, \quad |I_1| \leq \frac{\varepsilon}{3}. \]

由此，当 \( 0 < t < \eta \) 时，恒有

\[ |u(x, t) - f(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \]

故 (7) 式成立。证毕。

*) 编者注：本题原书把 \( \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \) 误写为

\[ \frac{\partial u}{\partial t} = \frac{1}{a^2} \frac{\partial^2 u}{\partial x^2}. \] 另外，原书只假定 \( f(x) \) 在 \((-\infty, +\infty)\) 上绝对可积，这是不够的。应加上假定 \( f(x) \) 在 \((-\infty, +\infty)\) 上连续。否则，结论

\[ \lim_{t \to 0} u(x, t) = f(x) \]

就可能不成立。例如，令

\[ f(x) = \begin{cases} 1, & \text{当 } x = 0 \text{ 时;} \\ 0, & \text{当 } x \neq 0 \text{ 时,} \end{cases} \]

则显然 \( f(x) \) 在 \((-\infty, +\infty)\) 绝对可积。这时

\[ u(x, t) = 0 \quad (t > 0, \quad -\infty < x < +\infty), \]

故

\[ \lim_{t \to 0} u(0, t) = 0 \neq 1 = f(0). \]
原书缺页

709-740
第六章 多变量函数的微分法

§1. 多变量函数的极限、连续性

1° 多变量函数的极限 设函数 \( f(P) = f(x_1, x_2, \ldots, x_n) \) 在以 \( P_0 \) 为中心的集含 \( E \) 上有定义，若对于任何的 \( e > 0 \) 存在有 \( \delta = \delta(e, P_0) > 0 \)，使得只要 \( P \in E \) 及 \( 0 < \rho(P, P_0) < \delta \) [其中 \( \rho(P, P_0) \) 为 \( P \) 和 \( P_0 \) 两点间的距离]，则

\[
|f(P) - A| < e,
\]

我们就说

\[
\lim_{P \to P_0} f(P) = A.
\]

2° 连续性 若

\[
\lim_{P \to P_0} f(P) = f(P_0),
\]

则称函数 \( f(P) \) 于 \( P_0 \) 点是连续的。

若函数 \( f(P) \) 于已知域内的每一点连续，则称函数 \( f(P) \) 于此域内是连续的。

3° 一致连续性 若对于每一个 \( e > 0 \) 都存在有仅与 \( e \) 有关的 \( \delta > 0 \)，使得对于域 \( G \) 中的任何点 \( P', P'' \)，只要是有

\[
\rho(P', P'') < \delta,
\]

便有不等式

\[
|f(P') - f(P'')| < e
\]

成立，则称函数 \( f(P) \) 于域 \( G \) 内是一致连续的。
于有界闭域内的连续函数于此域内是一致连续的。

确定并绘出下列函数存在的域。

3136. \( u = x + \sqrt{y} \)。

解 存在域为半平面，

\[ y \geq 0, \]

如图 6.1 阴影部分所示，包括整个 Ox 轴在内。

![图 6.1](image)

3137. \( u = \sqrt{1 - x^2} \)

\[ + \sqrt{y^2 - 1}. \]

解 存在域为满足不等式

\[ |x| \leq 1, \ |y| \geq 1 \]

的点集，如图 6.2 阴影部分所示，包括边界（粗实线）在内。

3138. \( u = \sqrt{1 - x^2 - y^2} \)。

解 存在域为圆
$$x^2 + y^2 \leq 1,$$

如图6.3阴影部分所示，包括圆周在内。

3139. \( u = \frac{1}{\sqrt{x^2 + y^2 - 1}} \).

解 存在域为满足不等式

$$x^2 + y^2 = 1$$

的点集，即圆 $$x^2 + y^2 = 1$$ 的外面，如图6.4所示，不包括圆周（虚线）在内。

3140. \( u = \frac{1}{\sqrt{(x^2 + y^2 - 1)(4 - x^2 - y^2)}} \).

解 存在域为满足不等式

$$1 \leq x^2 + y^2 \leq 4$$

的点集，如图6.5所示的环，包括边界在内。

3141. \( u = \sqrt{\frac{x^2 + y^2 - x}{2x - x^2 - y^2}} \).

解 存在域为满足不等式

$$x \leq x^2 + y^2 \leq 2x$$

的点集。由 $$x^2 + y^2$$ 
$x$得出

$\left(x-\frac{1}{2}\right)^2 + y^2 \geqslant \left(\frac{1}{2}\right)^2$,

由 $x^2 + y^2 < 2x$ 得出

$(x-1)^2 + y^2 < 1$,

两者组成一月形，

如图 6.6 阴影部分所示。

3142. $u = \sqrt{1 - (x^2 + y)^2}$。

解 存在域为满足

不等式

$-1 \leqslant x^2 + y \leqslant 1$

的点集，如图 6.7 阴影部分所示，包括边界在内。

3143. $u = \ln(-x-y)$。

解 存在域为半平面

$x+y \leqslant 0$，

如图 6.8 阴影部分所示，不包括直线 $x+y=0$ 在内。

3144. $u = \arcsin\frac{y}{x}$。

解 存在域为满足
不等式
\[ \left| \frac{y}{x} \right| \leq 1 \]
或 \[ |y| \leq |x| \quad (x \neq 0) \]
的点集，这是对顶的直角，如图6.9阴影部分所示，不包括原点在内。

3145. \( u = \arccos \frac{x}{x+y} \)。

图 6.9

解 存在域为满足不等式
\[ \left| \frac{x}{x+y} \right| \leq 1 \]
的点集。由 \( \left| \frac{x}{x+y} \right| \leq 1 \) 得 \( |x| \leq |x+y| \quad (x \neq -y) \)，
即 \( x^2 \leq x^2 + 2xy + y^2 \) 或 \( y(y+2x) \geq 0 \)，也即
\[
\begin{cases}
y \geq 0, \\
y \geq -2x,
\end{cases}
\quad \text{或} \quad
\begin{cases}
y \leq 0, \\
y \leq -2x.
\end{cases}
\]

但 \( x, y \) 不能同时为零。这是由直线：
\( y = 0 \) 和 \( y = -2x \) 所围成的一对对顶的角，如图6.10阴影部分所示，包括边界在内，但不包括公共顶点 \( O (0,0) \) 在内。
3146. \( u = \arcsin \frac{x}{y^2} + \arcsin(1 - y) \).

解 存在域为满足不等式
\[ | \frac{x}{y^2} | \leq 1 \text{ 及 } |1 - y| \leq 1 (y \neq 0) \]
的点集，即
\[
\begin{cases}
  y^2 \geq x, \\
  0 < y \leq 2
\end{cases}
\quad \text{和}
\begin{cases}
  y^2 \geq -x, \\
  0 < y \leq 2.
\end{cases}
\]
这是由抛物线：
\( y^2 = x, y^2 = -x \)
和直线 \( y = 2 \) 所围成的曲边三角形，如图6-11阴影部分所示，不包括原点在内。

3147. \( u = \sqrt{\sin(x^2 + y^2)} \).

解 存在域为满足不等式
\[ \sin(x^2 + y^2) \geq 0 \]
或 \( 2k\pi \leq x^2 + y^2 \leq (2k + 1)\pi (k = 0, 1, 2, \cdots) \) 的点集，如图6-12所示的同心环族。
解 存在域为满足不等式

$$\left| \frac{z}{\sqrt{x^2 + y^2}} \right| \leq 1$$

（x, y 不同时为零）

或

$$x^2 + y^2 - z^2 \geq 0$$

（x, y 不同时为零）

的点集，这是圆锥 $x^2 + y^2 - z^2 = 0$ 的外面，如图 6·13 阴影部分所示，包括边界在内，但要除去圆锥的顶点。

3149. $u = \ln(xyz)$。

解 存在域为满足不等式

$$xyz \geq 0$$

的点集，即

$$x > 0, y > 0, z > 0; \text{或} x > 0, y < 0, z < 0;$$

$$x < 0, y < 0, z > 0; \text{或} x < 0, y > 0, z < 0.$$
解 存在域为满足不等式
\[-x^2 - y^2 + z^2 \geq 1\]
的点集。这是双叶双曲面 \(x^2 + y^2 - z^2 = -1\) 的内部。如图6·14阴影部分所示，不包括界面在内。
作出下列函数的等位线:

3151. \(z = x + y\).
解 等位线为平行直
线族
\[x + y = k,\]
其中 \(k\) 为一切实数，
如图6·15所示。

3152. \(z = x^2 + y^2\).
解 等位线为曲线族
\[x^2 + y^2 = a^2\]
\((a \neq 0)\).
当 \(a = 0\) 时为原点，当
\(a \neq 0\) 时，等位线为以
原点为圆心的同心圆族。

3153. \(z = x^2 - y^2\).
解 等位线为曲线族
\[x^2 - y^2 = k,\]
当 \(k = 0\) 时为两条互相垂直的直线：\(y = x, y = -x\)。
当 $k \neq 0$ 时为以 $y = \pm x$ 为公共渐近线的等边双曲线族，
其中当 $k > 0$ 时顶点为 $( - \sqrt{k}, \ 0 ), ( \sqrt{k}, \ 0 )$，当
$k < 0$ 时顶点为 $( 0, - \sqrt{-k} ), ( 0, \sqrt{-k} )$。

3154. $z = (x + y)^2$。

解 等位线为曲线族

$$(x + y)^2 = a^2 \ (a \geq 0).$$

当 $a = 0$ 为直线 $x + y = 0$。当 $a \neq 0$ 时为与直线 $x + y = 0$ 平行的且等距的直线 $x + y = \pm a$。

3155. $z = \frac{y}{x}$。

解 等位线为以坐标原点为束心的直线束

$y = kx \ (x \neq 0)$，
不包括 $Oy$ 轴在内。

3156. $z = \frac{1}{x^2 + 2y^2}$。

解 等位线为椭圆族

$x^2 + 2y^2 = a^2 \ (a \geq 0)$。

长半轴为 $a$，短半轴为 $\frac{a}{\sqrt{2}}$，焦点为 $( -a \sqrt{\frac{3}{2}}, 0 )$
及 $(a \sqrt{\frac{3}{2}}, 0 )$。

3157. $z = \sqrt{xy}$。

解 等位线为曲线族

$xy = a^2 \ (a \geq 0)$。

当 $a = 0$ 时为坐标轴 $x = 0$ 及 $y = 0$。当 $a \neq 0$ 时为以两坐标轴为公共渐近线且位于第一、第三象限内的等
边双曲线族，顶点为
\((-a, -a)\)及\((a, a)\)。

3158. \(z = |x| + y\)。
解 等位线为曲线族
\(|x| + y = k\)，
其中\(k\)为一切实数。当
\(x \geq 0\)时为\(x + y = k\);
当\(x < 0\)时为\(-x + y = k\)。这是顶点在\(Oy\)
轴上两支互相垂直的
射线所构成的折线
族，如图6·16所示。

3159. \(z = |x| + |y| - |x+y|\)。
解 等位线为曲线族
\(|x| + |y| - |x+y| = a\)。
因为恒有\(|x| + |y| \geq |x+y|\)，所以\(a \geq 0\)。
当\(a = 0\)时，由\(|x| + |y| = |x+y|\)两边平方即得
\(xy \geq 0\)，
即为整个第一、第三象限，包括两坐标轴在内。
当\(a > 0\)时，\(xy < 0\)，分下面四组求解：
(1) \(x > 0, y < 0\)，\(x+y \geq 0\)，\(|x| + |y| - |x+y| = a\)，解之得
\(y = -\frac{a}{2}\)；
(2) \(x > 0, y < 0\)，\(x+y \leq 0\)，\(|x| + |y| - |x+y| = a\)，解之得
\(x = \frac{a}{2}\)；
(3) \( x<0, \ y>0, x+y \geq 0, |x| + |y| - |x+y| = a \), 解之得 \( x = -\frac{a}{2} \);

(4) \( x<0, \ y>0, x+y \leq 0, |x| + |y| - |x+y| = a \), 解之得 \( y = \frac{a}{2} \).

是项点位于直线 \( x+y=0 \) 上的两支互相垂直的折线族，它的各射线平行于坐标轴，如图 6.17 所示。

3.6.0. \( z = e^{\frac{-2x}{x^2+y^2}} \).

解 等位线为曲线族

\[
\frac{2x}{x^2+y^2} = k \quad (x, y \text{ 不同时为零})
\]

其中 \( k \) 为异于零的一切实数。上式可变形为

\[
(x - \frac{1}{k})^2 + y^2 = \left(\frac{1}{k}\right)^2 \quad (k \neq 0).
\]

当 \( k=0 \) 时，即得 \( e^{\frac{2x}{x^2+y^2}} = 1 \)，从而等位线为 \( x = 0 \) 即 \( Oy \) 轴，但不包括原点。

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当 $k = 0$ 时为中心在 $Ox$ 轴上且经过坐标原点（但不包括原点在内）的圆束，圆心在 $(\frac{1}{k}, 0)$，半径为 $|\frac{1}{k}|$。

如图6.18所示。

3161. $z = x^a$ ($x > 0$)。

解 等位线为曲线族

$x^a = a$ ($a > 0$)。

当 $a = 1$ 时为直线 $x = 1$ 及 $Ox$ 轴的正向半射线，但不包括原点在内。

当 $0 < a < 1$ 与 $a > 1$ 时的图象如图6.19所示。

3162. $z = x^a e^{-x}$ ($x > 0$)。

解 等位线为曲线族
\[ x^y e^{-x} = a \ (a > 0), \]

即

\[ y \ln x - x = \ln a. \]

当 \( a = e^{-1} \) 时为直线 \( x = 1 \)

和曲线 \( y = \frac{x - 1}{\ln x} \); 当 \( 0 < a < \frac{1}{e} \)

或 \( a > 1 \)

图象布满整个右半平面，如图6.20所示，不包括 \( Oy \) 轴。

3163. \( z = \ln \sqrt{\frac{(x-a)^2 + y^2}{(x+a)^2 + y^2}} \ (a > 0). \)

解 等位线为曲线族

\[ \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = k^2 \ (k > 0). \]

整理得

\[ (1-k^2)x^2 - 2a(1+k^2)x + (1-k^2)a^2 + (1-k^2)y^2 = 0. \]

当 \( k = 1 \) 时得 \( x = 0 \), 即 \( Oy \) 轴. 当 \( k \neq 1 \) 时，上述方程可变形为

\[ \left( x - \frac{a(1+k^2)}{1-k^2} \right)^2 + y^2 = \left( \frac{2ak}{1-k^2} \right)^2, \]

这是以点 \( \left( \frac{a(1+k^2)}{1-k^2}, 0 \right) \) 为圆心，半径为 \( \left| \frac{2ak}{1-k^2} \right| \)
的圆族。当 $0 < k < 1$ 时，圆分布在右半平面；当 $k > 1$ 时，圆分布在左半平面。

如果注意到圆心与原点距离的平方为

$$
\left[ \frac{a(1+h^2)}{1-h^2} \right]^2 = \frac{a^2((1-h^2)^2 + 4h^2)}{(1-h^2)^2}
$$

$$
= a^2 + \left( \frac{2ah}{1-h^2} \right)^2.
$$

即等位线圆族与圆 $x^2 + y^2 = a^2$ 在交点处的半径互相垂直（或圆心距与两圆的半径构成直角三角形），便知等位线圆族与圆 $x^2 + y^2 = a^2$ 成正交。如图6·21所示。

图 6·21

3164. $z = \arctg \frac{2ay}{x^2 + y^2 - a^2} (a > 0)$。

解 等位线为曲线族

$$
\frac{2ay}{x^2 + y^2 - a^2} = k,
$$

其中 $k$ 为一切实数，但要除去点 $(a, 0)$ 及 $(-a, 0)$。

当 $k = 0$ 时，$y = 0$，即为 $Ox$ 轴，但不包含上述两点；

d 当 $k \neq 0$ 时，方程可变形为
\[ x^2 + (y - \frac{a}{k})^2 = a^2 (1 + \frac{1}{k^2}) , \]

这是圆心在Oy轴上且经过点(-a, 0)及(a, 0)但不包括这两点在内的圆族，如图6.22所示。

3.6.5 \( z = \text{sgn}(\sin x \sin y) \)。

解 若 \( z = 0 \), 则 \( \sin x \cdot \sin y = 0 \), 此即直线族

\[ x = m\pi \text{和} y = n\pi \ (m, n = 0, \pm 1, \pm 2, \cdots) ; \]

若 \( z = -1 \) 或 \( z = 1 \), 则 \( \sin x \sin y < 0 \) 或 \( \sin x \sin y > 0 \), 此即正方形系

\[ m\pi < x < (m + 1)\pi, \ n\pi < y < (n + 1)\pi, \]

其中 \( z = (-1)^{n+m} \)。

如图6.23所示，\( z = 0 \) 时为图中网格直线；\( z = 1 \) 为图中带斜线的正方形；

\( z = -1 \) 为图中空白正方形，但后两者都不包括边界。

求下列函数的等位
3166. \( u = x + y + z \).

解释等位面为平行平面族
\[ x + y + z = k, \]
其中\( k \)为一切实数。

3167. \( u = x^2 + y^2 + z^2 \).

解释等位面为中心在原点的同心球族
\[ x^2 + y^2 + z^2 = a^2 \quad (a \geq 0), \]
其中当\( a = 0 \)时即为原点。

3168. \( u = x^2 + y^2 - z^2 \).

解释当\( u = 0 \)时等位面为圆锥\( x^2 + y^2 - z^2 = 0 \)；当\( u \gg 0 \)时等位面为单叶双曲面族\( x^2 + y^2 - z^2 = a^2 \quad (a \gg 0) \)；当\( u < 0 \)时等位面为双叶双曲面族\( -x^2 - y^2 + z^2 = a^2 \quad (a \gg 0) \)。

3169. \( u = (x + y)^2 + z^2 \).

解释等位面为曲面族
\[ (x + y)^2 + z^2 = a^2 \quad (a \geq 0), \]
当\( a = 0 \)时为\( x + y = 0 \)和\( z = 0 \)。当\( a \gg 0 \)时作坐标变换
\[
\begin{align*}
    x' &= x \cos \frac{\pi}{4} + y \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} (x + y), \\
    y' &= -x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} (x + y), \\
    z' &= z,
\end{align*}
\]
这是旋转变换，在新坐标系中原等位面方程转化为
\[ 2x'^2 + z'^2 = a^2, \]
即

\[ \frac{x^{'2}}{a^2} + \frac{z^{'2}}{a^2} = 1, \]

这是以 \( y' \)轴为公共轴的椭圆柱面，母线的方向平行于
\( y' \)轴，准线为 \( y' = 0 \) 平面上的椭圆

\[ \frac{x^{'2}}{a^2} + \frac{z^{'2}}{a^2} = 1, \]

长半轴为 \( a \) （\( z' \)轴方向），短半轴为 \( \frac{a}{\sqrt{2}} \) （\( x' \)轴方向）。

\( y' \)轴在新系 \( O-x'y'z' \) 中的方程为

\[ \begin{cases} x' = 0, \\ z' = 0, \end{cases} \]

而在旧系 \( O-xyz \) 中的方程为

\[ \begin{cases} x + y = 0, \\ z = 0, \end{cases} \]

即为所求的椭圆柱面族的公共对称轴。

3170. \( u = sgn \sin (x^2 + y^2 + z^2) \).

解 当 \( u = 0 \) 时等位面为球心在原点的同心球族
\[ x^2 + y^2 + z^2 = n\pi \quad (n = 0, 1, 2, \cdots) \]
当 \( u = -1 \) 或 \( u = 1 \) 时等位面为球层族
\[ n\pi < x^2 + y^2 + z^2 < (n + 1)\pi \quad (n = 0, 1, 2, \cdots), \]

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其中 $u = (-1)^n$。

根据曲面的已知方程研究其性质。

3171. $z = f(y - ax)$。

解：引入参数 $t, s$，将曲面方程 $z = f(y - ax)$ 表成参数方程

$$\begin{align*}
    x &= t, \\
    y &= ai + s, \\
    z &= f(s).
\end{align*}$$

今固定 $s$，得到以 $t$ 为参数的直线方程，其方向数为 $1, a, 0$。因此，曲面为以 $1, a, 0$ 为母线方向的一个柱面。令 $t = 0$，可得

$$\begin{align*}
    x &= 0, \\
    y &= s, \quad \text{或} \quad x = 0, \\
    z &= f(s), \quad z = f(y),
\end{align*}$$

这是 $x = 0$ 平面上的一条曲线，也是柱面 $z = f(y - ax)$ 的一条准线。

3172. $z = f(\sqrt{x^2 + y^2})$。

解：这是绕 $Oz$ 轴旋转的旋转曲面的标准形式。令 $y = 0$，得曲线

$$\begin{align*}
    y &= 0, \\
    z &= f(x) \quad (x \geq 0),
\end{align*}$$

它是旋转曲面的一条母线。

3173. $z = x f\left(\frac{y}{x}\right)$。
解  引入参数 $t,s$, 将曲面方程 $z = xf(y/x)$ 表成参数方程

$$
\begin{aligned}
\begin{cases}
x = t, \\
y = st (t \neq 0), \\
z = tf(s).
\end{cases}
\end{aligned}
$$

今固定 $s$, 这是以 $t$ 为参数的一条过原点的直线。因此，所给曲面为顶点在原点的一锥面，但不包括原点在内。令 $t = 1$, 得曲线

$$
\begin{aligned}
\begin{cases}
x = 1, \\
y = s, \\
z = f(s),
\end{cases}
\end{aligned}
\quad \text{或} \quad
\begin{aligned}
\begin{cases}
x = 1, \\
z = f(y),
\end{cases}
\end{aligned}
$$

这是 $x = 1$ 平面上的一条曲线，也是锥面 $z = xf(y/x)$ 的一条准线。

3174*. $z = f(y/x)$.

解  引入参数 $t,s$, 将曲面方程 $z = f(y/x)$ 表成参数方程

$$
\begin{aligned}
\begin{cases}
x = t, \\
y = st, \\
z = f(s).
\end{cases}
\end{aligned}
$$

* 题号右上角“*”号表示题解答案与原习题集中译本所附答案不一致；以后不再说明。译本基本是按俄文第二版翻译的。俄文第二版中有一些错误已在俄文第三版中改正。
今固定 $s$，这是一条过点 $(0, 0, f(s))$ 的直线，方向数为 $1, s, 0$。因此，它与 $Oz$ 轴垂直，与 $Oxy$ 平面平行，且其方向与 $s$ 有关。从而得知，曲面 $z = f\left(\frac{y}{x}\right)$ 表示一个直纹面。一般说来，它既不是柱面，又不是锥面。令 $t = 1$，得到直纹面的一条准线

$$\begin{cases} x = 1, \\ z = f(y). \end{cases}$$

从此曲线上每一点引一条与 $Oz$ 轴垂直且相交的直线。这样的直线的全体，便构成由 $z = f\left(\frac{y}{x}\right)$ 所表示的直纹面。

3175。作出函数

$$F(t) = f(\cos t, \sin t)$$

的图形。式中

$$f(x, y) = \begin{cases} 1, & \text{若 } y \geq x, \\ 0, & \text{若 } y < x. \end{cases}$$

解 按题设，当 $\sin t \geq \cos t$，即 $\frac{\pi}{4} + 2k\pi \leq t \leq \frac{5\pi}{4} + 2k\pi$ ($k = 0, \pm 1, \pm 2, \cdots$) 时，$F(t) = 1$；而当

![图 6.24](image-url)

20.
\[\sin t = \cos t, \text{ 即 } -\frac{3}{4} \pi + 2k\pi < t < \frac{\pi}{4} + 2k\pi \text{ 时，} F(t) = 0. \text{ 如图 6.24 所示。}\]

3176. 若

\[f(x, y) = \frac{2x y}{x^2 + y^2},\]

求 \(f(1, \frac{y}{x})\).

解  \(f(1, \frac{y}{x}) = \frac{2 \cdot 1 \cdot \frac{y}{x}}{1 + (\frac{y}{x})^2} = \frac{2xy}{x^2 + y^2} = f(x, y).\)

3177. 若

\[f\left(\frac{y}{x}\right) = \frac{\sqrt{x^2 + y^2}}{x} (x \geq 0),\]

求 \(f(x)\).

解  由 \(f\left(\frac{y}{x}\right) = \sqrt{1 + (\frac{y}{x})^2}\) 知 \(f(x) = \sqrt{1 + x^2}.\)

3178. 设

\[z = \sqrt{y} + f(\sqrt{x} - 1).\]

若当 \(y = 1\) 时 \(z = x\)，求函数 \(f\) 和 \(z\).

解  因为当 \(y = 1\) 时 \(z = x\)，所以

\[f(\sqrt{x} - 1) = x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1) = (\sqrt{x} - 1)((\sqrt{x} - 1) + 2),\]

从而得
\[ f(t) = t(t + 2) = t^2 + 2t, \]

且
\[ z = \sqrt{x} + x - 1 \quad (x > 0). \]

3179. 设
\[ z = x + y + f(x - y). \]
若当 \( y = 0 \) 时，\( z = x^2 \)，求函数 \( f \) 及 \( z \)。

解 因为当 \( y = 0 \) 时 \( z = x^2 \)，所以
\[ x^2 = x + f(x), \]
即
\[ f(x) = x^2 - x, \]
且
\[ z = x + y + (x - y)^2 = 2y + (x - y)^2. \]

3180. 若 \( f(x + y, \frac{y}{x}) = x^2 - y^2 \)，求 \( f(x, y) \)。

解 因为
\[ f(x + y, \frac{y}{x}) = x^2 - y^2 = (x + y)(x - y) \]

\[ = (x + y)^2 \frac{x - y}{x + y} = (x + y)^2 \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}. \]

所以
\[ f(x, y) = x^2 \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}. \]

3181. 证明：对于函数
\[ f(x, y) = \frac{x - y}{x + y} \]
有
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = 1; \quad \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = -1, \]
从而 \( \lim_{y \to 0} f(x, y) \) 不存在。

证
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{x \to 0} \frac{x - y}{x + y} \right\} = \lim_{x \to 0} \frac{x}{x} = 1, \]
\[ \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x - y}{x + y} \right\} \]
\[ = \lim_{y \to 0} \frac{-y}{y} = -1. \]

由于两个单极限都存在，而累次极限不等，故 \( \lim_{y \to 0} f(x, y) \) 不存在。

3182. 证明：对于函数
\[ f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}, \]
有
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = 0, \]
然而 \( \lim_{x \to 0} f(x, y) \) 不存在。

证
\[ \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{x^2 y^2}{x^2 y^2 + (x - y)^2} \right\} \]
\[ = \lim_{x \to 0} 0 = 0, \]

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\[
\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} \right\} = \lim_{y \to 0} 0 = 0.
\]

如果沿 \( y = kx \to 0 \) 方向取极限，则有
\[
\lim_{x \to 0} f(x, y) = \lim_{x \to 0} \frac{x^4 k^2}{x^4 k^2 + x^2 (1-k)^2}.
\]

特别地，分别取 \( k = 0 \) 及 \( k = 1 \)，便得到不同的极限 0 及 1。因此，\( \lim_{y \to 0} f(x, y) \)不存在。

3183. 证明：对于函数
\[
f(x, y) = (x+y) \sin \frac{1}{x} \sin \frac{1}{y}
\]

累次极限 \( \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} \) 和 \( \lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} \) 不存在，然而 \( \lim_{y \to 0} f(x, y) = 0 \)。

证 由不等式
\[
0 \leq |(x+y) \sin \frac{1}{x} \sin \frac{1}{y}| \leq |x+y| \leq |x| + |y|
\]
知 \( \lim_{y \to 0} f(x, y) = 0 \)。

但当 \( x \neq \frac{1}{k \pi} \), \( y \to 0 \) 时，\( (x+y) \sin \frac{1}{x} \sin \frac{1}{y} \) 的极限不存在，因此累次极限 \( \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} \) 不存在，同法可证累次极限 \( \lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} \) 也不存在。

3184. 求 \( \lim_{x \to a} \left\{ \lim_{y \to b} f(x, y) \right\} \) 及 \( \lim_{y \to b} \left\{ \lim_{x \to a} f(x, y) \right\} \)，设：
(a) \( f(x, y) = \frac{x^2 + y^2}{x^2 + y^4}, \quad a = \infty, \quad b = \infty; \)

(b) \( f(x, y) = \frac{x^y}{1 + x^y}, \quad a = +\infty, \quad b = +0; \)

(c) \( f(x, y) = \sin \frac{\pi x}{2x+y}, \quad a = \infty, \quad b = \infty; \)

(d) \( f(x, y) = \frac{1}{xy} \cdot \frac{x^y}{1 + x^y}, \quad a = 0, \quad b = \infty; \)

(e) \( f(x, y) = \log_2(x+y), \quad a = 1, \quad b = 0. \)

解

(a) \( \lim_{t \to -\infty} \left\{ \lim_{s \to -\infty} f(x, y) \right\} = \lim_{s \to -\infty} \left\{ \lim_{t \to -\infty} \frac{x^2 + y^2}{x^2 + y^4} \right\} \)

\( = \lim_{s \to -\infty} 0 = 0, \)

\( \lim_{s \to -\infty} \left\{ \lim_{t \to -\infty} f(x, y) \right\} = \lim_{t \to -\infty} \left\{ \lim_{s \to -\infty} \frac{x^2 + y^2}{x^2 + y^4} \right\} \)

\( = \lim_{t \to -\infty} 1 = 1; \)

(b) \( \lim_{x \to +\infty} \left\{ \lim_{y \to +0} f(x, y) \right\} = \lim_{y \to +0} \left\{ \lim_{x \to +\infty} \frac{x^y}{1 + x^y} \right\} \)

\( = \lim_{y \to +0} \frac{1}{2} = \frac{1}{2}, \)

\( \lim_{y \to +0} \left\{ \lim_{x \to +\infty} f(x, y) \right\} = \lim_{x \to +\infty} \left\{ \lim_{y \to +0} \frac{x^y}{1 + x^y} \right\} \)

\( = \lim_{x \to +\infty} 1 = 1; \)

(b) \( \lim_{y \to -\infty} \left\{ \lim_{x \to +\infty} f(x, y) \right\} = \lim_{x \to +\infty} \left\{ \lim_{y \to -\infty} \frac{x^y}{2x+y} \right\} \)

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\[
\lim_{x \to \infty} 0 = 0, \\
\lim_{y \to \infty} \left\{ \lim_{x \to \infty} f(x, y) \right\} = \lim_{y \to \infty} \left\{ \lim_{x \to \infty} \sin \frac{\pi x}{2x + y} \right\} = \lim_{y \to \infty} 1 = 1; \\
\lim_{x \to 0} \left\{ \lim_{y \to \infty} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to \infty} \frac{1}{x} \cdot \tan \frac{xy}{1 + xy} \right\} = \lim_{x \to 0} \left\{ 0 \cdot \tan 1 \right\} = 0, \\
\lim_{y \to \infty} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to \infty} \left\{ \lim_{x \to 0} \frac{1}{x} \cdot \tan \frac{xy}{1 + xy} \right\} = \lim_{y \to \infty} \left\{ \tan \frac{xy}{1 + xy} \cdot \lim_{x \to 0} \frac{1}{1 + xy} \right\} = \lim_{y \to \infty} 1 = 1; \\
\lim_{x \to 1} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 1} \left\{ \lim_{y \to 0} \log_x (x + y) \right\} = \lim_{x \to 1} \left\{ \frac{\ln (x + y)}{\ln x} \right\} = \lim_{x \to 1} \frac{\ln x}{\ln x} = 1, \\
\lim_{y \to 0} \left\{ \lim_{x \to 1} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 1} \frac{\ln (x + y)}{\ln x} \right\} = \infty.
\]

求下列极限；

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3185. \[ \lim_{(x, y) \to \infty} \frac{x + y}{x^2 - xy + y^2} \]

解：由不等式 \( x^2 + y^2 \geq 2 |xy| \) 得

\[
0 \leq \left| \frac{x + y}{x^2 - xy + y^2} \right| \leq \left| \frac{x + y}{x^2 + y^2 - |xy|} \right| \leq \frac{1}{|x|} + \frac{1}{|y|} ,
\]

而 \( \lim_{x \to \infty} \left( \frac{1}{|x|} + \frac{1}{|y|} \right) = 0 \)，故有

\[ \lim_{(x, y) \to \infty} \frac{x + y}{x^2 - xy + y^2} = 0 . \]

3186. \[ \lim_{(x, y) \to \infty} \frac{x^2 + y^2}{x^4 + y^4} \]

解：由不等式

\[
0 \leq \frac{x^2 + y^2}{x^4 + y^4} \leq \frac{x^2 + y^2}{2x^2 y^2} = \frac{1}{2} \left( \frac{1}{x^2} + \frac{1}{y^2} \right) ,
\]

及 \( \lim_{x \to \infty} \frac{1}{2} \left( \frac{1}{x^2} + \frac{1}{y^2} \right) = 0 \)，即得

\[ \lim_{(x, y) \to \infty} \frac{x^2 + y^2}{x^4 + y^4} = 0 . \]

3187. \[ \lim_{(x, y) \to (0, 0)} \frac{\sin xy}{x} \]

解：\( \lim_{(x, y) \to (0, 0)} \frac{\sin xy}{x} = \lim_{x \to 0} \left( \frac{\sin xy}{xy} \cdot y \right) = a . \)
318. \( \lim_{x \to +\infty} (x^2 + y^2) e^{-x+y} \).

解

\[
\lim_{x \to +\infty} (x^2 + y^2) e^{-x+y} = \lim_{x \to +\infty} \left[ \frac{(x+y)^3}{e^{x+y}} - 2 \cdot \frac{x}{e^x} \cdot \frac{y}{e^y} \right] = 0 \star)
\]

\star) 利用 564 题的结果。

3189. \( \lim_{y \to +\infty} \left( \frac{xy}{x^2 + y^2} \right)^2 \).

解

由不等式

\[
0 \leq \left( \frac{xy}{x^2 + y^2} \right)^2 \leq \left( \frac{1}{2} \right)^2
\]

及 \( \lim_{x \to +\infty} \left( \frac{1}{2} \right)^2 = 0 \)，即得

\[
\lim_{x \to +\infty} \left( \frac{xy}{x^2 + y^2} \right)^2 = 0.
\]

3790. \( \lim_{x \to 0} \frac{x+y^2}{x^2 y^2} \).

解

由不等式

\[
|x^2 y^2 \ln(x^2 + y^2)| \leq \frac{(x^2 + y^2)^2}{4} |\ln(x^2 + y^2)|
\]

及 \( \lim_{x \to 0} \frac{(x^2 + y^2)^2}{4} \ln(x^2 + y^2) = \lim_{t \to 0} \frac{1}{4} t^2 \ln t = 0 \)，即得

\[
\lim_{x \to 0} \frac{x+y^2}{x^2 y^2} = \lim_{x \to 0} e^{x^2 y^2 \ln(x^2 + y^2)} = e^0 = 1.
\]
3191. \( \lim_{x \to -\infty} \left( 1 + \frac{1}{x} \right)^{\frac{x^2}{x+y}} \).

解
\[
\lim_{x \to -\infty} \left( 1 + \frac{1}{x} \right)^{\frac{x^2}{x+y}} = \lim_{x \to -\infty} \left( 1 + \frac{1}{x} \right)^{\frac{x}{x+y}}
\]
\[
= \lim_{x \to -\infty} e^{\frac{x}{x+y}}
\]
\[
= \left( \lim_{x \to -\infty} \left( 1 + \frac{1}{x} \right) \right) \cdot \left( \lim_{y \to 0} \frac{x}{x+y} \right) = e \cdot 1 - e.
\]

3192. \( \lim_{x \to 0} \frac{\ln(\sqrt{x+e^x})}{\sqrt{x^2+y^2}} \).

解
\[
\lim_{x \to 0} \frac{\ln(\sqrt{x+e^x})}{\sqrt{x^2+y^2}} = \frac{\ln(1+e^0)}{1} = \ln 2.
\]

3193. 若 \( x = \rho \cos \varphi \), \( y = \rho \sin \varphi \), 问下列极限沿怎样的方向 \( \varphi \) 有确定的极限值存在:

(a) \( \lim_{\rho \to 0} e^{\frac{x}{\sqrt{x^2+y^2}}} \);  (b) \( \lim_{\rho \to 0} e^{\frac{x^2-y^2}{\sin 2\rho y}} \).

解
(a) \( \lim_{\rho \to 0} e^{\frac{x}{\sqrt{x^2+y^2}}} = \lim_{\rho \to 0} e^{\frac{\cos \varphi}{\rho}} \)

\[
= \begin{cases} 
0, & \text{当} \cos \varphi \leq 0; \\
1, & \text{当} \cos \varphi = 0; \\
+\infty, & \text{当} \cos \varphi > 0.
\end{cases}
\]

于是，仅当 \( \cos \varphi \leq 0 \) 即 \( \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \) 时，所给的极限
才有确定的值。

(6) \( e^{x^2 - y^2} \sin 2xy = e^{\rho^2 \cos 2\phi} \sin(\rho^2 \sin 2\phi) \).

当 \( \rho \to +\infty \) 时，\( \sin(\rho^2 \sin 2\phi) \) 有界，除 \( \phi = \frac{k\pi}{2} \)

\( (k = 0, 1, 2, 3) \) 外无极值，且

\[
\lim_{\rho \to +\infty} e^{\rho \cos 2\phi} = \begin{cases} 
0, & \text{当 } \cos 2\phi < 0; \\
1, & \text{当 } \cos 2\phi = 0; \\
+\infty, & \text{当 } \cos 2\phi > 0.
\end{cases}
\]

于是，仅当 \( \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4} \) 及 \( \frac{5\pi}{4} \leq \phi \leq \frac{7\pi}{4} \) 以及 \( \phi = 0, \phi = \pi \) 时才有确定的极值。

求下列函数的不连续点，

3194. \( u = \frac{1}{\sqrt{x^2 + y^2}} \).

解 函数 \( u = \frac{1}{\sqrt{x^2 + y^2}} \) 在点 \((0, 0)\) 无定义，故原点 \((0, 0)\) 为此函数的不连续点。以下各题类似情况，不再说明。

3195. \( u = \frac{xy}{x + y} \).

解 直线 \( x + y = 0 \) 上的一切点均为 \( u = \frac{xy}{x + y} \) 的不连续点。

3196. \( u = \frac{x + y}{x^2 + y^2} \).

解 对于任意不等于零的实数 \( a \)，有
\[
\lim_{x \to a \atop y \to b} \frac{x+y}{x^3+y^3} = \lim_{x \to a \atop y \to b} \frac{1}{x^2-xy+y^2} = \frac{1}{3a^2}.
\]

于是，对于直线 \(x+y=0\) 上除去原点 \(O\) 外的一切点均为可移去的不连续点。而原点 \(O(0,0)\) 为无穷型不连续点。

3197. \(u = \sin \frac{1}{xy}\).

解  \(xy=0\) 上的一切点即两坐标轴上的诸点均为 \(u = \sin \frac{1}{xy}\) 的不连续点。

3198. \(u = \frac{1}{\sin x \sin y}\).

解  直线 \(x=m\pi\) 及 \(y=n\pi\) (\(m, n = 0, \pm 1, \pm 2, \ldots\))上的各点均为 \(u = \frac{1}{\sin x \sin y}\) 的不连续点。

3199. \(u = \ln(1-x^2-y^2)\).

解  圆周 \(x^2+y^2=1\) 上各点是 \(u = \ln(1-x^2-y^2)\) 的不连续点。

3200. \(u = \frac{1}{xyz}\).

解  坐标面：\(x=0\), \(y=0\), \(z=0\) 上各点均为 \(u = \frac{1}{xyz}\) 的不连续点。

3201. \(u = \ln \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}\).
解：点 \((a, b, c)\) 为 

\[ u = \ln \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} \]

的不连续点。

3202. 证明：函数

\[ f(x, y) = \begin{cases} 
\frac{2xy}{x^2 + y^2}, & \text{若 } x^2 + y^2 \neq 0; \\
0, & \text{若 } x^2 + y^2 = 0,
\end{cases} \]

分别对于每一个变数 \(x\) 或 \(y\)（当另一变数的值固定时）是连续的，但并非对这些变数的总体是连续的。

证：先固定 \(y = a \neq 0\)，则得 \(x\) 的函数

\[ g(x) = f(x, a) = \begin{cases} 
\frac{2ax}{x^2 + a^2}, & x \neq 0; \\
0, & x = 0,
\end{cases} \]

即 \(g(x) = \frac{2ax}{x^2 + a^2} (-\infty < x < +\infty)\)，它是处处有定义的有理函数，又当 \(y = 0\) 时，\(f(x, 0) = 0\)，它显然是连续的。于是，当变数 \(y\) 固定时，函数 \(f(x, y)\) 对于变数 \(x\) 是连续的。同理可证，当变数 \(x\) 固定时，函数 \(f(x, y)\) 对于变数 \(y\) 是连续的。

作为二元函数，\(f(x, y)\) 虽在除点 \((0, 0)\) 外的各点均连续，但在点 \((0, 0)\) 不连续。事实上，当动点 \(P(x, y)\) 沿射线 \(y = kx\) 趋于原点时，有

\[ \lim_{x \to 0} f(x, y) = \lim_{x \to 0} \frac{2kx^2}{x^2(1 + k^2)} = \frac{2k}{1 + k^2}, \]

对于不同的 \(k\) 可得不同的极限值，从而知 \(\lim_{x \to 0} f(x, y)\) 不存在。因此，函数 \(f(x, y)\) 在原点不是二元连续。
3203. 证明：函数

\[ f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{若} x^2 + y^2 \neq 0, \\ 0, & \text{若} x^2 + y^2 = 0, \end{cases} \]

在点 \( O(0, 0) \) 沿着过此点的每一射线

\[ x = t \cos \alpha, \ y = t \sin \alpha \ (0 \leq t < +\infty) \]

连续，即

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0, 0); \]

但此函数在点 \((0, 0)\) 并非连续的。

证 当 \( \sin \alpha = 0 \) 时，\( \cos \alpha = 1 \) 或 \(-1\)。于是，当 \( t \neq 0 \) 时，

\[ f(t \cos \alpha, t \sin \alpha) = \frac{t^2 \cdot 0}{t^4 + 0} = 0, \quad \text{而} \quad f(0, 0) = 0, \]

故有

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0, 0). \]

当 \( \sin \alpha \neq 0 \) 时，有

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = \lim_{t \to 0} \frac{t \cos^2 \alpha \sin \alpha}{t^4 \cos^4 \alpha + t^2 \sin^2 \alpha} = \lim_{t \to 0} \frac{t \cos^2 \alpha \sin \alpha}{t^2 \cos^4 \alpha + \sin^2 \alpha} = 0, \]

故

\[ \lim_{t \to 0} f(t \cos \alpha, t \sin \alpha) = f(0, 0). \]

其次，设动点 \( P(x, y) \) 沿抛物线 \( y = x^2 \) 趋于原点，得

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \neq f(0, 0). \]

因此，函数 \( f(x, y) \) 在点 \((0, 0)\) 不连续。
3204. 证明：函数

\[ f(x, y) = x \sin \frac{1}{y}, \text{若 } y \neq 0 \text{ 及 } f(x, 0) = 0 \]

的不连续点的集合不是封闭的。

证 当 \( y_0 \neq 0 \) 时，函数 \( f(x, y) \) 在点 \((x_0, y_0)\) 显见是连续的，即 \( f(x, y) \) 在除去 \( Ox \) 轴以外的一切点均连续。又因 \(|f(x, y) - f(0, 0)| = |f(x, y)| \leq |x|\)，故知 \( f(x, y) \) 在原点也是连续的。

考虑当 \( x_0 \neq 0 \) 时，对于点 \((x_0, 0)\)，由于极限

\[ \lim_{y \to 0} f(x_0, y) = \lim_{y \to 0} x_0 \sin \frac{1}{y} \]

不存在，故知 \( f(x, y) \) 在点 \((x_0, 0)\) 不连续。

这样，我们证明了，函数 \( f(x, y) \) 的全部不连续点为 \( Ox \) 轴上除去原点外的一切点。显然，原点是不连续点集合的一个点，但它本身却不是 \( f(x, y) \) 的不连续点。因此，\( f(x, y) \) 的不连续点的集合不是封闭的。

3205. 证明：若函数 \( f(x, y) \) 在某域 \( G \) 内对变数 \( x \) 是连续的，而关于 \( x \) 对变数 \( y \) 是一致连续的，则此函数在所考虑的域内是连续的。

证 任意固定一点 \( P_0(x_0, y_0) \in G \)。

由于 \( f(x, y) \) 关于 \( x \) 对变数 \( y \) 一致连续，故对任给的 \( \varepsilon > 0 \)，存在 \( \delta_1 = \delta_1(\varepsilon) > 0 \)，使当 \((x, y') \in G, (x, y'') \in G \) 且 \(|y' - y''| < \delta_1 \) 时，就有

\[ |f(x, y') - f(x, y'')| < \frac{\varepsilon}{2}. \]
又因 \( f(x, y) \) 在点 \((x_0, y_0)\) 关于变数 \(x\) 是连续的，
故对上述的 \(\varepsilon\)，存在 \(\delta_2 \geq 0\)，使当 \(|x - x_0| < \delta_2\) 时，
就有
\[ |f(x, y_0) - f(x_0, y_0)| \leq \frac{\varepsilon}{2}. \]

取 \(0 < \delta \leq \min \{\delta_1, \delta_2\}\)，并使点 \((x_0, y_0)\) 的 \(\delta\) 邻域
全部包含在区域 \(G\) 内，则当点 \(P(x, y)\) 属于点 \((x_0, y_0)\)
的 \(\delta\) 邻域，即 \(|PP_0| < \delta\) 时，
\[ |x - x_0| < \delta \leq \delta_2, \quad |y - y_0| < \delta \leq \delta_1. \]
从而有
\[
|f(x, y) - f(x_0, y_0)| \leq |f(x, y) - f(x, y_0)| + |f(x, y_0) - f(x_0, y_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
因此，\(f(x, y)\) 在点 \(P_0\) 连续。由 \(P_0\) 的任意性知，函数
\(f(x, y)\) 在 \(G\) 内是连续的。

3206. 证明：若在某域 \(G\) 内函数 \(f(x, y)\) 对变数 \(x\) 是连续的，
并满足对变数 \(y\) 的里普什兹条件，即
\[ |f(x, y') - f(x, y'')| \leq L|y' - y''|, \]
式中 \((x, y') \in G, (x, y'') \in G\) 而 \(L\) 为常数，则此函数在
已知域内是连续的。
证 由于 \(f(x, y)\) 在 \(G\) 内满足对 \(y\) 的里普什兹条件，
故知 \(f(x, y)\) 在 \(G\) 内关于 \(x\) 对变数 \(y\) 是一致连续的。
因此，由 3205 题的结果，即知 \(f(x, y)\) 在 \(G\) 内是连续的。

3207. 证明：若函数 \(f(x, y)\) 分别地对每一个变数 \(x\) 和 \(y\) 是
连续的并对于其中的一个是单调的，则此函数对两个变数的总体是连续的（尤格定理）。

证 不妨设 \( f(x, y) \) 关于 \( x \) 是单调的。

设 \( (x_0, y_0) \) 为函数 \( f(x, y) \) 的定义域 \( G \) 内的任一点。由于 \( f(x, y) \) 关于 \( x \) 连续，故对任给的 \( \varepsilon > 0 \)，存在 \( \delta_1 > 0 \)（假定 \( \delta_1 \) 足够小，使我们所考虑的点都落在 \( G \) 内），使当 \( |x - x_0| \leq \delta_1 \) 时，就有

\[
|f(x_0, y_0) - f(x_0, y_0)| \leq \frac{\varepsilon}{2}.
\]

对于点 \( (x_0 - \delta_1, y_0) \) 及 \( (x_0 + \delta_1, y_0) \)，由于 \( f(x, y) \) 关于 \( y \) 连续，故对上述的 \( \varepsilon \)，存在 \( \delta_2 > 0 \)（也要求 \( \delta_2 \) 足够小，使所考虑的点落在 \( G \) 内），使当 \( |y - y_0| \leq \delta_2 \) 时，就有

\[
|f(x_0 - \delta_1, y) - f(x_0 - \delta_1, y_0)| \leq \frac{\varepsilon}{2}
\]

及

\[
|f(x_0 + \delta_1, y) - f(x_0 + \delta_1, y_0)| \leq \frac{\varepsilon}{2}.
\]

令 \( \delta = \min \{\delta_1, \delta_2\} \)，则当 \( |\Delta x| \leq \delta, |\Delta y| \leq \delta \) 时，

由于 \( f(x, y) \) 关于 \( x \) 单调，故有

\[
|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)| \\
\leq \max \{|f(x_0 + \delta_1, y_0 + \Delta y) - f(x_0, y_0)|, \|f(x_0 - \delta_1, y_0 + \Delta y) - f(x_0, y_0)|\|\}.
\]

但是

\[
|f(x_0 \pm \delta_1, y_0 + \Delta y) - f(x_0, y_0)| \\
\leq |f(x_0 \pm \delta_1, y_0 + \Delta y) - f(x_0 \pm \delta_1, y_0)| \\
+ |f(x_0 \pm \delta_1, y_0) - f(x_0, y_0)|
\]
\[ \frac{e}{2} + \frac{e}{2} = e, \]

故当 \(|\Delta x| \leq \delta, |\Delta y| \leq \delta\) 时，就有

\[ \left| f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \right| \leq \epsilon, \]

即 \(f(x, y)\) 在点 \((x_0, y_0)\) 是连续的。由点 \((x_0, y_0)\) 的任意性知，\(f(x, y)\) 是 \(G\) 内的二元连续函数。

3208. 设函数 \(f(x, y)\) 于域 \(a \leq x \leq A; b \leq y \leq B\) 上是连续的，
而函数列 \(\varphi_n(x)\) (\(n = 1, 2, \ldots\)) 在 \([a, A]\) 上一致收敛并满足条件 \(b \leq \varphi_n(x) \leq B\)。证明：函数列

\[ F_n(x) = f(x, \varphi_n(x)) \]

也在 \([a, A]\) 上一致收敛。

证：由于 \(b \leq \varphi_n(x) \leq B\) ，故 \(F_n(x) = f(x, \varphi_n(x))\) 有意义。

由题设 \(f(x, y)\) 在域 \(a \leq x \leq A, b \leq y \leq B\) 上连续，
故在此域上一致连续。即对任给的 \(\epsilon > 0\)，存在 \(\delta = \delta(\epsilon) > 0\) ，使对于此域中的任意两点 \((x_1, y_1),
(x_2, y_2)\)，只要 \(|x_1 - x_2| \leq \delta, |y_1 - y_2| \leq \delta\) 时，就有

\[ \left| f(x_1, y_1) - f(x_2, y_2) \right| \leq \epsilon. \]

特别地，当 \(|y_1 - y_2| \leq \delta\) 时，对于一切的 \(x \in [a, A]\)，
均有

\[ \left| f(x, y_1) - f(x, y_2) \right| \leq \epsilon. \]

对于上述的 \(\delta = \delta(\epsilon) > 0\)，因为 \(\varphi_n(x)\) 在 \([a, A]\) 上一致收敛，
故存在自然数 \(N\)，使当 \(m > N, n > N\) 时，对于一切的 \(x \in [a, A]\)，均有

\[ \left| \varphi_n(x) - \varphi_m(x) \right| \leq \delta. \]

于是，对任给的 \(\epsilon > 0\) ，存在自然数 \(N\)，使当 \(m > N, n > N\) 时，
均有

\[ \left| \varphi_n(x) - \varphi_m(x) \right| \leq \delta. \]
$N, n \geq N$ 时，对于一切的 $x \in [a, A)$，均有
\[ |F_n(x) - F_n(x)| = |f(x, \varphi_n(x)) - f(x, \varphi_n(x))| < \varepsilon. \]
因此，$F_n(x)$ 在 $(a, A)$ 上一致收敛。

3209. 设：1）函数 $f(x, y)$ 于域 $R (a \leq x < A; b \leq y < B)$ 内是连续的；2）函数 $\varphi(x)$ 于区间 $(a, A)$ 内连续并有属于区间 $(b, B)$ 内的值。证明：函数
\[ F(x) = f(x, \varphi(x)) \]
于区间 $(a, A)$ 内是连续的。
证 设点 $(x_0, y_0)$ 为域 $R$ 中的任一点。由题设知函数 $f(x, y)$ 于域 $R$ 中连续，故对任给的 $\varepsilon > 0$，存在 $\delta > 0$，使当 $|x - x_0| < \delta, |y - y_0| < \delta ((x, y) \in R)$ 时，就有
\[ |f(x, y) - f(x_0, y_0)| < \varepsilon. \]
再由 $\varphi(x)$ 在 $(a, A)$ 中的连续性可知，对上述的 $\delta > 0$，存在 $\eta > 0$（可取 $\eta = \delta$），使 当 $|x - x_0| < \eta (x \in (a, A))$ 时，恒有
\[ |\varphi(x) - \varphi(x_0)| = |y - y_0| < \delta. \]
于是，
\[ |f(x, \varphi(x)) - f(x_0, \varphi(x_0))| < \varepsilon, \]
即
\[ |F(x) - F(x_0)| < \varepsilon. \]
因此，$F(x)$ 在点 $x_0$ 处连续。由 $x_0$ 的任意性知函数 $F(x)$ 在 $(a, A)$ 内是连续的。

3210. 设：1）函数 $f(x, y)$ 于域 $R (a \leq x < A; b \leq y < B)$ 内是连续的；2）函数 $x = \varphi(u, v)$ 及 $y = \psi(u, v)$ 于域 $R'$
\[(a' \leq u \leq A';\; b' \leq v \leq B')\] 内是连续的并有分别属于区间 \((a, A)\) 和 \((b, B)\) 的值。证明：函数
\[F(u, v) = f(\varphi(u, v), \psi(u, v))\]
于域 \(R'\) 内连续。
证 以下假定所取的 \(\delta\) 或 \(\eta\) 足够小，使点的 \(\delta\) 或 \(\eta\) 邻域都在所给的域内。
设点 \((x_0, y_0)\) 为域 \(R\) 中的任一点。由于 \(f(x, y)\) 在 \(R\) 内连续，故对任给的 \(\varepsilon > 0\)，存在 \(\delta > 0\)，使当
\[|x - x_0| \leq \delta,\; |y - y_0| \leq \delta\] 时，就有
\[|f(x, y) - f(x_0, y_0)| \leq \varepsilon.\]
再由 \(\varphi\) 及 \(\psi\) 的连续性知，对于上述的 \(\delta\)，存在 \(\eta > 0\)，使当
\[|u - u_0| \leq \eta,\; |v - v_0| \leq \eta\] 时，就有
\[|x - x_0| \leq \delta,\; |y - y_0| \leq \delta,\]
其中 \(x_0 = \varphi(u_0, v_0),\; y_0 = \psi(u_0, v_0)\)。
于是，对任给的 \(\varepsilon > 0\)，存在 \(\eta > 0\)，使当 \(|u - u_0| \leq \eta,\; |v - v_0| \leq \eta\) 时，就有
\[|f(\varphi(u, v), \psi(u, v)) - f(\varphi(u_0, v_0), \psi(u_0, v_0))| \leq \varepsilon,\]
即
\[|F(u, v) - F(u_0, v_0)| \leq \varepsilon.\]
因此，\(F(u, v)\) 在点 \((u_0, v_0)\) 连续，由 \((u_0, v_0)\) 的任意性知，函数 \(F(u, v)\) 于域 \(R'\) 内连续。

§2. 偏导函数。多变量函数的微分

1° 偏导函数  若所论及的多变数的函数的一切偏导函
数是连续的，则微分的结果与微分的次序无关。

2° 多变量函数的微分 若自变量 $x, y, z$ 的函数 $f(x, y, z)$ 的全增量可写为下形

$$
\Delta f(x, y, z) = A \Delta x + B \Delta y + C \Delta z + o(\rho),
$$

式中 $A, B, C$ 与 $\Delta x, \Delta y, \Delta z$ 无关而 $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$，则称函数 $f(x, y, z)$ 可微分，而增量的线性主部 $A \Delta x + B \Delta y + C \Delta z$ 等于

$$
df(x, y, z) = f_x'(x, y, z) \Delta x + f_y'(x, y, z) \Delta y + f_z'(x, y, z) \Delta z,
$$

(其中 $dx = \Delta x, dy = \Delta y, dz = \Delta z$) 称为此函数的微分。

当变数 $x, y, z$ 为其他自变数的可微分的函数时，公式(1)仍有其意义。

若 $x, y, z$ 为自变数，则对于高阶的微分，有符号公式

$$
d^n f(x, y, z) = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^n f(x, y, z).
$$

3° 复合函数的导函数 若 $w = f(x, y, z)$，其中 $x = \varphi(u, v)$, $y = \psi(u, v)$, $z = \chi(u, v)$ 且函数 $\varphi, \psi, \chi$ 可微分，则

$$
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u},
$$

$$
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}.
$$

计算函数 $w$ 的二阶导函数时最好用下列符号公式

$$
\frac{\partial^2 w}{\partial u^2} = \left( P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z} \right)^2 w + \frac{\partial P_1}{\partial u} \frac{\partial w}{\partial x}
$$
\[ + \frac{\partial Q_1}{\partial u} \frac{\partial w}{\partial x} + \frac{\partial R_1}{\partial u} \frac{\partial w}{\partial z} \]

及 \[ \frac{\partial^2 w}{\partial u \partial v} = \left( P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z} \right) \left( P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} + R_2 \frac{\partial}{\partial z} \right) w + \frac{\partial P_1}{\partial v} \frac{\partial w}{\partial x} + \frac{\partial Q_1}{\partial v} \frac{\partial w}{\partial y} + \frac{\partial R_1}{\partial v} \frac{\partial w}{\partial z}, \]

其中 \[ P_1 = \frac{\partial x}{\partial u}, \quad Q_1 = \frac{\partial y}{\partial u}, \quad R_1 = \frac{\partial z}{\partial u} \]

及 \[ R_2 = \frac{\partial x}{\partial v}, \quad Q_2 = \frac{\partial y}{\partial v}, \quad R_2 = \frac{\partial z}{\partial v}. \]

4° 在已知方向上的导函数 若用方向余弦 \( \{ \cos \alpha', \cos \beta', \cos \gamma' \} \) 表示 \( Oxyz \) 空间的方向 \( l \)，且函数 \( u = f(x, y, z) \) 可微分，则沿方向 \( l \) 的导函数按下式来计算

\[ \frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos \alpha' + \frac{\partial u}{\partial y} \cos \beta' + \frac{\partial u}{\partial z} \cos \gamma'. \]

在已知点函数增加最迅速的速度之大小与方向 用矢量函数的梯度

\[ \text{grad } u = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} + \frac{\partial u}{\partial z} \mathbf{k} \]

来表示，它的大小等于

\[ |\text{grad } u| = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2}. \]

3211. 证明.
\[ f'_x(x, b) = \frac{d}{dx}[f(x, b)]. \]

证 令 \( \varphi(x) = f(x, b) \)，则

\[ \frac{d}{dx}[f(x, b)] = \varphi'(x) = \lim_{dx \to 0} \frac{\varphi(x + dx) - \varphi(x)}{dx} = \lim_{dx \to 0} \frac{f(x + dx, b) - f(x, b)}{dx} = f'_x(x, b). \]

注 在求某一固定点的导数及微分时，用本题的结果常可减少运算量。在本节中，我们就多次利用本题的结果来简化运算。

3212. 设

\[ f(x, y) = x + (y - 1) \arcsin \sqrt{\frac{x}{y}}, \]

求 \( f'_x(x, 1) \)。

解 由于 \( f(x, 1) = x \)，故 \( f'_x(x, 1) = 1 \)。

求下列函数的一阶和二阶偏导函数:

3213. \( u = x^4 + y^4 - 4x^2y^2 \)。

解 \( \frac{\partial u}{\partial x} = 4x^3 - 8xy^2, \frac{\partial u}{\partial y} = 4y^3 - 8x^2y, \)

\[ \frac{\partial^2 u}{\partial x^2} = 12x^2 - 8y^2, \quad \frac{\partial^2 u}{\partial y^2} = 12y^2 - 8x^2, \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -16xy. \]

*) 以下各题不再写 \( \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \)，而仅写 \( \frac{\partial^2 u}{\partial x \partial y} \)

因为当它们连续时是相等的，并且在今后各题中均把
\[ \frac{\partial^2 u}{\partial x \partial y} \text{理为} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right). \]

3214. \( u = xy + \frac{x}{y} \).

解 \[ \frac{\partial u}{\partial x} = y + \frac{1}{y}, \quad \frac{\partial u}{\partial y} = x - \frac{x}{y^2}, \]
\[ \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 2x, \quad \frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{1}{y^2}. \]

3215. \( u = \frac{x}{y^2} \).

解 \[ \frac{\partial u}{\partial x} = \frac{1}{y^2}, \quad \frac{\partial u}{\partial y} = -\frac{2x}{y^3}, \]
\[ \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{6x}{y^4}, \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{2}{y^3}. \]

3216. \( u = \frac{x}{\sqrt{x^2 + y^2}} \).

解 \[ \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} - \frac{2x \cdot x}{2(x^2 + y^2)^{\frac{3}{2}}} = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}, \]
\[ \frac{\partial u}{\partial y} = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}}, \]
\[ \frac{\partial^2 u}{\partial x^2} = -\frac{3}{2} \frac{y^2}{(x^2 + y^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2x}{(x^2 + y^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{3x y^2}{(x^2 + y^2)^{\frac{5}{2}}}. \]
\[ u = \frac{x(2y^2 - x^2)}{(x^2 + y^2)^{\frac{8}{3}}} \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y} \left[ \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}} \right] = \frac{2y}{(x^2 + y^2)^\frac{3}{2}} - \frac{3y^3}{(x^2 + y^2)^\frac{6}{2}} = \frac{y(2x^2 - y^2)}{(x^2 + y^2)^\frac{8}{3}} \]

3217. \( u = x \sin(x + y) \).

解

\[ \frac{\partial u}{\partial x} = \sin(x + y) + x \cos(x + y) \]

\[ \frac{\partial u}{\partial y} = x \cos(x + y) \]

\[ \frac{\partial^2 u}{\partial x^2} = \cos(x + y) + \cos(x + y) - x \sin(x + y) = 2 \cos(x + y) - x \sin(x + y) \]

\[ \frac{\partial^2 u}{\partial y^2} = -x \sin(x + y) \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \cos(x + y) - x \sin(x + y) \]

3218. \( u = \frac{\cos x^2}{y} \).

解

\[ \frac{\partial u}{\partial x} = -\frac{2x \sin x^2}{y}, \quad \frac{\partial u}{\partial y} = -\frac{\cos x^2}{y^2} \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{2 \sin x^2 + 4x^2 \cos x^2}{y} \]
\[
\frac{\partial^2 u}{\partial y^2} = \frac{2 \cos x^2}{y^3},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{2x \sin x^2}{y^2},
\]

3219. \( u = \tan \frac{x^2}{y}. \)

解
\[
\frac{\partial u}{\partial x} = \frac{2x}{y} \sec^2 \frac{x^2}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x^2 \sec x^2}{y^2},
\]
\[
\frac{\partial^2 u}{\partial x^2} = \frac{2}{y} \sec^2 \frac{x^2}{y} + \frac{2x^2}{y^2} \cdot \sec^2 \frac{x^2}{y} \cdot \tan \frac{x^2}{y} \cdot \frac{2x}{y},
\]
\[
= \frac{2}{y} \sec^2 \frac{x^2}{y} + \frac{3x^2}{y^2} \sec^2 \frac{x^2}{y} \sin \frac{x^2}{y},
\]
\[
\frac{\partial^2 u}{\partial y^2} = \frac{2x^2}{y^3} \sec^2 \frac{x^2}{y} + \frac{2x^4}{y^4} \sec^2 \frac{x^2}{y} \sin \frac{x^2}{y},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = -\frac{2x^2}{y^3} \sec^2 \frac{x^2}{y} - \frac{4x^3}{y^4} \sec^2 \frac{x^2}{y} \sin \frac{x^2}{y},
\]

3220. \( u = x^y. \)

解
由 \( u = x^y = e^{y \ln x} \) 即得
\[
\frac{\partial u}{\partial x} = y x^{y-1}, \quad \frac{\partial u}{\partial y} = e^{y \ln x} \cdot \ln x = x^y \ln x,
\]
\[
\frac{\partial^2 u}{\partial x^2} = y(y-1) x^{y-2}, \quad \frac{\partial^2 u}{\partial y^2} = x^y \ln^2 x,
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + y x^{y-1} \ln x
\]
= x^{n-1}(1 + y \ln x) \quad (x \gg 0).

3221. \ u = \ln(x + y^2).

解
\[ \frac{\partial u}{\partial x} = \frac{1}{x + y^2}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x + y^2}, \]
\[ \frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x + y^2)^2}, \]
\[ \frac{\partial^2 u}{\partial y^2} = -\frac{2y^2}{x^2 + y^2} - \frac{2y \cdot 2y}{(x + y^2)^2} = \frac{2(x - y^2)}{(x + y^2)^2}, \]
\[ \frac{\partial^2 u}{\partial x \partial y} = -\frac{2y}{(x + y^2)^2}. \]

3222. \ u = \arctg \frac{y}{x}.

解
\[ \frac{\partial u}{\partial x} = -\frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}, \]
\[ \frac{\partial u}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}, \]
\[ \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{2xy}{(x^2 + y^2)^2}, \]
\[ \frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{x^2 + y^2} + \frac{y \cdot 2y}{(x^2 + y^2)^2} \]
\[ = -\frac{x^2 - y^2}{(x^2 + y^2)^2}. \]

3223. \ u = \arctg \frac{x + y}{1 - xy}.

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解 由776题知

\[ \arctg \frac{x+y}{1-xy} = \arctg x + \arctg y - \pi, \]

其中 \( \varepsilon = 0, 1 \) 或 \(-1\)。于是，

\[ \frac{\partial u}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{1+y^2}, \]

\[ \frac{\partial^2 u}{\partial x^2} = -\frac{2x}{(1+x^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{2y}{(1+y^2)^2}, \]

\[ \frac{\partial^2 u}{\partial x \partial y} = 0. \]

本题如不用776题的结果，直接求导数也可获解。

例如，

\[ \frac{\partial u}{\partial x} = \frac{1}{1+(\frac{x+y}{1-xy})^2} \cdot \frac{1-xy+y(x+y)}{(1-xy)^2} \]

\[ = \frac{1}{1+x^2}. \]

3224. \( u = \arcsin\frac{x}{\sqrt{x^2+y^2}}. \)

解

\[ \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{x}{\sqrt{x^2+y^2}}. \]

\[ = \frac{\sqrt{x^2+y^2}}{|y|} \cdot \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} \]

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\[
\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \left( \frac{x}{\sqrt{x^2 + y^2}} \right),
\]

\[
= \frac{x}{x^2 + y^2} \cdot \frac{y}{|y|} = -\frac{x \text{sgn} y}{x^2 + y^2},
\]

\[
\frac{\partial^2 u}{\partial x^2} = -\frac{2x|y|}{(x^2 + y^2)^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left[ -\frac{xy}{|y|(x^2 + y^2)} \right]
\]

\[
= -\frac{x|y|(x^2 + y^2) - xy\left[ \frac{|y|}{y} (x^2 + y^2) + 2y|y| \right]}{y^2(x^2 + y^2)^2}
\]

\[
= -\frac{2x|y|}{(x^2 + y^2)^2},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{|y|}{y} \frac{(x^2 + y^2) - 2y|y|}{(x^2 + y^2)^2}
\]

\[
= \frac{x^2 \text{sgn} y - y|y|}{(x^2 + y^2)^2} = \frac{(x^2 - y^2) \text{sgn} y}{(x^2 + y^2)^2} \quad (y \neq 0).
\]

*) 利用3216题的结果。

3225. \( u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \).
解

\[
\frac{\partial u}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},
\]

\[
\frac{\partial u}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},
\]

\[
\frac{\partial u}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},
\]

\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{\frac{7}{2}}}
\]

\[
= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{7}{2}}},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{3xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},
\]

利用对称性，即得

\[
\frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \frac{\partial^2 u}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},
\]

\[
\frac{\partial^2 u}{\partial y \partial z} = \frac{3yz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},
\]

\[
\frac{\partial^2 u}{\partial z \partial x} = \frac{3xz}{(x^2 + y^2 + z^2)^{\frac{5}{2}}},
\]

3226. \ u = \left( \frac{x}{y} \right)^{\prime}.

\[
\frac{\partial u}{\partial x} = x^2 y^{-2}.
\]
\[
\frac{\partial u}{\partial x} = z x^{t-1} y^{-z} = \frac{z}{x} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial u}{\partial y} = -z x^t y^{-z-1} = -\frac{z}{y} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial u}{\partial z} = \left( \frac{x}{y} \right)^z \ln \frac{x}{y},
\]
\[
\frac{\partial^2 u}{\partial x^2} = z(z-1) x^{t-2} y^{-z} = \frac{z(z-1)}{x^2} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial^2 u}{\partial y^2} = (-z)(-z-1) x^t y^{-z-2} = \frac{z(z+1)}{y^2} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial^2 u}{\partial z^2} = \left( \frac{x}{y} \right)^z \ln^2 \frac{x}{y},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{(z u_x)'}{y} = \frac{z}{x} \left[ -\frac{z}{y} \left( \frac{x}{y} \right)^z \right]
\]
\[
= -\frac{z^2}{xy} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial^2 u}{\partial y \partial z} = \left( \frac{z u_y}{} \right)' = -\frac{z}{y} \left( \frac{x}{y} \right)^z \ln \frac{x}{y} - \frac{1}{y} \left( \frac{x}{y} \right)^z
\]
\[
= \frac{1 + z \ln \frac{x}{y}}{y} \left( \frac{x}{y} \right)^z,
\]
\[
\frac{\partial^2 u}{\partial z \partial x} = \left( u \ln \frac{x}{y} \right)' = \frac{z}{x} \left( \frac{x}{y} \right)^z \ln \frac{x}{y} + \frac{1}{x} \left( \frac{x}{y} \right)^z
\]
\[
= \frac{1 + z \ln \frac{x}{y}}{x} \left( \frac{x}{y} \right)^z \left( \frac{x}{y} \gg 0 \right),
\]
3227. \( u = x^y \).

解

\[
\frac{\partial u}{\partial x} = \frac{y}{z} x^{y-1} = \frac{yu}{xz},
\]

\[
\frac{\partial u}{\partial y} = \frac{1}{z} x^y \ln x = \frac{u \ln x}{z},
\]

\[
\frac{\partial u}{\partial z} = -\frac{y}{z^2} x^y \ln x = -\frac{yu \ln x}{z^2},
\]

\[
\frac{\partial^2 u}{\partial x^2} = \frac{xyz \frac{\partial u}{\partial x} - yzu}{x^2 z^2} = \frac{y(y-z)u}{x^2 z^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\ln x}{z} \frac{\partial u}{\partial y} = \frac{u \ln x}{z^2},
\]

\[
\frac{\partial^2 u}{\partial z^2} = -\frac{yu \ln x}{z^4} \left[ \frac{z^2 \frac{\partial u}{\partial z} - 2uz}{z^4} \right],
\]

\[
= -\frac{yu \ln x \cdot (2z + y \ln x)}{z^4},
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{xz} \left( u + y \frac{\partial u}{\partial y} \right) = \frac{u(z + y \ln x)}{xz^2},
\]

\[
\frac{\partial^2 u}{\partial y \partial z} = \ln x \cdot \left( \frac{1}{z} \frac{\partial u}{\partial z} - \frac{u}{z^2} \right)
\]

\[
= -\frac{u \ln x \cdot (z + y \ln x)}{z^3},
\]

\[
\frac{\partial^2 u}{\partial z \partial x} = -\frac{y}{z^2} \left( \ln x \frac{\partial u}{\partial x} + \frac{u}{x} \right) = -\frac{yu(z + y \ln x)}{xyz}.
\]
解 \[ \frac{\partial u}{\partial x} = y^x x^{y^x - 1} = \frac{uy^x}{x}, \]

\[ \frac{\partial u}{\partial y} = z y^{x-1} x^{y^x} \ln x = zu y^{x-1} \ln x, \]

\[ \frac{\partial u}{\partial z} = x^{y^x} y^x \ln x \cdot \ln y = uy^x \ln x \cdot \ln y, \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{dy}{y^x} \left( -\frac{u}{x^2} + \frac{1}{x} \frac{\partial u}{\partial x} \right) = \frac{uy^x (y^x - 1)}{x^2}, \]

\[ \frac{\partial^2 u}{\partial y^2} = 2 \ln x \cdot \left( y^{x-1} \frac{\partial u}{\partial y} + (x-1) y^{x-2} u \right) \]
\[ = uz y^{x-2} \ln x \cdot (z y^x \ln x - z - 1), \]

\[ \frac{\partial^2 u}{\partial z^2} = (y^x \frac{\partial u}{\partial z} + uz y^x \ln y) \ln x \cdot \ln y \]
\[ = uy^x \ln x \cdot \ln^2 y \cdot (1 + y^x \ln x), \]

\[ \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{x} \left( y^x \frac{\partial u}{\partial y} + uz y^{x-1} \right) \]
\[ = \frac{uz y^{x-1} (y^x \ln x + 1)}{x}, \]

\[ \frac{\partial^2 u}{\partial y \partial z} = \left( y^{x-1} u + uz y^{x-1} \ln y + z y^{x-1} \frac{\partial u}{\partial z} \right) \ln x \]
\[ = uy^{x-1} \ln x \cdot (1 + z y^x \ln y \cdot (1 + y^x \ln x)), \]
\[
\frac{\partial^2 u}{\partial x \partial y} = y^2 \ln y \cdot \left( \frac{\partial u}{\partial x} \ln x + \frac{u}{x} \right)
\]

\[= \frac{uy^2 \ln y \cdot (y^2 \ln x + 1)}{x} \quad (x > 0, y > 0).
\]

3228. 设 (a) \( u = x^2 - 2xy - 3y^2 \); (b) \( u = x^2 \); (c) \( u = \arccos \sqrt{\frac{x}{y}} \)，验证等式

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
\]

证 (a) \( \frac{\partial u}{\partial x} = 2x - 2y \), \( \frac{\partial u}{\partial y} = -2x - 6y \),

\[
\frac{\partial^2 u}{\partial x \partial y} = -2, \quad \frac{\partial^2 u}{\partial y \partial x} = -2,
\]

于是，\( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \).

(b) \( \frac{\partial u}{\partial x} = y^2 x^{n-1} \), \( \frac{\partial u}{\partial y} = 2y x^n \ln x \quad (x > 0) \),

\[
\frac{\partial^2 u}{\partial x \partial y} = 2y x^n x^{n-1} + 2y x^n x^{n-1} \ln x,
\]

\[
\frac{\partial^2 u}{\partial y \partial x} = 2y x^n x^{n-1} \ln x + 2y x^{n-1},
\]

于是，\( \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \).
(b) 当 $0 < x \leq y$ 时，我们有

$$
u = \arccos \sqrt{\frac{x}{y}} = \arccos \sqrt{\frac{x}{y}}.
$$

$$
\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{1 - \frac{x}{y}}} \cdot \frac{1}{2 \sqrt{x \sqrt{y}}} = -\frac{1}{2 \sqrt{x (y-x)}},
$$

$$
\frac{\partial u}{\partial y} = -\frac{1}{\sqrt{1 - \frac{x}{y}}} \left(-\frac{\sqrt{y}}{2y^2}\right) = \frac{\sqrt{y}}{2 \sqrt{y^2 (y-x)}},
$$

$$
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4 \sqrt{x (y-x)^\frac{3}{2}}},
$$

$$
\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{4 \sqrt{x} \sqrt{y^2 (y-x)}} + \frac{\sqrt{y}}{4y (y-x)^\frac{3}{2}}
$$

$$
= \frac{1}{4 \sqrt{x} (y-x)^\frac{3}{2}},
$$

于是，当 $0 < x \leq y$ 时，有

$$
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
$$

当 $y \leq x < 0$ 时，$u = \arccos \sqrt{-\frac{x}{\sqrt{-y}}}$. 

$$
\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{1 - \frac{x}{y}}} \left(-\frac{1}{2 \sqrt{-x \sqrt{-y}}}\right)
$$
\[
\frac{1}{2 \sqrt{1 - \frac{x}{y}} y^3} = \frac{1}{\sqrt{1 - \frac{x}{y}}} \frac{1}{2(-y)^\frac{3}{2}} = \frac{\sqrt{1 - \frac{x}{y}}}{2 \sqrt{xy^2 - y^3}}
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4 \sqrt{1 - \frac{x}{y}} (x - y)^\frac{3}{2}}
\]

\[
\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{4 \sqrt{1 - \frac{x}{y}} x y^2 - y^3} + \frac{\sqrt{1 - \frac{x}{y}}}{4 \sqrt{y^2} (x + y)^\frac{3}{2}}
\]

\[
= \frac{1}{4 \sqrt{1 - \frac{x}{y}} (x - y)^\frac{3}{2}}
\]

于是，当 \( y \leq x < 0 \) 时，也有

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.
\]

仔细观察可以看到，在不同的区域上，一阶偏导数相差一个符号，但二阶混合偏导数却是相等的。

3230. 设 \( f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2} \)，若 \( x^2 + y^2 \neq 0 \) 及 \( f(0, 0) = 0 \)。证明

\[
f''_{xx}(0, 0) \neq f''_{yy}(0, 0).
\]

证 由于

\[
\lim_{x \to 0} f(x, y) - f(0, 0) = \lim_{x \to 0} \frac{xy}{x} \frac{x^2 - y^2}{x^2 + y^2} - 0 = -y,
\]

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故 $f'_x(0, y) = -y$，从而

$$f''_{xy}(0, 0) = \frac{d}{dy} \left[ f'_x(0, y) \right] \bigg|_{y=0} = -1$$

同法可求得 $f'_y(x, 0) = x$，从而

$$f''_{yx}(0, 0) = \frac{d}{dx} \left[ f'_y(x, 0) \right] \bigg|_{x=0} = 1.$$ 于是，$f''_{xy}(0, 0) \neq f''_{yx}(0, 0)$.

3231. 设 $u = f(x, y, z)$ 为 $n$ 次齐次函数，就下列各题验证关于齐次函数的尤拉定理:

(a) $u = (x - 2y + 3z)^2$；(b) $u = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$；

(b) $u = (\frac{x}{y})^\frac{1}{3}$.

证 关于 $n$ 次齐次函数的尤拉定理如下:

设 $n$ 次齐次函数 $f(x, y, z)$ 在域 $A$ 中关于所有变量均有连续偏导函数，则下述等式成立

$$\begin{align*}
x f'_x(x, y, z) + y f'_y(x, y, z) + z f'_z(x, y, z) \\
eq n f(x, y, z).
\end{align*}$$

(a) 由于 $(tx - 2ty + 3tz)^2 = t^2 u$，故 $u$ 为二次齐次函数。又因

* 为了书写的简便，在这里我们仅限于讨论三个变量的情形。
\[
\frac{\partial u}{\partial x} = 2(x - 2y + 3z), \quad \frac{\partial u}{\partial y} = -4(x - 2y + 3z), \\
\frac{\partial u}{\partial z} = 6(x - 2y + 3z),
\]

故得
\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = (x - 2y + 3z)(2x - 4y + 6z) = 2u,
\]
即函数 \( u \) 满足尤拉定理。

（6）由于对任何的 \( t \geq 0 \),
\[
\frac{tx}{\sqrt{(tx)^2 + (ty)^2 + (tz)^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = t^0 \cdot u,
\]
故 \( u \) 为零次齐次函数。又因
\[
\frac{\partial u}{\partial x} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \quad \frac{\partial u}{\partial y} = -\frac{xy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \\
\frac{\partial u}{\partial z} = -\frac{xz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},
\]
故得
\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 1 \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} (xy^2 \\
+ xz^2 - xy^2 - xz^2) = 0 \cdot u,
\]
即函数 \( u \) 满足尤拉定理。

（6）由于
\[
\left( \frac{t x}{t y} \right)^{\frac{r}{z}} = \left( \frac{x}{y} \right)^{\frac{r}{z}} = t^0 \cdot u \quad (t>0),
\]

故函数 \(u\) 为零次齐次函数，又因

\[
\frac{\partial u}{\partial x} = \frac{1}{y} \cdot \frac{y^2}{z} \left( \frac{x}{y} \right)^{\frac{r}{z} - 1} = \frac{yu}{xz},
\]

\[
\frac{\partial u}{\partial y} = \left( e^{-\frac{y}{x}} \right) \left( \frac{x}{y} \right)^{\frac{r}{z}} \cdot \left[ \frac{1}{z} \ln \frac{x}{y} - \frac{y}{z} \cdot \frac{1}{y} \right]
\]

\[
= \frac{u}{z} \left( \ln \frac{x}{y} - 1 \right),
\]

\[
\frac{\partial u}{\partial z} = \left( \frac{x}{y} \right)^{\frac{r}{z}} \ln \frac{x}{y} \cdot \left( -\frac{y}{z^2} \right) = -\frac{yu}{z^2} \ln \frac{x}{y},
\]

故得

\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = x \cdot \frac{yu}{xz} + y \cdot \frac{u}{z} \left( \ln \frac{x}{y} - 1 \right)
\]

\[
- z \cdot \frac{yu}{z^2} \ln \frac{x}{y} = 0 \cdot u,
\]

即函数 \(u\) 满足尤拉定理。

3232. 证明：若可微函数 \(u = f(x, y, z)\) 满足方程式

\[
x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu,
\]

则它为 \(n\) 次齐次函数。

证 任意固定域中一点 \((x_0, y_0, z_0)\)，考察下面的 \(i\) 的函数 \((t>0)\)。
\[ F(t) = \frac{f(tx_0, ty_0, tz_0)}{t^n} \]

它当 \( t > 0 \) 时有定义且是可微的。应用复合函数的求导法则，对 \( t \) 求导数即得

\[
F'(t) = \frac{1}{t^n} \left\{ \begin{array}{l}
x_0 f'_x(tx_0, ty_0, tz_0) + y_0 f'_y(tx_0, ty_0, tz_0) + z_0 f'_z(tx_0, ty_0, tz_0) \\
- \frac{n}{t^{n+1}} f(tx_0, ty_0, tz_0)
\end{array} \right\}
\]

\[
= \frac{1}{t^{n+1}} \left\{ \begin{array}{l}
(tx_0 f'_x(tx_0, ty_0, tz_0) + ty_0 \\
+ y_0 f'_y(tx_0, ty_0, tz_0) + tz_0 f'_z(tx_0, ty_0, tz_0) \\
- nf(tx_0, ty_0, tz_0) \end{array} \right\},
\]

由于 \( tx_0 f'_x(tx_0, ty_0, tz_0) + ty_0 f'_y(tx_0, ty_0, tz_0) + tz_0 f'_z(tx_0, ty_0, tz_0) = nf(tx_0, ty_0, tz_0) \)

故

\[ F'(t) = 0. \]

从而当 \( t > 0 \) 时，\( F(t) = c \)，其中 \( c \) 为常数。现在确定 \( c \)。为此，在定义 \( F(t) \) 的等式中令 \( t = 1 \)，则得

\[ c = f(x_0, y_0, z_0). \]

于是，
\[ F(t) = \frac{f(tx_0, ty_0, tz_0)}{t^n} = f(x_0, y_0, z_0), \]

即

\[ f(tx_0, ty_0, tz_0) = t^n f(x_0, y_0, z_0). \]

上式说明函数 \( f(x, y, z) \) 为一个 \( n \) 次的齐次函数，这就是所要证明的。

3233. 证明：若 \( f(x, y, z) \) 是可微分的 \( n \) 次齐次函数，则其偏导函数 \( f'_x(x, y, z), f'_y(x, y, z), f'_z(x, y, z) \) 是 \( (n-1) \) 次的齐次函数。

证 由等式

\[ f(tx, ty, tz) = t^n f(x, y, z) \]

两端分别对 \( x, y, z \) 求偏导函数，则得

\[ tf'_1(tx, ty, tz) = t^n f'_1(x, y, z), \]
\[ tf'_2(tx, ty, tz) = t^n f'_2(x, y, z), \]
\[ tf'_3(tx, ty, tz) = t^n f'_3(x, y, z), \]

其中 \( f'_1(\cdot, \cdot, \cdot), f'_2(\cdot, \cdot, \cdot), f'_3(\cdot, \cdot, \cdot) \) 分别代表
\( f(\cdot, \cdot, \cdot) \) 对第一个，第二个，第三个变量的偏导数。

于是，

\[ f'_1(tx, ty, tz) = t^{n-1} f'_1(x, y, z), \]
\[ f'_2(tx, ty, tz) = t^{n-1} f'_2(x, y, z), \]
\[ f'_y(tx, ty, tz) = t'^{n-1} f'_y(x, y, z), \]

即偏导函数 \( f_x(x, y, z) \), \( f'_y(x, y, z) \) 及 \( f'_z(x, y, z) \)
均为 \((n-1)\)次的齐次函数。

3234. 设 \( u = f(x, y, z) \)是可微分两次的 \( n \) 次齐次函数。证明

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^2 u = n(n-1)u.
\]

证 由3233题知：\( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \)及\( \frac{\partial u}{\partial z} \)均为 \((n-1)\) 次齐次函数。应用尤拉定理，即得

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial x} = (n-1) \frac{\partial u}{\partial x}, \quad (1)
\]

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial y} = (n-1) \frac{\partial u}{\partial y}, \quad (2)
\]

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial z} = (n-1) \frac{\partial u}{\partial z}. \quad (3)
\]

将(1)式两端乘以 \( x \)，(2)式两端乘以 \( y \)，(3)式两端乘以 \( z \)，然后相加，即得

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)^2 u = (n-1) \left( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right) = n(n-1)u,
\]

这就是所要证明的等式。
求下列函数的一阶和二阶微分（x, y, z 为自变数）：

3235. \( u = x^n y^m \)。

解

\[
du = x^{n-1} y^{m-1} (m y dx + n x dy),
\]

\[
d^2 u = m(n-1) x^{n-2} y^{m-1} dx^2 + 2mn x^{n-1} y^{m-1} dx dy + n(n-1) x^n y^{m-2} d y^2.
\]

\[
= x^{n-2} y^{m-2} (m(m-1) y^2 d x^2 + 2mn x y d x d y + n(n-1) x^2 d y^2).
\]

3236. \( u = \frac{x}{y} \)。

解

\[
du = y d x - x d y
\]

\[
d^2 u = \frac{y^2 (d x d y - d x d y) - 2 y dy (y d x - x d y)}{y^4}
\]

\[
= -\frac{2}{y^3} (y d x - x d y) d y.
\]

3237. \( u = \sqrt{x^2 + y^2} \)。

解

\[
du = \frac{x d x + y d y}{\sqrt{x^2 + y^2}},
\]

\[
d^2 u = \frac{d(x d x + y d y)}{\sqrt{x^2 + y^2}} + (x d x + y d y)
\]

\[
\cdot d\left(\frac{1}{\sqrt{x^2 + y^2}}\right) = \frac{d x^2 + d y^2}{\sqrt{x^2 + y^2}} - (x d x + y d y)^2
\]

\[
= \frac{(y d x - x d y)^2}{(x^2 + y^2)^{\frac{3}{2}}}.
\]
3238. \( u = \ln \sqrt{x^2 + y^2} \).

解

\[
du = \frac{x dx + y dy}{x^2 + y^2},
\]

\[
d^2u = \frac{d(x dx + y dy)}{x^2 + y^2} - \frac{2(x dx + y dy)^2}{(x^2 + y^2)^2}
\]

\[
= \frac{dx^2 + dy^2}{x^2 + y^2} - \frac{2(x dx + y dy)^2}{(x^2 + y^2)^2}
\]

\[
= \frac{(y^2 - x^2)(dx^2 - dy^2) - 4x y dx dy}{(x^2 + y^2)^2}.
\]

3239. \( u = e^{xy} \).

解

\[
du = e^{xy}(y dx + x dy),
\]

\[
d^2u = e^{xy}[(y dx + x dy)^2 + 2dx dy]
\]

\[
= e^{xy}[y^2 dx^2 + 2(1 + xy) dx dy + x^2 dy^2].
\]

3240. \( u = xy + yz + zx \).

解

\[
du = (y + z) dx + (z + x) dy + (x + y) dz,
\]

\[
d^2u = 2(dx dx + d y dz + d z dx).
\]

3241. \( u = \frac{z}{x^2 + y^2} \).

解

\[
du = -\frac{2z}{(x^2 + y^2)^2} (x dx + y dy) + \frac{dz}{x^2 + y^2}
\]

\[
= \frac{(x^2 + y^2) dz - 2z(x dx + y dy)}{(x^2 + y^2)^2},
\]

\[
d^2u = \frac{1}{(x^2 + y^2)^2} \left\{ (x^2 + y^2)^2 (2(x dx + y dy) dz
\]

\[ - 2(x dx + y dy) dz - 2z(dx^2 + dy^2) \right\}.\]

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